[11] D. Bünermann, *Auslander–Reiten quivers of exact one-parameter partially ordered sets*, in: Proc. ICRA III (Puebla 1980), Lecture Notes in Math. 903, Springer, 1981, 55–61.
[12] P. Dowbor, *Representations of hereditary rings* (in Polish), Ph. D. thesis, Nicholas Copernicus Univ., Toruń 1981.
[13] P. Dowbor and D. Simson, *Quasi-Artin species and rings of finite representation type*, J. Algebra 63 (1980), 435–443.
[14] P. Gabriel, *Auslander–Reiten sequences and representation-finite algebras*, Proc. ICRA II (Ottawa 1979), Lecture Notes in Math. 831, Springer, 1981, 1–71.
[15] — *The universal cover of a representation finite algebra*, Proc. ICRA III (Puebla 1980), Lecture Notes in Math. 903, Springer, 1981, 68–105.
[16] D. Happel, *Composition factors for indecomposable modules*, Proc. Amer. Math. Soc. 86 (1) (1982), 29–31.
[17] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (2) (1982), 399–443.
[18] K. Igusa and G. Todorov, *Radical layers of representable functors*, J. Algebra 89 (1984), 105–147.
[19] — — *A characterization of finite Auslander–Reiten quivers*, J. Algebra 89 (1984), 148–177.
[20] B. Klemp and D. Simson, *A diagrammatic characterization of schurian vector space PI-categories of finite type*, Bull. Polish Acad. Sci. Math. 32 (1984), 11–18.
[21] — — *Schurian sp-representation-finite right peak PI-rings and their indecomposable socle projective modules*, J. Algebra, 1990, in print.
[22] L. A. Nazarova and A. V. Roiter, *Representations of partially ordered sets*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 28 (1972), 5–31 (in Russian).
[23] C. M. Ringel, *On algorithms for solving vector space problems. I. Report on the Brauer–Thrall conjectures: Roiter's theorem and the theorem of Nazarova and Roiter*, in: Proc. Workshop ICRA II (Ottawa 1979), Lecture Notes in Math. 831, Springer, 1980, 104–136.
[24] — *Kawada's theorem, Abelian group theory*, Lecture Notes in Math. 874, Springer, 1981, 431–447.
[25] — *Tame algebras and integral quadratic forms*, Lecture Notes in Math. 1099, Springer, 1984.
[26] K. W. Roggenkamp, *Indecomposable representations of orders*, in: *Topics in Algebra, Part I: Rings and Representations of Algebras*, Banach Center Publications 26, PWN, Warszawa 1990, 449–491.
[27] D. Simson, *Partial Coxeter functors and right pure semisimple hereditary rings*, J. Algebra 71 (1981), 195–218.
[28] — *Vector space categories, right peak rings and their socle projective modules*, J. Algebra 92 (1985), 532–571.
[29] — *Socle reductions and socle projective modules*, J. Algebra 103 (1986), 18–68.
[30] — *On Representations of Partially Ordered Sets*, Paderborn, July 1986, 188 p.
[31] — *A narrow over-ring adjustification functor*, J. Algebra, to appear.
[32] — *Representations of bounded stratified posets, coverings and socle projective modules*, in: *Topics in Algebra, Part I: Rings and Representations of Algebras*, Banach Center Publications 26, PWN, Warszawa 1990, 499–453.
[33] — *A splitting theorem for multipeak path algebras*, Fund. Math. 137 (2) (1990).
[34] D. Simson and A. Skowroński, *Extensions of artinian rings by hereditary injective modules*, in: Proc. ICRA III (Puebla 1980), Lecture Notes in Math. 903 (1981), 315–330.

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On supercomplete uniform spaces IV: Countable products

by

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Abstract. We show that the product of countably many supercomplete C-scattered spaces is supercomplete. The result implies similar but weaker theorems of [1], [17] and [4].

1. Introduction. It is well known that the product of paracompact spaces is in general not paracompact. It was proved by Z. Frolík in [5] that a countable product of locally compact paracompact spaces is paracompact. The same is true for the larger class of paracompact p-spaces of Arkhangel'skii [2]. Recently a weaker structural condition of being scattered or C-scattered has been used by K. Alster [1], M. E. Rudin and S. Watson [17], and by L. M. Friedler, H. W. Martin and S. W. Williams in [4], to obtain similar results. We prove in this paper a natural extension of their results by showing that a countable product of supercomplete C-scattered spaces is supercomplete. The notion of supercompleteness was defined by J. R. Isbell in [13]; by his result — we can take it as a definition — a uniform space μX is *supercomplete* iff X is topologically paracompact and the Ginsburg–Isbell locally fine coreflection ([6]) $\lambda\mu$ is the fine uniformity of $\mathscr{F}(X)$ of X. By using the concept of metric-fine coreflections, we show at the end of the paper that a countable product of σ-C-scattered paracompact spaces is paracompact.

