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End-extending models of $I\Delta_0+\exp+B\Sigma_1$

by

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Abstract. Given a suitable theory T we characterize models of $I\Delta_0+\exp+B\Sigma_1$ having a proper end-extension to a model of T.

Introduction. We consider the language of arithmetic where $+$ and $\cdot$ are treated as relations. We consider the following theories T: $I\Delta_0$, $I\Delta_0+\exp$, $I\Delta_0+\Omega_n$ where Ω_n is the axiom introduced in [WP II] or $I\Delta_0+$"F_α is total" where $\alpha<\varepsilon_0$ and F_α is the αth function in the Wainer hierarchy. We assume that T is presented in the form: $I\Delta_0+\forall x\,\exists y\;\Phi(x, y)$ where Φ is a Δ_0 formula.

We prove the following theorem:

THEOREM. *There is a Π_1 formula $\varphi(x)$ such that for every $n\in N$, $T+\exp\vdash\varphi(\underline{n})$ ($\underline{n}$ denotes the appropriate numeral) and such that for every non-standard model M of $I\Delta_0+\exp+B\Sigma_1$ which is cofinal with ω the following conditions are equivalent*:

(1) *M has a proper end-extension to a model of T.*

(2) *M has a proper end-extension to a model of $T+B\Sigma_1$.*

(3) *There is a non-standard r in M such that $M\models\varphi(r)$.*

The problem of the existence of the formula φ and of the equivalence of (1)–(3) was considered in [WP I] for the theory $T=I\Delta_0$ and for countable models M of $I\Delta_0+B\Sigma_1$. The infinitary notion of "fullness" is an attempt to find a counterpart of φ in that case and some partial results towards the equivalence of (1)–(3) are proved in that paper. In our case, where we consider ground models M satisfying the exp axiom, we are able to characterize their extendability to a model of T in a very regular way — this is the content of the theorem. The same regularity occurs if instead of exp we assume that a certain strong form of the bounded Matiyasevich conjecture holds in M (see the author's forthcoming paper).

The technique of the paper is related to that of [WP I] but it is in some aspects different. For instance it enables us to deal with models cofinal with ω instead of with countable ones. On the other hand, it cannot be directly applied to models not satisfying exp. For models satisfying exp it can be treated as an alternative technique to that of [WP I]. Similar ideas were developed also in [WP II] [W], [K I], [K II].

The sense of the theorem is a bit different in the case where T is one of the theories $I\Delta_0+\Omega_n$ or $I\Delta_0$ itself than in the case where T extends $I\Delta_0+\exp$.

In the first case the conditions (1)–(3) are just true for any model M satisfying the assumptions (see § 2). Thus we get an extension of the Wilkie–Paris theorem from the case of countable M to the case of M cofinal with ω.

In the second case we get a criterion for a model to be extendable. It can be shown that there is a model which satisfies none of (1)–(3) (see [WP I]).

The most difficult part of the proof of our theorem is the construction of φ and the proof of (3)⇒(2). Let us sketch the proof. We extend our language by a new constant $\underline{a}$. Let $\varphi(x)$ be the formula $\mathrm{Con}(T_x)$ where Con is a certain semantical notion of consistency described in § 1 (related to "Tableaux consistency" studied in [WP II]) and T_x is the theory consisting of:

(i) The first x axioms of a suitable axiomatization of $I\Delta_0$,

(ii) The axiom $\exists y$ "y is the value of the xth iteration of the function defined by Φ at $\underline{a}$".

Then $\varphi(\underline{n})$ is very much related to the sentence $\mathrm{Con}(T, \underline{n})$ introduced in [WP II]. We show that it is provable in $T+\exp$. The rest of the proof is in showing that if $M \models \varphi(r)$ for a non-standard r then a version of the arithmetized completeness theorem holds in M and there is a model $M' \supseteq_e M$ with $M < a \in M'$ such that the rth iteration of the function determined by Φ at a exists in M' (e.g. $2^{2^{\cdot^{\cdot^{2^a}}}}$, r times, if Φ defines exponentiation) and $M' \models I\Delta_0$. But then the initial segment M'' of M' defined by standard iterations of Φ at a (e.g. by $2^{2^{\cdot^{\cdot^{2^a}}}}$, n times, for $n \in N$) is the required model of $T+B\Sigma_1$.

The proof of the theorem is given in § 1.

Our main technical notion is the notion of l-closure of a number a w.r.t. a theory T in a model M of PA^-. This notion is related to Herbrand's theorem. Let $l \in N$. Let $M \models \mathrm{PA}^-$ and let a be an element of M. An l-closure of a w.r.t. T in M is a finite approximation to a Skolem hull of a in M w.r.t. a certain family of Skolem functions for the axioms of T and for their subformulas which is ordered in a certain very special way. We close a under the Skolem functions l times and order the set so obtained in such a way that given the index of an element in this ordering we can easily decode which subformula of which axiom the element satisfies. For technical reasons we also include a finite initial segment of M into the Skolem closure. The following simple examples are to explain how we order the closures. Later we give a precise general definition.

Assume that T consists of just one sentence of the form $\forall x \exists y\ \psi(x, y)$ where ψ is an open formula. Let $M \models \mathrm{PA}^-$ and let f denote a Skolem function (possibly partial) for $\exists y\ \psi$ in M. Let a be an element of M.

