THEOREM 3.5. *Let $(\mu_i X_i\colon i \in N)$ be a countable family of σ-$\mathscr{C}$-scattered supercomplete spaces. Then $m(\prod_N \mu_i X_i)$ is supercomplete.*

COROLLARY 3.6. *A countable product of σ-$\mathscr{C}$-scattered paracompact spaces is paracompact.*

References

[1] K. Alster, *A class of spaces whose Cartesian product with every hereditarily Lindelöf space is Lindelöf*, Fund. Math. 114 (1981), 173–181.
[2] A. Arkhangel'skii, *On a class of spaces containing all metric and all locally bicompact spaces*, Soviet Math. Dokl. 4 (1963), 1051–1055.
[3] H. Corson, *Determination of paracompactness by uniformities*, Amer. J. Math. 80 (1958), 185–190.
[4] L. M. Friedler, H. W. Martin and S. W. Williams, *Paracompact C-scattered spaces*, Pacific J. Math. 129 (1987), 277–296.
[5] Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. Math. 8 (1960), 747–750.
[6] — *A note on metric-fine spaces*, Proc. Amer. Math. Soc. 46 (1974), 111–119.
[7] S. Ginsburg and J. R. Isbell, *Some operators on uniform spaces*, Trans. Amer. Math. Soc. 93 (1959), 145–168.
[8] A. Hohti, *On supercomplete uniform spaces*, Proc. Amer. Math. Soc. (3) 87 (1983), 557–560.
[9] — *On supercomplete uniform spaces* II, Czechoslovak Math. J. 37 (1987), 376–385.
[10] — *On supercomplete uniform spaces* III, Proc. Amer. Math. Soc. 97 (1985), 339–342.
[11] A. Hohti and J. Pelant, *On complexity of metric spaces*, Fund. Math. 25 (1985), 133–142.
[12] M. Hušek and J. Pelant, *Extensions and restrictions in products of metric spaces*, Topology Appl. 25 (1987), 245–252.
[13] J. Isbell, *Supercomplete spaces*, Pacific J. Math. 12 (1962), 287–290.
[14] — *Uniform spaces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R. I., 1964.
[15] J. Pelant, *Locally fine uniformities and normal covers*, Czechoslovak Math. J. 37 (112) (1987), 181–187.
[16] M. D. Rice, *A note on uniform paracompactness*, Proc. Amer. Math. Soc. 62 (1977), 359–362.
[17] M. E. Rudin and S. Watson. *Countable products of scattered paracompact spaces*, ibid. 89 (1983), 551–552.
[18] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math. 73 (1971), 59–74.
[19] — *Spaces defined by topological games*, ibid. 88 (1975), 193–223.

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Received 7 June 1989

On supercomplete uniform spaces V: Tamano's product problem

by

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Abstract. In this paper we solve the analogue of Tamano's problem [8] for supercomplete spaces. We show that a supercomplete space μX has the property that its product with every supercomplete space is again supercomplete if, and only if, X is C-scattered [19].

1. Introduction. This is the last member in our series of papers [4]–[7] on supercomplete uniform spaces. These spaces were introduced and characterized by J. R. Isbell in [11]. By definition, μX is *supercomplete* if the uniform hyperspace $H(\mu X)$, equipped with the Hausdorff uniformity, is a complete uniform space. By [11], supercompleteness is a uniform form of paracompactness: μX is supercomplete iff (1) X is (topologically) paracompact and (2) the Ginsberg–Isbell locally fine coreflection $\lambda\mu$ [3], [11] is the fine uniformity of X. (In this case, every open cover of X can be analyzed combinatorially by using uniform covers as a starting point.) This notion has also been studied in the context of linear spaces and closed graph theorems [2], [15]; [10] gives an application to homogeneous spaces. Several results concerning product spaces and supercompleteness have been obtained in [4]–[7] and [8]; closely related questions on uncountable products are dealt with in [17].