Our proof uses a simple recursive technique based on well-founded (or Noetherian) trees, applied e.g. in [11], [12], [15] in the context of uniform spaces.

2. Preliminaries. This section consists of preliminary definitions. We refer the reader to [14] for basic information on uniform spaces. For the definition of the Ginsburg–Isbell locally fine coreflection λ, the reader is referred to the first three papers [8], [9], [10] in our study on supercomplete spaces. A well-founded tree is a partially ordered set $\mathscr{T} = (T, \leqslant)$ with a unique minimal element $\mathrm{Root}(\mathscr{T})$ such that every branch, i.e., maximal linearly ordered subset, of $\mathscr{T}$ is finite. We denote by $\mathrm{End}(\mathscr{T})$ the set of all $\leqslant$-maximal elements of $\mathscr{T}$. Given $p \in T$, the set of all immediate $\leqslant$-successors of p is denoted by $S(p)$. Thus, $S(p) = \{q \in T\colon\ q > p \text{ and } q > r > p \text{ for no } r \in T\}$. Furthermore,

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for every $p \in \mathrm{End}(\mathcal{T})$ there is a unique branch $\{p_0, \ldots, p_n\}$ such that $p_0 = \mathrm{Root}(\mathcal{T})$, $p_n = p$ and $p_{i+1} \in S(p_i)$ for $0 \leqslant i < n$.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two well-founded trees. Given $p \in \mathrm{End}(\mathcal{T}_1)$, the symbol $\mathcal{T}_1 \vee_p \mathcal{T}_2$ denotes the tree obtained from $\mathcal{T}_1$ by "hanging" $\mathcal{T}_2$ below p, i.e., it is the tree $\mathcal{T} = (T, \leqslant)$ defined by

$$T = T_1 \cup (T_2 \setminus \{\mathrm{Root}(\mathcal{T}_2)\}),$$

with the partial order satisfying $r \leqslant q$ iff either

(1) $q, r \in T_i$ and $q \leqslant_i r$, $i = 1, 2$, or

(2) $q \in T_2$, $r \in T_1$ and $r \leqslant p$.

Obviously $\mathcal{T}_1 \vee_p \mathcal{T}_2$ is well-founded.

There is a strict connection between well-founded trees and locally fine coreflections. Indeed, if a μX is a uniform space and $\mathcal{U} \in \lambda\mu$, then there is a well-founded tree $\mathcal{T}$ and a natural map φ: $T \to 2^X$ satisfying the following properties:

(1) $\varphi(\mathrm{Root}(\mathcal{T})) = X$;

(2) $\varphi[S(p)]$ is a uniform cover of $\varphi(p)$ for each $p \in T \setminus \mathrm{End}(\mathcal{T})$;

(3) the sets $\cap\{\varphi(p)\colon p \in B\}$, where $B \in \mathcal{B}(\mathcal{T})$, the collection of all branches of $\mathcal{T}$, refine the cover $\mathcal{U}$.

Let $\boldsymbol{K}$ be a closed-hereditary class of topological spaces. A space X is called *$\boldsymbol{K}$-scattered* ([19]) if every nonempty closed subset of X has a point with a $\boldsymbol{K}$-neighbourhood. A *$\boldsymbol{K}$-exhaustion* of a $\boldsymbol{K}$-scattered space X is a sequence $(S_\alpha)_{\alpha<\beta}$ of open subsets of X such that (i) if $\gamma < \tau < \beta$, then $S_\gamma \subset S_\tau$, (ii) $\overline{S_\gamma \setminus \bigcup\{S_\alpha\colon \alpha < \gamma\}}$ is in $\boldsymbol{K}$ for all $\gamma < \beta$ and (ii) $\bigcup\{S_\alpha\colon \alpha < \beta\} = X$. The least such $\beta \in \mathrm{Ord}$ for X will be called the *$\boldsymbol{K}$-length* of X, written $\mathrm{length}_{\boldsymbol{K}}(X)$. The existence of $\boldsymbol{K}$-exhaustions can be proved by induction. Indeed, given a $\boldsymbol{K}$-scattered space X, let $S_0 = \emptyset$. Assuming that (S_α) has been defined for $\alpha < \tau$, let $Y_\tau = X \setminus \bigcup\{S_\alpha\colon \alpha < \tau\}$. In case $Y_\tau = \emptyset$, $(S_\alpha)_{\alpha<\tau}$ is an exhaustion of X; otherwise the closed subspace Y_τ contains a point with an (in X) open neighbourhood U such that $\overline{U \cap Y_\tau}$ is in $\boldsymbol{K}$; we put $S_\tau = \bigcup\{S_\alpha\colon \alpha < \tau\} \cup U$.

In the sequel we consider only the class $\boldsymbol{K} = \mathcal{C}$ of compact spaces, although our arguments can be extended to some other classes. (For $\mathcal{C}$-scattered paracompact spaces, see [18].)