We define the 0-*closure* of a w.r.t. T in M to be $\emptyset$, the 1-*closure* to be $\langle 0, a\rangle$, the 2-*closure* to be $\langle 0, a, 1, f(0), 2, f(a)\rangle$, the 3-*closure* to be $\langle 0, a, 1, f(0), 2, f(a), 3, f(0), 4, f(a), 5, f(1), 6, f(f(0)), 7, f(2), 8, f(f(a))\rangle$. If the appropriate values of f are not defined in M then the corresponding closure does not exist.

In general, if $\langle x_1, \ldots, x_{2L}\rangle$ is the l-closure of a in M w.r.t. T then the $(l+1)$-*closure* is the sequence

$$\langle x_1, \ldots, x_{2L}, L, f(x_1), L+1, f(x_2), \ldots, L+2L-1, f(x_{2L})\rangle$$

provided that the appropriate values of f exist.

Thus, to build the l-closure we close $\{a\}$ under f and the operation of taking consecutive elements of M $(0, 1, 2, \ldots)$ $(l-1)$ times. Then we order the set so obtained by putting the consecutive numbers $0, 1, 2, \ldots$ at odd-numbered places and the appropriate values of f at even-numbered places.

It is not hard to see what is the number $L(l)$ — the half of the length of any l-closure — in this case. Indeed, we have

$$L(0) = 0, \quad L(1) = 1, \quad L(l+1) = L(l)+2L(l) \quad \text{for every } l.$$

The appropriate version of the Herbrand theorem in this case can be formulated as follows: $\mathrm{PA}^- \vdash \exists x \forall y\ \neg\psi(x, y)$ iff there is an $l \in N$ such that for every model M of PA^- and every a in M there is no l-closure of a w.r.t. T in M.

If in a model M we can make an l-closure of a for every $l \in N$ and if those closures extend one another then the union of those closures is a model of T.

The same idea works for an arbitrary theory T. Before considering the general case, let us consider another example. Assume that T consists of two sentences:

$$\varphi_1:\ \forall x_1 \exists y_1 \forall x_2 \exists y_2\ \psi_1(x_1, y_1, x_2, y_2) \quad \text{and}$$

$$\varphi_2:\ \forall x_1 \exists y_1 \forall x_2 \exists y_2\ \psi_2(x_1, y_1, x_2, y_2)$$

where ψ_1, ψ_2 are open.

Let M be a model of PA^-, $a \in M$, and assume that there are (partial) functions $f_1, f_2\colon M \to M$ and $\hat{f}_1, \hat{f}_2\colon M \times M \to M$ such that

$$M \models \forall x_2 \exists y_2\ \psi_1(x_1, f_1(x_1), x_2, y_2),$$

$$M \models \forall x_2 \exists y_2\ \psi_2(x_1, f_2(x_1), x_2, y_2)$$

for every x_1 for which f_1, f_2 are defined, and

$$M \models \psi_1(x_1, f_1(x_1), x_2, \hat{f}_1(x_1, x_2)),$$

$$M \models \psi_2(x_1, f_2(x_1), x_2, \hat{f}_2(x_1, x_2))$$

whenever $f_1, f_2, \hat{f}_1, \hat{f}_2$ are defined.

Then the 0-closure of a w.r.t. T in M is $\emptyset$, the 1-closure is $\langle 0, a\rangle$, the 2-closure is

$$\langle 0, a, 1, f_1(0), 2, f_1(a), 3, f_2(0), 4, f_2(a)\rangle,$$

the 3-closure is

$$\begin{aligned}\langle 0, a, 1, f_1(0), 2, f_1(a), 3, f_2(0), 4, f_2(a),\\ 5, f_1(0), 6, f_1(a), \ldots, 13, f_1(4), 14, f_1(f_2(a)),\\ 15, f_2(0), 16, f_2(a), \ldots, 23, f_2(4), 24, f_2(f_2(a)),\\ 25, \hat{f}_1(0, 0), 26, \hat{f}_1(0, a), 27, \hat{f}_1(0, 1), 28, \hat{f}_1(0, f_1(0)), \ldots\rangle\end{aligned}$$

To build the 3-closure we first list the 2-closure; then we keep putting the consecutive numbers at odd-numbered places, and at even-numbered ones we first put values of f_1 at consecutive terms of the 2-closure; then we put values of f_2 at those terms; then we put the values of $\hat{f}_1$ at all pairs $\langle x_i, x_j\rangle$ where x_i is the ith term of the 1-closure and x_j is the jth term of the 2-closure and the pairs $\langle i, j\rangle$ are ordered lexicographically; then we put the values of $\hat{f}_2$ at all pairs $\langle x_i, x_j\rangle$ such as before. Similarly we pass from the l-closure to the $(l+1)$-closure.

In this case $f_1, \hat{f}_1$ and $f_2, \hat{f}_2$ are Skolem functions for the axioms φ_1 and φ_2 and for their subformulas, respectively. If those functions are total in M then we can build the l-closure for every $l\in N$ and the union of all those closures is the Skolem hull of $\{a\}$ and a collection of consecutive numbers in M. It follows that such a union is a model of T.

