An n-dimensional compactum which remains n-dimensional after removing all Cantor n-manifolds

by

Roman Pol (Warszawa)

Abstract. For each natural $n, n \geqslant 2$, a compactum X is constructed such that $\dim X = \dim(X \backslash K_X) = n$, where K_X is the union of all Cantor n-manifolds in X. This answers a question asked by P. S. Aleksandrov.

1. The examples. Our terminology follows Kuratowski [Ku]. We denote by I the unit interval $[0, 1]$, $\operatorname{Bd} A$ is the boundary of the set A in a topological space and compactum means a compact metrizable space.

Let us recall that an n-dimensional Cantor manifold is a compact n-dimensional space which cannot be separated by any closed subset of dimension $\leqslant n-2$.

The following example provides an answer to a question asked by P. S. Aleksandrov, cf. [A; 42], [A–P; Ch. 5, §9, 3], [F_2; §3, 3.2, Question 7].

EXAMPLE A. *For each natural $n \geqslant 2$ there exists an n-dimensional compact metrizable space X such that the complement $X \backslash K_X$ of the union K_X of all n-dimensional Cantor manifolds in X has dimension n.*

More specifically, we shall construct in this note compacta with the following properties.

EXAMPLE B. *For each natural $n \geqslant 2$ there exists an n-dimensional metrizable continuum X and a continuous map $q\colon X \to I$ onto the unit interval, X being irreducible between $q^{-1}(0)$ and $q^{-1}(1)$, such that*

(i) *the image $q(K)$ of the union K of all n-dimensional Cantor manifolds in X has empty interior in I,*

(ii) *there exists a σ-compact zero-dimensional set $C \subset X \backslash K$ with $q(C) = q(K)$ such that whenever $C \subset E \subset X$ and $q(E)$ has nonempty interior in I then $\dim(E) = n$.*

In particular, $\dim(X \backslash K) = n$.

We shall obtain these examples starting with certain peculiar n-dimensional compacta Z in I^{n+1} and getting X from Z by replacing some disjoint collections of n-balls in Z by disjoint collections of umbrellas. To get the spaces X described in Example A it is enough to use the compacta Z defined by Lelek [L], or Rubin, Schori and Walsh

[R–S–W] (actually, the constructions of the spaces Z we need are based on ideas going back to Mazurkiewicz [M] and Knaster [Kn]); the spaces X described in Example B are based on compacta Z obtained by some results from [P].

Remark. A. V. Ivanov [I] constructed, using the continuum hypothesis, perfectly normal compact (non-metrizable) spaces X with $\dim(X\backslash K_X) = \dim X = n$, K_X being as in Example A. Some striking examples concerning the sets K_X in the class of hereditarily normal compact spaces X were constructed, also under the continuum hypothesis, by V. V. Fedorčuk [F_1].

2. The compacta Z. The examples will be obtained by a modification of certain compacta Z in I^{n+1}. For Example A one can use the compacta Z defined by Lelek [L; Example, p. 80] or Rubin, Schori and Walsh [R–S–W; Example 4.5]:

PROPOSITION A ([L], [R–S–W]). *Let $p\colon I^{n+1}\to I$ be the projection onto the first coordinate and let $L\subset I$ be a Cantor set. For each $n\geqslant 1$ there exists a compactum $Z\subset I^{n+1}$ such that $p(Z)=L$ and for each $M\subset Z$ with $p(M)=L$, $\dim M = n$.*

For Example B we need compacta Z with slightly stronger properties:

PROPOSITION B. *For each $n\geqslant 1$ there exists an n-dimensional continuum Z in I^{n+1} which joins the opposite faces $\{0\}\times I^n$ and $\{1\}\times I^n$ such that whenever the projection of $M\subset Z$ onto the first coordinate has nonempty interior in I, then $\dim M = n$.*

To get such a continuum Z, let us choose pairwise disjoint Cantor sets $T_1, T_2, \ldots$ in I such that each non-degenerate interval in I contains some T_k and let, for each k, $G_k\subset T_k\times I^n$ be an $(n-1)$-dimensional set such that whenever $M\subset I^{n+1}\backslash G_k$ projects onto T_k, $\dim M = n$. The sets G_k can be taken from [P]: one can consider the zero-dimensional sets $N_1, N_2, \ldots$ defined in Section 3.1 of [P], where T is the set of the irrationals of T_k, and let $G_k = (N_1\cup\ldots\cup N_n)\cap I^{n+1}$. Now, the union $G=\bigcup_{k=1}^{\infty} G_k$ is $(n-1)$-dimensional, each G_k being closed in G, and by a theorem of Mazurkiewicz [Ku; § 59, II] there exists a continuum Z in $I^{n+1}\backslash G$ which joins the opposite faces $\{0\}\times I^n$ and $\{1\}\times I^n$; this continuum has the required properties.