In [18], H. Tamano asked for a characterization of paracompact spaces the product of which with every paracompact space is paracompact. While it is known [16] that in the class of p-spaces of Arkhangel'skii [1], such paracompact spaces are σ-locally compact, the general problem has proved to be difficult. In this paper we solve the analogous question for supercomplete spaces, with a relatively simple proof.

2. Preliminaries. The basic reference to uniform spaces is [12]. For a completely regular space X, $\mathscr{F}(X)$ denotes the fine uniformity of X, consisting of all the normal covers of X, and βX denotes the Čech–Stone compactification of X. The basic properties of the Čech–Stone compactification can be found e.g. in [20]. We repeat here the definition of (slowed-down) Ginsburg–Isbell derivatives (see [9]) of uniformities. Let $\mathscr{C}(X) \subseteq P(P(X))$ denote the collection of all covers of X. Then $\mathscr{C}(X)$ is ordered by the relation $\prec$ of refinement. Let μ, ν be filters in $\mathscr{C}(X)$ with respect to $\prec$. The symbol ν/μ

1985 *Mathematics Subject Classification*: Primary 54E15; Secondary 54B10.

denotes the collection of the elements of $\mathscr{C}(X)$ refined by a cover of the form $\{U_i \cap V_j^i\}$, where $\{U_i\} \in \mu$ and for each i, $\{V_j^i\} \in \nu$. In general the filter ν/μ is not a uniformity, even when μ and ν are uniformities of X. However, let μ be a uniformity, and define, by transfinite induction, a family of filters in $\mathscr{C}(X)$ by setting $\mu^{(0)} = \mu$, $\mu^{(\alpha+1)} = \mu^{(\alpha)}/\mu$ for a successor ordinal and let $\mu^{(\beta)} = \bigcup\{\mu^{(\alpha)}: \alpha < \beta\}$ in case β is a limit ordinal. There (obviously) is α such that $\mu^{(\alpha+1)} = \mu^{(\alpha)}$, by [3] this $\mu^{(\alpha)}$ is a uniformity, called the locally fine coreflection of μ and denoted by $\lambda\mu$. For technical reasons, we also introduce the (-1)th derivative $\mu^{(-1)}$. An element $\mathscr{U} \in \mathscr{C}(X)$ is called *trivial* if $X \in \mathscr{U}$. Then we define $\mu^{(-1)}$ as the collection of all the trivial covers of X. Clearly $\mu^{(0)} = \mu^{(-1)}/\mu$.

If μ is any set of elements of $\mathscr{C}(X)$, and $A \subset X$, then $\mu \upharpoonright A$ denotes the restriction $\{\mathscr{U} \upharpoonright A: \mathscr{U} \in \mu\}$ to A, where $\mathscr{U} \upharpoonright A = \{U \cap A: U \in \mathscr{U}\}$. If μ is a uniformity of X, then $\mu \upharpoonright A$ is the relative (induced) subspace uniformity of A. We recall that the operations of taking Ginsburg–Isbell derivatives and forming restrictions to subsets commute; i.e., for all $A \subset X$ and all α we have $\mu^{(\alpha)} \upharpoonright A = (\mu \upharpoonright A)^{(\alpha)}$.

3. The result. In this section we prove the result promised in Introduction, extending the analogous theorem for paracompact p-spaces obtained in [5]. However, we notice that the result of [5] is used here to handle a subcase.