The l-closure is the appropriate "$(l-1)$-fold" Skolem hull which is ordered in such a way that given the index i of a term x_i of the l-closure $\langle x_1, \ldots, x_{2L}\rangle$ we can tell what role x_i plays in the hull. In particular, there are recursive functions $g_1(l, 1, h)$, $g_1(l, 2, h)$, $g_2(l, 1, h_1, h_2)$, $g_2(l, 2, h_1, h_2)$ such that if h is an index of a term of the l-closure and h_1, h_2 are indexes of terms of the $(l-1)$-closure and the l-closure respectively then:

$x_{g_1(l,1,h)} = f_1(x_h)$ is the $g_1(l, 1, h)$th term of the $(l+1)$-closure,
$x_{g_1(l,2,h)} = f_2(x_h)$ is the $g_1(l, 2, h)$th term of the $(l+1)$-closure,
$x_{g_2(l,1,h_1,h_2)} = \hat{f}_1(x_{h_1}, x_{h_2})$ is the $g_2(l, 1, h_1, h_2)$th term of the $(l+1)$-closure,
$x_{g_2(l,2,h_1,h_2)} = \hat{f}_2(x_{h_1}, x_{h_2})$ is the $g_2(l, 2, h_1, h_2)$th term of the $(l+1)$-closure.

The exact form of these functions is not very important.

In this case the numbers $L(l)$ satisfy

$$L(0) = 0, \quad L(1) = 1,$$

$$L(l+1) = L(l)+2\cdot 2L(l)+2\cdot 2L(l)\cdot 2L(l-1) \quad \text{for } l \geqslant 1.$$

Moreover, g_1, g_2 are the following functions:

$$g_1(l, 1, h) = 2(L(l)+h),$$

$$g_1(l, 2, h) = 2(L(l)+2L(l)+h),$$

$$g_2(l, 1, h_1, h_2) = 2(L(l)+2\cdot 2L(l)+\langle h_1, h_2\rangle_l),$$

$$g_2(l, 2, h_1, h_2) = 2(L(l)+2\cdot 2L(l)+2L(l-1)\cdot 2L(l)+\langle h_1, h_2\rangle_l),$$

where $\langle h_1, h_2\rangle_l$ is the number of the pair $\langle h_1, h_2\rangle$ in the lexicographical ordering of pairs of numbers $\langle h_1, h_2\rangle$ such that $h_1 \leqslant 2L(l-1)$, $h_2 \leqslant 2L(l)$.

Now we pass to the general case. We define the function $L(l)$ and the functions $g_s(l, j, h_1, \ldots, h_s)$ for an arbitrary theory T. The idea has been described above and the definitions are technical.

Thus the reader may have a quick look at the next section and pass to §1.

§0. Definition 0.1. Let $\varphi(x)$ be of the form

$$\forall y_1 \exists z_1 \ldots \forall y_k \exists z_k \ \theta(x, y_1, z_1, \ldots, y_k, z_k)$$

where θ is open. Later we assume about any formula that it is in this form.

We define a *derivative* of φ in the following way: $\varphi'(x, y_1, z_1)$ is

$$\forall y_2 \exists z_2 \ldots \forall y_k \exists z_k \ \theta(x, y_1, z_1, y_2, z_2, \ldots, y_k, z_k).$$

Then $\varphi''(x, y_1, z_1, y_2, z_2)$ is

$$\forall y_3 \exists z_3 \ldots \forall y_k \exists z_k \ \theta(x, y_1, z_1, \ldots, y_k, z_k).$$

Generally, we let $\varphi^{(i)}(x, y_1, z_1, \ldots, y_i, z_i)$ be

$$\forall y_{i+1} \exists z_{i+1} \ldots \forall y_k \exists z_k \ \theta(x, y_1, z_1, \ldots, y_k, z_k) \quad \text{for } i<k$$

and

$$\theta(x, y_1, z_1, \ldots, y_k, z_k) \& z_{k+1} = \ldots = z_i = 0 \quad \text{for } i \geqslant k.$$

Definition 0.2. Let the function $L(l)$ be defined as:

$$L(0) = 0, \quad L(1) = 1,$$

$$L(l+1) = L(l)+\sum_{1\leqslant s\leqslant l} (l+1-s)\cdot 2L(l)\cdot 2L(l-1)\cdot \ldots \cdot 2L(l+1-s)$$

in the case where we are dealing with an infinite theory

$$T = \{\varphi_1, \varphi_2, \ldots\}$$

and let

$$L(0) = 0, \quad L(1) = 1,$$

$$L(l+1) = L(l)+\sum_{1\leqslant s\leqslant k} n\cdot 2L(l)\cdot 2L(l-1)\cdot \ldots \cdot 2L(l+1-s)$$

for a finite theory $T = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ where every φ_i has at most k existential quantifiers.

Moreover, we let

$$\begin{aligned} g_s(l, j, h_1, \ldots, h_s) = 2\big(&L(l)+l\cdot 2L(l)+(l-1)\cdot 2L(l-1)\cdot 2L(l) \\ &+(l-2)\cdot 2L(l-2)\cdot 2L(l-1)\cdot 2L(l)+\ldots+\big(l+1-(s-1)\big)\cdot 2L\big(l+1-(s-1)\big)\cdot \ldots \cdot 2L(l) \\ &+(j-1)\cdot 2L(l+1-s)\cdot \ldots \cdot 2L(l)+\langle h_1, \ldots, h_s\rangle_l\big) \end{aligned}$$

in the first case and

$$\begin{aligned} g_s(l, j, h_1, \ldots, h_s) = 2\big(&L(l)+n\cdot 2L(l)+n\cdot 2L(l-1)\cdot 2L(l)+n\cdot 2L(l-2)\cdot 2L(l-1)\cdot 2L(l) \\ &+\ldots+n\cdot 2L\big(l+1-(s-1)\big)\cdot \ldots \cdot 2L(l)+(j-1)\cdot 2L(l+1-s)\cdot \ldots \cdot 2L(l)+\langle h_1, \ldots, h_s\rangle_l\big) \end{aligned}$$

in the second case, where $h_1 \leqslant 2L(l+1-s), \ldots, h_s \leqslant 2L(l)$ and $1 \leqslant s \leqslant l$ in the first case and $1 \leqslant s \leqslant k$ in the second case.