Remark 1. Let Z be a continuum described in Proposition B and let $p\colon Z\to I$ be the restriction to Z of the projection onto the first coordinate. Then the set $\{t\in I\colon \dim p^{-1}(t)=0\}$ is residual in I.

To check this let us consider a countable base $V_1, V_2, \ldots$ in Z with $\operatorname{Bd} V_i\leqslant n-1$, $i=1,2,\ldots$ (recall that $\dim Z = n$). The sets $B_i = p(\operatorname{Bd} V_i)$ are compact and have empty interior in I and hence $A = I\backslash\bigcup_{l=1}^{\infty} B_l$ is a dense G_δ-set in I. If $t\in A$, the fiber $p^{-1}(t)$ is disjoint from every boundary $\operatorname{Bd} V_i$, which means that the intersections $V_i\cap p^{-1}(t)$ form a closed-and-open base in $p^{-1}(t)$, i.e. $\dim p^{-1}(t)=0$.

Actually, the set $\{t\colon \dim p^{-1}(t)=0\}$ is G_δ, cf. [Ku; § 45, IV].

Remark 2. Compacta similar to those described in Proposition B, but with properties falling somewhat short of our needs, are also defined in Krasinkiewicz [Kr; Corollary 3.4] and in [P; Corollary 5.2(i)].

3. The compacta X. Let us fix a natural number $n\geqslant 2$ and let Z be an n-dimensional compactum in I^{n+1} defined either in Proposition A or Proposition B. We shall describe a modification of Z which yields a compactum X with properties listed in Example A or Example B, respectively. From now on we shall assume that Z is given by Proposition B — in case A one just neglects certain details; we can assume that the continuum Z is irreducible between the opposite faces $\{0\}\times I^n$ and $\{1\}\times I^n$, cf. [Ku; § 48].

Let $p\colon Z\to I$ be the projection onto the first coordinate restricted to the continuum Z.

Let $U_1, U_2, \ldots$ be an open base in I^n and let, for each i, $L_i = \{t\in I\colon \{t\}\times U_i\subset p^{-1}(t)\}$. The sets L_i are compact and since $L_i\times U_i\subset Z$, L_i has empty interior in I (recall that $\dim Z = n$), therefore, splitting each set $L_i\backslash(L_1\cup\ldots\cup L_{i-1})$ into countably many sufficiently small compact pieces, one can find pairwise disjoint compact sets $T_1, T_2, \ldots$ such that each T_j is contained in some L_i, $\bigcup_{j=1}^{\infty} T_j = \bigcup_{i=1}^{\infty} L_i$ and $\operatorname{diam} T_j\to 0$, diam standing for the diameter, cf. [Ku; § 26, II]. Given an index j, fix any i with $T_j\subset L_i$ and choose inside U_i a closed ball $D_j = \{x\in I^n\colon \|x-c_j\|\leqslant r_j\}$ with center c_j and positive radius $r_j\leqslant 1/j$, disjoint from the boundary of the cube I^n. The sets $T_j\times D_j\subset Z$ are pairwise disjoint (as $T_i\cap T_j=\emptyset$ for $i\neq j$) and $\operatorname{diam}(T_j\times D_j)\to 0$.

The compactum X is obtained from Z by replacing each ball $\{t\}\times D_j$, where $t\in T_j$, by an arc — this transforms the fiber $p^{-1}(t)$ into an umbrella. More precisely, we define in Z an upper semi-continuous decomposition $\boldsymbol{D}$ into singletons and the $(n-1)$-dimensional spheres

$$S(t, r) = \{t\}\times\{x\in I^n\colon \|x-c_j\| = r\},$$

where $t\in T_j$ and $0<r\leqslant r_j$, and we let $X = Z/\boldsymbol{D}$ be the factor space and $d\colon Z\to X$ the quotient map. Let $q\colon X\to I$ be the continuous map induced by the projection p, i.e. $p = q\circ d$.

Thus, for $t\in T_j$, $J_t = d(\{t\}\times D_j)$ is an arc whose end point $d(S(t, r_j))$ is the only point in common of J_t and the closure of $q^{-1}(t)\backslash J_t$. It follows that

(1) $e_t = d(t, c_j)$, where $t\in T_j$, does not belong to any n-dimensional Cantor manifold in $q^{-1}(t)$,

e_t being the other end point of the arc J_t. We set

$$C = \{e_t\colon t\in T\},\quad \text{where } T = \bigcup_{j=1}^{\infty} T_j. \tag{2}$$

Let us notice that since $d^{-1}(e_t) = \{(t, c_j)\}$, where $t\in T_j$,

$$d^{-1}(C) = \bigcup_{j=1}^{\infty} T_j\times\{c_j\} \tag{3}$$

and d maps $d^{-1}(C)$ homeomorphically onto C, hence C is σ-compact and zero-dimensional. Let us also notice that the continuum X is irreducible between $q^{-1}(0)$ and $q^{-1}(1)$, the fibers of the map d being connected [Ku; § 48, I, Th. 3].