THEOREM 3.1. *Let μX be a supercomplete uniform space. Then $\mu X \times \nu Y$ is supercomplete for every supercomplete space νY if and only if X is C-scattered.*

Proof. Sufficiency has been proved in [5]. For necessity, suppose that X is not C-scattered. It will be enough to find a paracompact space Y such that $\lambda(\mathscr{F}(X) \times \mathscr{F}(Y)) \neq \mathscr{F}(X \times Y)$, because $\lambda(\mu \times \nu) = \lambda(\lambda\mu \times \lambda\nu)$. As X is not C-scattered, it contains a non-empty closed subset F such that F is nowhere locally compact. Furthermore, we can assume that F is nowhere locally Čech-complete. Indeed, suppose that $p \in F$ has a Čech-complete neighbourhood U in F. A closed neighbourhood V of p contained in U is Čech-complete and paracompact, but not C-scattered. Hence, by Theorem 4.1 in [5] there is a separable metrizable space Z such that $\mathscr{F}V \times \mathscr{F}Z$ is not supercomplete. Now V is P-embedded in X, which implies that $\mathscr{F}(V) = \mathscr{F}(X) \upharpoonright V$ and hence that $\mathscr{F}X \times \mathscr{F}Z$ is not supercomplete.

As F is nowhere locally compact, $\beta F - F$ is a dense subset of βF. By a result of Isbell [3], there is a paracompact space Y and a open continuous onto map $\varphi: Y \to \beta F - F$. Denote by $\bar{\varphi}$ the Stone extension of φ to a continuous map $\beta Y \to \beta F$. Let Δ denote the inverse of the graph of $\bar{\varphi}$ in $\beta Y \times \beta F$, i.e., $\Delta = \{(p, q) \in \beta F \times \beta Y: \bar{\varphi}(q) = p\}$. Then Δ is a closed subspace of $\beta F \times \beta Y$ and $(F \times Y) \cap \Delta = \emptyset$. By regularity, there is for each $(x, y) \in F \times Y$ an open neighbourhood G_{xy} in $\beta F \times \beta Y$ such that

$$(\mathrm{cl}_{\beta F \times \beta Y}(G_{xy})) \cap \Delta = \emptyset.$$

Let $\mathscr{G} = \{G_{xy} \cap (F \times Y): x \in F, y \in Y\}$. We claim that the open cover $\mathscr{G}$ of $F \times Y$ is not in $\lambda(\mathscr{F}(F) \times \mathscr{F}(Y))$. On the other hand, we obviously can (and will) assume that $F \times Y$ is paracompact. Thus, $\mathscr{G}$ is an element of $\mathscr{F}(F \times Y)$, and this will show that $\mathscr{F}F \times \mathscr{F}Y$, and hence $\mathscr{F}X \times \mathscr{F}Y$, is not supercomplete.

We shall proceed by the method of contradiction and assume that $\mathscr{G} \in \lambda(\mathscr{F}(F) \times \mathscr{F}(Y))$. Then $\mathscr{G} \in [\mathscr{F}(F) \times \mathscr{F}(Y)]^{(\alpha_0)}$ for some α_0. It is clear that $\mathscr{G}$ is

not a trivial cover of $F \times Y$ (because otherwise the closure of some member equals $\beta F \times \beta Y$); consequently $\alpha_0 \geqslant 0$. Thus, there is a uniform cover $\mathscr{U}$ of $\mathscr{F}F \times \mathscr{F}Y$ and $\alpha_1 < \alpha_0$ such that

$$\mathscr{G} \upharpoonright U \in [\mathscr{F}(F) \times \mathscr{F}(Y)]^{(\alpha_1)} \upharpoonright U$$