Definition 0.3. Let $M \models PA^-$, $a \in M$. Let $l \in N$. A sequence $\langle x_1, \ldots, x_{2L(l)}\rangle$ of elements of M is called an *l-closure* of a w.r.t. T in M if

(1) $x_1 = 0$, $x_2 = a$,

(2) if $1 \leqslant s \leqslant l$ (or $s \leqslant k$ if T is finite), $j \leqslant l+1-s$ (or $j \leqslant n$ if T is finite), $j \leqslant i_1 \leqslant i_2 < \ldots < i_s \leqslant l$, $h_1 \leqslant 2L(i_1)$, $h_2 \leqslant 2L(i_2), \ldots, h_s \leqslant 2L(i_s)$, then

$$M \models \varphi_j^{(s)}(x_{h_1}, x_{g_1(i_1,j,h_1)}, x_{h_2}, x_{g_2(i_2,j,h_1,h_2)}, \ldots, x_{h_s}, x_{g_s(i_s,j,h_1,\ldots,h_s)}),$$

(3) if g is odd then $x_g = h$ where $g = 2h+1$.

Remark 0.1. If M is a model of T then for every $l \in N$ and every $a \in M$ there is an l-closure of a w.r.t. T in M.

Remark 0.2. If $x^1, x^2, \ldots, x^l, \ldots$ are a 1-closure, a 2-closure, ..., an l-closure, ... of a in M w.r.t. T and if

$$x^1 \subseteq x^2 \subseteq \ldots \subseteq x^l \subseteq \ldots$$

then $\bigcup_l x^l$ is a submodel of M which is a model of T.

We prove this remark similarly to the proof of the Claim in the proof of Lemma 1.3.

§1. In this section we prove the theorem.

Assume that T is given. We shall first reformulate it in the language with an additional constant $\underline{a}$.

Let $\Phi_n(x, y)$ be a Δ_0 formula such that $I\Delta_0 \vdash \forall x, y\ [\Phi_n(x, y) \equiv \exists y_1, \ldots, y_n\ (\Phi(x, y_1) \& \Phi(y_1, y_2) \& \ldots \& \Phi(y_{n-1}, y_n) \& y_n = y)]$.

In the case of the theories that we consider, Φ_n defines the nth iteration of the function in question, i.e. of the exponential function or of the ω_m-function or of F_α. If T is $I\Delta_0$ let us assume that Φ is the formula $y = x^2$.

Instead of the original theory T consider the following theory: The collection of Π_1 axioms axiomatizing the Δ_0 induction + the collection of axioms of the form "$\exists y\ \Phi_m(\underline{a}, y)$" for $m \in N$ + the collection of Π_1 axioms ensuring the existence of products and sums: "$\forall y, y'\ \forall z, z'\ ((\Phi_m(\underline{a}, y) \& \Phi_{m+1}(\underline{a}, y') \& z', z \leqslant y) \to \exists v, w \leqslant y'\ (v = z \cdot z' \& w = z + z'))$" for $m \in N$.

Let now T denote the new theory and let T_n denote the collection of the first n axioms of T — in any natural enumeration of T, T_n contains the axioms of the second and third group only for $m \leqslant n$.

Idea of the proof. Assume that we have found the suitable formula φ. The hardest part of the proof is to construct an end-extension of M under the condition (3), i.e. to deduce (1) or (2) from (3).

We look for an extension M' of M with the following property: there is an element $a \in M' \setminus M$ and a sequence $\{x^l\colon l \in M\}$ of l-closures of a in the sense of M' w.r.t. T such that $M' = \bigcup \{x^l\colon l \in M\}$. The notion of l-closure easily generalizes to the case where T is in the language with the constant $\underline{a}$ (we repeat the definitions requiring that $\underline{a}$ is interpreted as a) and this notion can be formalized in any reasonable theory T — hence we can speak about its non-standard version.

To construct M' we construct its complete diagram in M. Thus we choose in M constants to denote every element of M'. Since M' is going to consist of terms of l-closures of a for $l \in M$ and every such term has its index in M as a term of an l-closure, it is most convenient to use as constants the indexes. Hence the number 2 is used as a constant to denote a (as well as the constant $\underline{a}$), the odd number $2h+1$ is used as a constant to denote the number h for $h \in M$ and the number $g_s(l, j, h_1, \ldots, h_s)$ for $h_1, \ldots, h_s \in M$ is to denote an element $x \in M'$ satisfying in M' the formula

$$\varphi_j^{(s)}(x_{h_1}, x_{g_1(l,j,h_1)}, x_{h_2}, x_{g_2(l,j,h_1,h_2)}, \ldots, x_{h_s}, x)$$

where $x_{h_1}, \ldots, x_{h_s}$ are elements of M' denoted by $h_1, \ldots, h_s$ and $x_{g_1(l,j,h_1)}, \ldots, x_{g_{s-1}(l,j,h_1,\ldots,h_{s-1})}$ are elements of M' denoted by $g_1(l, j, h_1), \ldots, g_{s-1}(l, j, h_1, \ldots, h_{s-1})$ respectively.