To check the properties of X we begin with the following two observations (cf. (2)):

(4) $T = \{t \in I: \dim q^{-1}(t) = n\}$, and if $t \notin T$, $\dim q^{-1}(t) \leqslant n-1$,

(5) each k-dimensional Cantor manifold in X with $k \geqslant 2$ is contained in some $q^{-1}(t)$.

Recall that $T = \bigcup_{i=1}^{\infty} L_i$. Therefore, if $t \notin T$, $p^{-1}(t)$ has empty interior in $\{t\} \times I^n$, hence $\dim p^{-1}(t) \leqslant n-1$ and since d maps in this case $p^{-1}(t)$ onto $q^{-1}(t)$ in a one-to-one way, we get $\dim q^{-1}(t) \leqslant n-1$. On the other hand, if $t \in T$, i.e. $t \in L_i$ for some i, $\{t\} \times U_i \subset p^{-1}(t)$, and since the closed ball D_j is inside U_i, d embeds the n-dimensional region $\{t\} \times (U_i \backslash D_j)$ into $q^{-1}(t)$; the inequality $\dim q^{-1}(t) \leqslant n$ follows from the fact that $q^{-1}(t)$ is a union of a σ-compact set homeomorphic to $p^{-1}(t) \backslash (\{t\} \times D_j)$ and an arc.

To see (5), let us consider a k-dimensional Cantor manifold F in X with $k \geqslant 2$. The set $q(F)$ is then a singleton $\{t\}$. Otherwise, using Remark 1 in Section 2 we could find an $s \in I \backslash T$, strictly between some two points in $q(F)$, such that $p^{-1}(s)$ is zero-dimensional and the zero-dimensional set $q^{-1}(s)$ would separate the compactum F.

From (4) and (5) we get

(6) $$\dim X = n$$

as X does not contain any $(n+1)$-dimensional Cantor manifold, cf. [Ku; §46, XI].

It remains to check that the set C defined in (2) has the properties stated in Example B, (ii). By (1) and (5), C is disjoint from the union K of all n-dimensional Cantor manifolds in X and, by (4), $q(K) = q(C) = T$. Let $C \subset E \subset X$, where $q(E)$ has nonempty interior in I, let $H = E \backslash q^{-1}(T)$ and let $M = d^{-1}(C \cup H)$. The map d restricted to M is a homeomorphism onto $C \cup H$ (cf. (3)) and since $p(M) = q(C \cup H) = q(E)$, the properties of Z yield $\dim M = n$, hence $\dim E = n$.

References

[A] P. S. Aleksandrov, *On dimension of closed sets*, Uspekhi Mat. Nauk 4 (1949), 17–88.

[A–P] P. S. Aleksandrov and B. A. Pasynkov, *Introduction to Dimension Theory*, Moskva 1973 (in Russian).

[F_1] V. V. Fedorčuk, *On the dimension of hereditarily normal spaces*, Proc. London Math. Soc. 36 (1978), 163–175.

[F_2] — *Foundations of dimension theory*, Itogi Nauki i Tekh. (Sovremennye Problemy Mat.) 17 (1988) (in Russian).

[I] A. V. Ivanov, *On the internal dimensional kernel in the sense of P. S. Aleksandrov*, in: *Seminar in General Topology*, Izd. Moskov. Univ., 1981, 46–51.

[Kn] B. Knaster, *Sur les coupures biconnexes des espaces euclidiens de dimension $n > 1$ arbitraire*, Mat. Sb. 19 (1946), 9–18.

[Kr] J. Krasinkiewicz, *Homotopy separators and mappings into cubes*, Fund. Math. 131 (1988), 149–154.

[Ku] K. Kuratowski, *Topology*, vols. I, II, PWN, Warszawa 1966, 1968.

[L] A. Lelek, *Dimension inequalities for unions and mappings of separable metric spaces*, Colloq. Math. 23 (1971), 69–91.

[M] S. Mazurkiewicz, *Sur les problèmes $\varkappa$ et λ de Urysohn*. Fund. Math. 10 (1926), 311–319.

[P] R. Pol, *Countable dimensional universal sets*, Trans. Amer. Math. Soc. 297 (1986), 255–268.

[R–S–W] L. R. Rubin, R. M. Schori and J. J. Walsh, *New dimension-theory techniques for constructing infinite-dimensional examples*, Gen. Topology Appl. 10 (1979), 93–102.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WARSAW
PKiN, IXp.
00-901 Warszawa
Poland

Received 3 July 1989