for all $U \in \mathscr{U}$. The cover $\mathscr{U}$ being uniform we can find (uniform) open covers $\mathscr{U}_0$ and $\mathscr{V}_0$ of $\mathscr{F}F$ and $\mathscr{F}Y$, respectively, such that $\mathscr{U}_0 \times \mathscr{V}_0 \prec \mathscr{U}$. The open cover $\mathscr{U}_0$ of F can be extended over a Čech-complete subset G of βF. (It was shown during the proof of 3.4 in [5] that a uniform cover $\mathscr{W}$ of a Tychonoff space Z can be extended over a Čech-complete paracompact subspace of βZ. For completeness, we succinctly mention here how the extension can be obtained. Since $\mathscr{W}$ is uniform, there is a compatible pseudometric σ such that $\mathscr{W}$ is uniform in σZ, and $\mathscr{W}$ can be extended (uniformly) over the completion $\pi\sigma Z$, which is a Čech-complete and paracompact subspace of $\beta\pi\sigma Z$. Extend $\mathscr{W}$ over $f^{-1}[\pi\sigma Z]$, where f is the Stone extension of the embedding $Z \to \pi\sigma Z$.) Now $G_0 - F$ is dense in βF. In fact, let G be any Čech-complete subset of βF containing F and let $p \in F$. Then for any (in βF) open neighbourhood V of p, the set $G \cap V$ is an open subset of G and hence Čech-complete. If $G \cap V \subset F$, then it would be a Čech-complete neighbourhood of p in F, which is impossible by our assumption on F. Thus, $G \cap V \cap (\beta F - F) \neq \emptyset$. This implies $p \in \mathrm{cl}_{\beta F}(G - F)$, and so $G - F$ is dense in βF, and hence also in $\beta F - F$. Finally, it follows that for any such a Čech-complete subset G, and for any nonempty open subset V of βF or of $\beta F - F$, the set $V \cap (G - F)$ is a dense subset of V.

Now let $\tilde{\mathscr{U}}_0$ be an open cover of G_0, extending $\mathscr{U}_0$, and let $U_0 \in \mathscr{U}_0$ be arbitrary. We write $\tilde{\mathscr{U}}_0 = \{\tilde{U}: U \in \mathscr{U}_0\}$ where $\tilde{U} \cap F = U$. Since $\varphi[Y] = \beta F - F$, and since by the above $\tilde{U}_0 \cap (G - F) \neq \emptyset$, there is $V_0 \in \mathscr{V}_0$ such that $\varphi[V_0] \cap \tilde{U}_0 \neq \emptyset$. It follows that

$$(\tilde{U}_0 \times V_0) \cap \Delta \neq \emptyset.$$

Since U_0 is open, we have $\tilde{U}_0 \subset \mathrm{cl}_{\beta F} U_0$. Thus, $\tilde{U}_0 \times V_0 \subset \mathrm{cl}_{\beta F \times \beta Y}(U_0 \times V_0)$, which implies that $\mathscr{G} \upharpoonright (U_0 \times V_0)$ is not a trivial cover of $U_0 \times V_0$, since the closures (in $\beta F \times \beta Y$) of the elements of $\mathscr{G}$ do not meet Δ. Therefore, we have $\alpha_1 \geqslant 0$. Thus, there exists a uniform cover of $U_0 \times V_0$, with respect to the induced uniformity

$$(\mathscr{F}(F) \times \mathscr{F}(Y)) \upharpoonright (U_0 \times V_0) = (\mathscr{F}(F) \upharpoonright U_0) \times (\mathscr{F}(Y) \upharpoonright V_0),$$

call it $\mathscr{U}$, and $\alpha_2 < \alpha_1$ such that

$$\mathscr{G} \upharpoonright U \in [(\mathscr{F}(F) \upharpoonright U_0) \times (\mathscr{F}(Y) \upharpoonright V_0)]^{(\alpha_2)} \upharpoonright U$$