The diagram of M' is constructed in M piecewise. The pieces are called *preconditions*. An M-long sequence of preconditions determines the whole diagram of M'. In preconditions we encode the information that the constant $g_s(l, j, h_1, \ldots, h_s)$ is to denote an element of M' with the above property — this implies that the constructed model M' is a model of T, that the interpretation of the constant 2 (and of $\underline{a}$) is larger in M' than all elements of M, that the odd constants $2h+1$ denote in M' elements of M so that $2h+1$ denotes h and finally that M' is an end-extension of M.

We have the following definition for an arbitrary theory T:

Definition 1.1. Assume that the jth axiom of T is of the form

$$\forall y_1 \exists z_1 \ldots \forall y_k \exists z_k \bigwedge_{m \leqslant m_j} \varphi_{mj}(u_{mj}(y_1, z_1, \ldots, y_k, z_k))$$

where φ_{mj} is atomic or negated atomic and u_{mj} is the operation of choosing an appropriate subsequence of length $\leqslant 3$.

Let $x \in M$.

An *x-precondition* p is a function whose domain consists of all pairs $\langle t, \varphi\rangle$ where t is a sequence of length $\leqslant 3$ of numbers $\leqslant x$ and φ is an atomic or negated atomic formula in the language with the constant $\underline{a}$ with the number of free variables equal to the length of t and p: $\operatorname{dom} p \to 2$ so that the following holds:

(I) If $t = \langle g_1, g_2, g_3\rangle$, φ does not contain $\underline{a}$ and g_1, g_2, g_3 are of the form $2h_1+1$, $2h_2+1$, $2h_3+1$ respectively, then $p(\langle t, \varphi\rangle) = 0$ iff $\varphi(h_1, h_2, h_3)$ holds. Similarly for t of length 2.

(II) $p(\langle\langle 2\rangle, x = \underline{a}\rangle) = 0$.

(III) For any sequence $s, j, i_1, \ldots, i_s, h_1, \ldots, h_s$ such that $1 \leqslant s$, $i_1 < i_2 < \ldots < i_s$, $2L(i_s) \leqslant x$, $h_1 \leqslant 2L(i_1)$, $h_2 \leqslant 2L(i_2), \ldots, h_s \leqslant 2L(i_s)$ if $\varphi_j^{(s)}$ is open then $p(\langle u_{mj}(h_1, g_1(i_1, j, h_1), h_2, g_2(i_2, j, h_1, h_2), \ldots, h_s, g_s(i_s, j, h_1, \ldots, h_s)$, $\varphi_{mj}\rangle) = 0$ for $m \leqslant m_j$, provided that p is defined for the pair in question.

(IV) $p(\langle\langle 2h+1, 2\rangle, x_1 \leqslant x_2\rangle) = 0$ if defined.

(V) If $p(\langle\langle g, 2h+1\rangle, x_1 \leqslant x_2\rangle) = 0$ then there is $h' \leqslant h$ such that

$$p(\langle\langle g, 2h'+1\rangle, x_1 = x_2\rangle) = 0.$$

(VI) $p(\langle t, \varphi\rangle) = 0$ iff $p(\langle t, \neg\varphi\rangle) = 1$, where we identify $\neg\neg\varphi$ with φ.

(VII) If $p(\langle\langle g_1, g_2\rangle, x_1 = x_2\rangle) = 0$ then

$$p(\langle u^\frown\langle g_1\rangle^\frown v, \varphi\rangle) = p(\langle u^\frown\langle g_2\rangle^\frown v, \varphi\rangle)$$

if defined.

Comment on the definition. A precondition is supposed to be a fragment of the atomic diagram of M'. An x-precondition is supposed to contain a copy of the atomic diagram of M up to x — a fragment of the atomic diagram of M'. This is the reason why we require an x-precondition to decide the atomic and negated atomic formulas of the form $\varphi(2h_1+1, 2h_2+1, 2h_3+1)$ following their satisfaction in M by h_1, h_2, h_3 (condition I). Condition II is to ensure that the constant 2 is interpreted as the constant $\underline{a}$. Condition III ensures that M' is a model of T. Condition IV is to ensure that $M < a$. Finally, condition V is to ensure that $M \subseteq_e M'$. Conditions VI and VII are natural logical conditions. Condition VI states that a precondition, as a finite (in the sense of M) theory, does not contain a contradiction. However, we are in fact interested in those fragments of the atomic diagram of M' which not only do not contain a contradiction but are consistent in the sense of M. The semantical notion of the consistency of a precondition in M is expressed by means of M-extendability of a precondition in the following definition:

DEFINITION 1.2. An x-precondition p is called *arbitrarily extendable* if

$$\forall y \geqslant x\ \exists q\ (q \text{ is a } y\text{-precondition and } q \supseteq p).$$

An arbitrarily extendable precondition will be called a *condition* (following forcing terminology) — this is the notion which is our main tool.

Now we shall show how under a suitable assumption on M the construction of M' can be carried out.

Assume that the empty set $\emptyset$ is a condition in M. Under this assumption we shall show that there is an end-extension M' of M which is a model of T. Here T is arbitrary (recursive). The assumption is a version of semantical consistency of T in M. Later we shall reformulate the assumption for our particular T defined at the beginning of this section in terms of the condition (3) of the theorem for a suitable formula φ.

We need the following lemmas:

LEMMA 1.1. *Let p be an x-precondition which is a condition. Let $z \geqslant x$. Then there is a z-precondition q such that $q \supseteq p$ and q is again a condition.*

LEMMA 1.2. *Let $x_1 < x_2 < \ldots$ be a sequence of elements of M cofinal with M. Then there is a chain of conditions in M $p_1 \subseteq p_2 \subseteq \ldots$ such that p_i is an x_i-precondition.*

LEMMA 1.3. *If $p_1 \subseteq p_2 \subseteq \ldots$ are as above then there is a model M' of T which is a (proper) end-extension of M.*

Proof of Lemma 1.1. In this lemma we essentially use the fact that $M \models B\Sigma_1$. Consider the following easy observations.

Observation 1. There is a function $f(x)$ which is provably total in $I\Delta_0+\exp$ and such that every x-precondition is $\leqslant f(x)$.

Indeed, for f we can take the function 2^{2^x}.

Observation 2. Assume that T is $I\Delta_0$-provably recursive and identify T with its Δ_1 definition in M. Then the notion $\theta(x, p)$: "p is an x-precondition w.r.t. T" is Δ_1 in M.

Observation 3. Let $x \leqslant z \leqslant y$, let p be an x-precondition, q a y-precondition and $q \supseteq p$. Define $q_{\restriction z}$ as the restriction of q to those pairs $\langle t, \varphi\rangle$ which belong to the domain of a z-precondition. Then $q_{\restriction z}$ is a z-precondition and $p \subseteq q_{\restriction z} \subseteq q$.

Now assume that p, z are given and satisfy the assumptions of the lemma. Let $v = 2^{2^z}$. Suppose that there is no precondition q as in the conclusion of the lemma. Then we have

$$\forall q \leqslant v\ \{(q \text{ is a } z\text{-precondition } \&\ q \supseteq p) \Rightarrow \exists y \geqslant z$$
$$\exists u\ (u = 2^{2^y} \&\ \forall r \leqslant u\ (r \text{ is not a } y\text{-precondition or } \neg(r \supseteq q)))\}.$$

We can present the formula in curly brackets in a Σ_1 form. By $B\Sigma_1$ we can find a w such that for all $q \leqslant v$ the formula in $\{\ldots\}$ is bounded by w. We can assume that w is of the form 2^{2^y} for a y.

We shall show that there is no y-precondition extending p. So suppose that r is a y-precondition, $r \supseteq p$. Consider $q = r_{\restriction z}$. Then $p \subseteq q \subseteq r$. But, by the choice of w, $\exists y'\, \exists u' \leqslant w\ (y' \geqslant z \& u' = 2^{2^{y'}}$ & there is no y'-precondition extending q).

We have $y' \leqslant y$. Now $r_{\restriction y'}$ is a y'-precondition extending q, a contradiction. Thus p is not arbitrarily extendable, which contradicts our assumptions and completes the proof.

Proof of Lemma 1.2. We just have to iterate the use of Lemma 1.1.

Proof of Lemma 1.3. We build a Henkin model for the theory

$$\{\varphi(t)\colon \exists i \in \omega\ (p_i(\langle t, \varphi\rangle) = 0)\}.$$

For $g, g' \in M$ define the following equivalence relation:

$$g \sim g' \Leftrightarrow \exists i \in \omega\ (g, g' \leqslant x_i \& p_i(\langle\langle g, g'\rangle, x_1 = x_2\rangle) = 0).$$

Let $[g]$ be the equivalence class of g. Let the universe of M' consist of classes $[g]$ for $g \in M$. Let $a = [2]$. Define atomic relations in M' as follows: $[g_1]+[g_2] = [g_3]$ in M iff there is an i such that $p_i(\langle\langle g_1, g_2, g_3\rangle, x_1+x_2 = x_3\rangle) = 0$. Similarly we define other atomic relations. They are well defined in view of Condition VII of the definition of a precondition.

Let us show that $M' \models T$. Consider an axiom φ_j of T of the form

$$\forall y_1\, \exists z_1 \ldots \forall y_k\, \exists z_k \bigwedge_{m \leqslant m_j} \varphi_{mj}(u_{mj}(y_1, z_1, \ldots, y_k, z_k)).$$

By induction on $t \leqslant k$ we show

CLAIM. *Whenever $i_1 < i_2 < \ldots < i_{k-t}$, $h_1 \leqslant 2L(i_1)$, $h_2 \leqslant 2L(i_2), \ldots, h_{k-t} \leqslant 2L(i_{k-t})$ then*

$$\varphi_j^{(k-t)}([h_1], [g_1(i_1, j, h_1)], [h_2], [g_2(i_2, j, h_1, h_2)], \ldots$$
$$\ldots, [h_{k-t}], [g_{k-t}(i_{k-t}, j, h_1, \ldots, h_{k-t})])$$

holds in M'.

For $t = 0$ this follows from Condition III of the definition of a precondition and from the construction of M'. The inductive step is straightforward. For $t = k$, $\varphi_j^{(k-t)}$ is φ_j and hence $M' \models \varphi_j$, i.e. $M' \models T$.

Define a substructure $\hat{M} \subseteq M'$ as follows:

$$\hat{M} = \{[g]\colon \exists h\ (g = 2h+1)\}.$$

By Condition I of Def. 1.1, $\hat{M}$ is isomorphic to M and by Condition V, M' is an end-extension of $\hat{M}$. By Condition IV, a is larger than every element of $\hat{M}$ in M'. Thus the proof of Lemma 1.3 has been completed.