for all $U \in \mathscr{U}$. We can find uniform covers $\mathscr{U}'_1$ and $\mathscr{V}'_1$ of U_0 and V_0, respectively, such that $\mathscr{U}'_1 \times \mathscr{V}'_1 \prec \mathscr{U}$. Choose open covers $\mathscr{U}_1$ and $\mathscr{V}_1$ of F and Y, respectively, such that $\mathscr{U}_1 \upharpoonright U_0 \prec \mathscr{U}'_1$ and $\mathscr{V}_1 \upharpoonright V_0 \prec \mathscr{V}'_1$. As above, extend $\mathscr{U}_1$ to an open cover $\tilde{\mathscr{U}}_1$ of a Čech-complete subspace $G_1 \subset G_0$ of βF. The sets $U_0 \cap U$, where $U \in \mathscr{U}_1$, lie densely in $\tilde{U}_0$, and a fortiori so do the sets $\tilde{U}_0 \cap \tilde{U}$, $\tilde{U} \in \tilde{\mathscr{U}}_1$. For each $\tilde{U} \in \tilde{\mathscr{U}}_1$, $\tilde{U}_0 \cap \tilde{U} \cap (G_1 - F)$ is a dense subset of $\tilde{U}_0 \cap \tilde{U}$, so the sets $\tilde{U}_0 \cap \tilde{U} \cap (G_1 - F)$ lie densely in $\tilde{U}_0$. Therefore,

$\varphi[V_0] \cap \tilde{U}_0 \neq \emptyset$ implies that $\varphi[V_0] \cap (\tilde{U}_0 \cap \tilde{U}_1 \cap (G_1 - F)) \neq \emptyset$ for some $\tilde{U}_1 \in \tilde{\mathscr{U}}_1$. Choose $V_1 \in \mathscr{V}_1$ such that $\varphi[V_0 \cap V_1] \cap (\tilde{U}_0 \cap \tilde{U}_1) \neq \emptyset$. As above, we get

$$[(\tilde{U}_0 \cap \tilde{U}_1) \times (V_0 \cap V_1)] \cap \Delta \neq \emptyset,$$

and we see that $\mathscr{G} \restriction [(\tilde{U}_0 \cap \tilde{U}_1) \times (V_0 \cap V_1)]$ is not a trivial cover, which implies that $\alpha_2 \geqslant 0$. Thus, we have found a sequence $0 \leqslant \alpha_2 < \alpha_1 < \alpha_0$ of ordinal numbers.

For the inductive step, suppose that we have found sequences $(\alpha_i\colon i \in [n+1])$, $(\mathscr{U}_i \times \mathscr{V}_i\colon i \in [n])$, $(\tilde{\mathscr{U}}_i\colon i \in [n])$, $(\tilde{U}_i\colon i \in [n])$, $(V_i\colon i \in [n])$, and $(G_i\colon i \in [n])$ (where $[n]$ denotes the set $\{0, \ldots, n\}$) such that

(1) $\mathscr{U}_i \times \mathscr{V}_i$ is an open uniform cover of $\mathscr{F}F \times \mathscr{F}Y$ for all $i \in [n]$;

(2) $\tilde{\mathscr{U}}_i$ is an extension of $\mathscr{U}_i$ to an open cover of the Čech-complete subset G_i of βF for all $i \in [n]$;

(3) $\tilde{U}_i \in \tilde{\mathscr{U}}_i$, $U_i = \tilde{U}_i \cap F$ and $V_i \in \mathscr{V}_i$ for all $i \in [n]$;

(4) $F \subset G_n \subset \ldots \subset G_0 \subset \beta F$;

(5) for each $i \in [n]$, $\mathscr{G} \restriction (U_i \times V_i)$ is not a trivial cover;

(6) for all $i \in [n]$, $\mathscr{G} \restriction (U_i \times V_i) \in [\mathscr{F}(F) \times \mathscr{F}(Y)]^{(\alpha_{i+1})} \restriction (U_i \times V_i)$;

(7) $\alpha_{i+1} < \alpha_i$ for all $i \in [n]$;

(8) $\varphi[\bigcap_{i=0}^{n} V_i] \cap (\bigcap_{i=0}^{n} \tilde{U}_i) \neq \emptyset$.