We now proceed to the case of our particular theory T and to the proof of the theorem.

Let T be the reformulation of the original theory defined at the beginning of this section. Let $\varphi(x)$ be the formula

$$\text{“}\emptyset \text{ is a condition w.r.t. } T_x\text{”}$$

where T_x is the collection of the first x axioms of the theory T considered in M. Using the bounds from the proof of Lemma 1.1 we can formulate φ in a Π_1 form. Namely, φ is the formula

$$\forall y, u\ [u = 2^{2^y} \Rightarrow \exists p \leqslant u\ (p \text{ is a } y\text{-precondition w.r.t. } T_x)].$$

Let us show that $T+\exp \vdash \varphi(\underline{n})$ for $n \in N$ (here T is the original theory).

To show this we first need the following claim:

Claim. *Let n be given. For a number y let $l(y)$ denote the least number l for which $L(l) \geqslant y$, where L is the appropriate function from Definition 0.2 for the theory T_n (the reformulated version). Let K be an arbitrary model of the original theory T plus* exp. *Then*

$$K \models \forall a, y\ (\text{there is an } l(y)\text{-closure of } a \text{ w.r.t. } T_n).$$

Proof of the claim. The proof is by induction on y. In the proof we essentially use the fact that any l-closure of any number a w.r.t. T_n can be bounded by $f(a, l)$ where f is a Δ_0 definable function with the property that $f(a, l)$ exists in K whenever $L(l)$ exists.

Indeed, suppose that an l-closure of a w.r.t. T_n exists in K. Then, since all axioms of T_n except those of the form "$\exists y\ \Phi_m(\underline{a}, y)$" are Π_1 and since T_n contains axioms of the form "$\exists y\ \Phi_m(\underline{a}, y)$" only for $m \leqslant n$ we infer that every term of that l-closure is bounded by the least y satisfying $\Phi_n(a, y)$. Let this y be denoted by $\Phi_n(a)$. Then the closure as a sequence number is bounded by $(\Phi_n(a))^{2L(l)}$. Define $f(a, l) = (\Phi_n(a))^{2L(l)}$. Since Φ_n is total in K and since $K \models \exp$ we infer that $f(a, l)$ exists in K provided that $L(l)$ exists.

Now we can prove the claim by induction using the satisfaction relation to define the notion of an l-closure uniformly in l and then using the suitable bounds and the Δ_0 induction.

Now let us show that $K \models \Phi(n)$.

Let y be given. We want to construct a y-precondition w.r.t. T_n in K. Take any element $a \in K$ such that $L(l(y)) \leqslant a$. Let $x^{l(y)}$ be an $l(y)$-closure of a w.r.t. T_n in K. Define p as follows: if $t = \langle g_1, g_2, g_3\rangle$ is a triple of numbers $\leqslant y$ and φ is an atomic or negated atomic formula then define $p(\langle t, \varphi\rangle) = 0$ iff $\varphi(x_{g_1}, x_{g_2}, x_{g_3})$ holds in M, where x_{g_i} is the g_ith term of $x^{l(y)}$.

It is easy to see that p is a y-precondition w.r.t. T_n in K.

Thus we have proved that the original formulation of T together with exp proves $\varphi(\underline{n})$ for $n \in N$.

Now let us prove the required equivalence. We shall show (3)$\Rightarrow$(2)$\Rightarrow$(1)$\Rightarrow$(3).

Assume (3). Then $\emptyset$ is a condition w.r.t. the theory T_r in M for a non-standard r. Since T_r is a finite theory in M we can apply our previous considerations to infer that there is a structure M' which is an end-extension of M and which satisfies every standard axiom of T_r.

Let r' be a non-standard number in M such that T_r contains the axioms "$\exists y\ \Phi_s(\underline{a}, y)$" for all $s \leqslant r'$. Assume that for $s \leqslant r'$ the axiom "$\exists y\ \Phi_s(\underline{a}, y)$" is the $j(s)$th axiom of T_r and that it is syntactically of the form $\forall y' \exists y\ \Phi_s(\underline{a}, y)$ where Φ_s is presented in a suitable form (see Def. 0.1). Then $g_1(1, j(s), 0) \leqslant 2L(2)$ where g_1, L are the functions defined in M by Definition 0.2 for T_r. Hence $[g_1(1, j(s), 0)]$ is an element of M' for all $s \leqslant r'$. By our assumptions about the original theory T, every x-precondition w.r.t. T_r in M for $x > N$ decides positively all the formulas $g_1(1, j(m), 0) < g_1(1, j(m'), 0)$ (takes the value 0 at the suitable pairs) for all $m, m' \in N$ such that $m < m'$. Fix an x-precondition p for a non-standard x in the sequence determining M'. Then by Δ_0 overspill we infer that p decides positively all the formulas

$$g_1(1, j(s), 0) < g_1(1, j(s'), 0)$$

for all $s, s' \leqslant s_0$ for a non-standard $s_0 \leqslant r'$ and $s < s'$. Hence, in M' we have

$$[g_1(1, j(m), 0)] < [g_1(1, j(s_0), 0)] \quad \text{for all } m \in N.$$

But $[g_1(1, j(m), 0)]$ is equal to $\Phi_m(a)$ in M'. Hence $\Phi_m(a) < [g_1(1, j(s_0), 0)]$ in M' for $m \in N$. Let M'' be the initial segment of M' determined by the elements of the form $\Phi_m(a)$ for $m \in N$ in M'. Then M'' is a model of the original theory T. Since it is a proper initial segment of M' it is a model of $B\Sigma_1$ as well. So it is as required in (2).