Exactly as in the case $n = 1$, we find that $\alpha_{n+1} \geqslant 0$, and we find open covers $\mathscr{U}_{n+1}$, $\mathscr{V}_{n+1}$, a Čech-complete extension $G_{n+1} \subset G_n$ of F, an open extension $\tilde{\mathscr{U}}_{n+1}$ of $\mathscr{U}_{n+1}$ over G_{n+1}, $\alpha_{n+1} < \alpha_{n+1}$ and elements $\tilde{U}_{n+1} \in \tilde{\mathscr{U}}_{n+1}$, $V_{n+1} \in \mathscr{V}_{n+1}$ satisfying the above conditions (1)–(8) with n replaced by $n+1$. By complete induction we obtain an infinite decreasing sequence

$$\ldots < \alpha_{n+2} < \alpha_{n+1} < \alpha_n < \ldots < \alpha_0$$

of ordinal numbers, which is a contradiction. Thus, we conclude that $\mathscr{G}$ is not a member of $\lambda(\mathscr{F}(F) \times \mathscr{F}(Y))$, as required, and hence that $\mathscr{F}F \times \mathscr{F}Y$ is not supercomplete. This finishes the proof. ∎

Remark. In the above proof we found for a given non-C-scattered space X a space Y such that $\mathscr{F}X \times \mathscr{F}Y$ is not supercomplete. However, we did not check whether $X \times Y$ is paracompact or not. In case the paracompactness of $X \times Y$ is needed, Y can be replaced by a weakly σ-discrete stratifiable space provided by Junnila's construction [14], the product of which with X can be shown to be paracompact.

References

[1] A. Arkhangel'skii, *On a class of spaces containing all metric and all locally bicompact spaces*, Soviet Math. Dokl. 4 (1963), 1051–1055.

[2] I. M. Dektjarev, *A closed graph theorem for ultracomplete spaces*, Dokl. Akad. Nauk SSSR. 154 (1964), 771–773 (in Russian).

[3] S. Ginsburg and J. R. Isbell, *Some operators on uniform spaces*, Trans. Amer. Math. Soc. 93 (1959), 145–168.

[4] A. Hohti, *On supercomplete uniform spaces*, Proc. Amer. Math. Soc. 87 (3) (1983), 557–560.

[5] — *On supercomplete uniform spaces* II, Czechoslovak Math. J. 37 (1987), 376–385.

[6] — *On supercomplete uniform spaces* III, Proc. Amer. Math. Soc. 97 (1985), 339–342.

[7] A. Hohti and J. Pelant, *On supercomplete uniform spaces* IV, this volume, 119–124.

[8] A. Hohti, *On relative ω-cardinality and locally fine coreflections of products*, to appear in Topology Proc.

[9] — *On Ginsburg–Isbell derivatives and ranks of metric spaces*, Pacific J. Math. 111 (1) (1984), 39–48.

[10] — *Another alternative proof of Effros' theorem*, to appear in Topology Proc.

[11] J. Isbell, *Supercomplete spaces*, Pacific J. Math. 12 (1962), 287–290.

[12] — *Uniform Spaces*, Math. Surveys 12, Amer. Math. Soc., Providence, Rhode Island, 1964.

[13] — *A note on complete closure algebras*, Math. Systems Theory 3 (4) (1969), 310–312.

[14] H. J. K. Junnila, *Stratifiable pre-images of topological spaces*, Coll. Math. Soc. János Bolyai 23 (1980), 689–703.

[15] J. L. Kelley, *Hypercomplete linear topological spaces*, Michigan J. Math. 5 (1958), 235–246.

[16] K. Morita, *On the product of paracompact spaces*, Proc. Japan Acad. 39 (1963), 559–563.

[17] J. Pelant, *Locally fine uniformities and normal covers*, Czechoslovak Math. J. 37 (112), 1987, 181–187.

[18] H. Tamano, *On compactifications*, J. Math. Kyoto Univ. 1 (1962), 162–193.

[19] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math. 73 (1971), 59–74.

[20] R. C. Walker, *The Stone–Čech Compactification*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 83, Springer-Verlag, Berlin–Heidelberg–New York 1974.

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Received 7 June 1989