Thus we have proved (3)$\Rightarrow$(2).

Evidently (2)$\Rightarrow$(1).

Now assume (1). Let M' be a model of the original theory T end-extending M, $a \in M' \setminus M$. Consider two cases.

Case I (in this case the proof is due to Wilkie). Assume $T \vdash \exp$. Then M' is a model of $T+\exp$ and hence M' satisfies $\varphi(n)$ for all standard n. We infer that $M' \models \forall w\ \varphi'(w, n)$ where φ is of the form $\forall w\ \varphi'(w, x)$, $\varphi' \in \Delta_0$, for $n \in N$. Hence in particular, $M' \models \forall w \leqslant a\ \varphi'(w, n)$ for $n \in N$. Thus, by Δ_0 overspill $M' \models \forall w \leqslant a\ \varphi'(w, r)$ for a non-standard r. We can assume that $r \in M$. Then in particular, $M \models \forall w\ \varphi'(w, r)$ and thus $M \models \varphi(r)$, i.e. (3).

Case II. $I\Delta_0 + \exp \vdash T$. Then M is a model of $T+\exp$. Let b be the element of M' such that $b = \log\log\log a$ in M'. Since $M \models \exp$, $b > M$. We have $\Phi_m(b) < 2^b$ for all $m \in N$. Hence, if L^n is the L-function for the theory T_n then

$$(\Phi_n(b))^{2L^n(l(b))} \leqslant (\Phi_n(b))^{2^b} \leqslant 2^{b \cdot 2^b} \leqslant 2^{2^{2^b}} \leqslant a \quad \text{for } n \in N.$$

From the proof of the Claim it follows that

$$M' \models \forall y \leqslant b\ (\text{there is a } y\text{-precondition w.r.t. } T_n)$$

for every $n \in N$. The above formula is bounded in M' by a. Hence, applying Δ_0 overspill we infer that there is a non-standard r in M' such that

$$M' \models \forall y \leqslant b\ (\text{there is a } y\text{-precondition w.r.t. } T_r).$$

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Again we can assume that $r \in M$ and now we infer that $M \models \varphi(r)$ as in Case I. Thus we have proved (3).

So the proof of the theorem has been completed.

§ 2. Final remarks. By methods similar to the method of proof of Lemma 8.10 of [WP II] we can show that if T is one of our theories which is a subtheory of $I\Delta_0 + \exp$ then the condition (3) of the theorem is satisfied for our formula φ corresponding to T. We can even prove a stronger result, namely

$$M \models \forall x \ \varphi(x)$$

for any model M of $I\Delta_0 + \exp + B\Sigma_1$.

Thus we have the following

COROLLARY. *If M is a model of $I\Delta_0 + \exp + B\Sigma_1$ and M is cofinal with ω then M has a proper end-extension to a model M' of $I\Delta_0 + \Omega_n + B\Sigma_1$, for any $n \in N$.*

This corollary is an extension of an analogous theorem for countable models M proved in [WP I].

The notion of l-closure of a number w.r.t. a theory in a model of PA^- for a standard l is general and can concern any theory T in our language. Also, Remarks 0.1 and 0.2 are generally true. In general, an l-closure of a number a is not uniquely determined. Hence, although for every l we can make an l-closure of a, for a given a, in a model M of T, it is not necessarily true that those l-closures extend each other, as required in Remark 0.2. Now, let us briefly describe those theories T for which any l-closure of any a is uniquely determined in any model of T.

DEFINITION 2.1. Let φ be of the form considered in Def. 0.1, i.e.

$$\forall y_1 \exists z_1 \ldots \forall y_k \exists z_k \ \theta(x, y_1, z_1, \ldots, y_k, z_k)$$

where θ is open.

We say that φ is *univocal* in T if

$$T \vdash \forall x \ [\varphi(x) \Leftrightarrow \forall y_1 \exists! z_1 \ \forall y_2 \exists! z_2 \ldots \forall y_k \exists! z_k \ \theta(x, y_1, z_1, y_2, z_2, \ldots, y_k, z_k)].$$

The following remark is easy:

Remark 2.1. If T is $I\Sigma_n$ for $n \geqslant 0$ then for every Π_m formula φ there is a Π_m formula φ^* such that φ^* is univocal in T and $T \vdash \varphi \Leftrightarrow \varphi^*$ for $m \leqslant n+2$.

Indeed, if $k = 0$ we take $\varphi^* = \varphi$. If the remark is true for $k-1$ and φ is of the above form then we let φ^* be the normal form of

$$\forall y_1 \exists z_1 \ (\varphi'^*(x, y_1, z_1) \& \forall z \ (z < z_1 \Rightarrow \neg \varphi'(x, y_1, z))^*).$$

Hence it follows that $I\Delta_0$ and the theories that we consider in the paper can be axiomatized by univocal sentences. It can be noted (although this is not important for our considerations) that if M is a model of the form $M = \bigcup_{l \in N} x^l(a)$ where a is an element of M and $x^l(a)$ is the (uniquely determined) l-closure of a in M w.r.t. T where T is axiomatized by univocal Π_1 sentences and sentences of the form $\exists y \ \Phi_m(\underline{a}, y)$ presented in a univocal form then every element of M is Σ_1 definable in M from the parameter a.

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