Let $(X_n\colon n \in N)$ be a countable family of topological spaces. Then $\boldsymbol{T}(\prod_N X_i)$ denotes the set of all trees $\mathcal{T}$ with the following property:

$(*)$ the elements of $\mathcal{T}$ are subsets of $\prod_N X_i$ of the form $\prod_N Z_i$ where each $Z_i \subset X_i$ is closed, and where the set $\{i \in N\colon Z_i \neq X_i\}$ is an initial segment of N. The partial order of $\mathcal{T}$ is the set inclusion order.

For every element P of such a tree $\mathcal{T}$, define $\mathrm{seg}(P) = \max\{n \in N\colon \pi_n[P] \neq X_n\}$, where $\pi_n\colon \prod_N X_i \to X_n$ is the standard projection.

Now assume that the spaces X_i are $\mathcal{C}$-scattered and paracompact. Let $\mathcal{T} \in T(\prod_N X_i)$. For each $j \in N$, the *j-reduct* of $\mathcal{T}$ at $P \in \mathrm{End}(\mathcal{T})$, where $j \leqslant \mathrm{seg}(P)+1$, written $\mathrm{red}_j(\mathcal{T}, P)$, is defined as follows. Put $\alpha = \mathrm{length}_{\mathcal{C}}(\pi_j[P])$. If α is a successor ordinal, and $j \leqslant \mathrm{seg}(P)$, let $\mathrm{red}_j(\mathcal{T}, P) = \mathcal{T}$; if $j = \mathrm{seg}(P)+1$, then let $\{W_0, W_1\}$ be a normal closed cover of X_j such that $X_j \neq W_i$ for $i = 0, 1$, and let $\mathrm{red}_j(\mathcal{T}, P)$ be obtained from $\mathcal{T}$ by adding the sets $\pi_j^{-1}(W_i) \cap P$ below P, i.e., let

$$\mathrm{red}_j(\mathcal{T}, P) = \mathcal{T} \vee_P \mathcal{T}',$$

where $\mathcal{T}'$ is the member of $\boldsymbol{T}(\prod_N X_2)$ consisting of $P(= \mathrm{Root}(\mathcal{T}'))$ and the sets $\pi_j^{-1}[W_i] \cap P$, $i = 0, 1$. Otherwise, when α is a limit ordinal, $\pi_j[P]$ has an open cover by subsets S_β with $\beta < \alpha$; with respect to the fine uniformity of X_j, this cover is uniform and can be refined by a uniform closed cover $\mathcal{W}$. Let $\mathrm{red}_j(\mathcal{T}, P)$ be obtained by adding all the elements $\pi_j^{-1}[W] \cap P$ below P in $\mathcal{T}$. (Notice that $\mathrm{length}_{\mathcal{C}}(W) < \alpha$ for all $W \in \mathcal{W}$.)

Let X be $\mathcal{C}$-scattered. If $\mathrm{length}_{\mathcal{C}}(X)$ is a successor ordinal, then there is a compact subset of X, denoted by $\mathrm{top}_{\mathcal{C}}(X)$, such that if U is any neighbourhood of $\mathrm{top}_{\mathcal{C}}(X)$ in X, then $\mathrm{length}_{\mathcal{C}}(X \setminus U) < \mathrm{length}_{\mathcal{C}}(X)$. In case $\mathrm{length}_{\mathcal{C}}(X)$ is a limit ordinal, we simply define $\mathrm{top}_{\mathcal{C}}(X) = \emptyset$. (Notice that the functions red_j and $\mathrm{top}_{\mathcal{C}}$ are defined by using the axiom of choice.) We conclude this preliminary section by a simple lemma.

LEMMA 2.1. *Let $(P_n\colon n \in N)$ be a decreasing family of subsets of $\prod_N X_i$ such that for each $i \in N$ there is n_i such that $\pi_i[P_{n_i}]$ is compact. Given an open cover $\mathcal{V}$ of $\prod_N X_i$, there is j such that P_j is covered by finitely many elements from $\mathcal{V}$.*

Proof. Indeed,

$$P = \prod_N \pi_k[P_{n_k}]$$

is a product of compact sets. Thus, there is a finite $\mathcal{F} \subset \mathcal{V}$ such that $P \subset \bigcup\{V\colon V \in \mathcal{F}\}$. Write

$$P^{(r)} = \prod_{k=0}^{r} \pi_k[P_{n_k}] \times \prod_{k>r} X_k.$$

Then $P = \bigcap\{P^{(r)}\colon r \in N\}$. Hence, there is r such that already $P^{(r)}$ is covered by $\mathcal{F}$. Choose j with $P_j \subset P^{(r)}$. ■

3. The result. In this section we prove that the product of a countable family of $\mathcal{C}$-scattered supercomplete spaces is supercomplete. In the proof we use well-founded trees and $\mathcal{C}$-exhaustions, defined in Section 2, together with the following principle. Let X be a set, let $\mathcal{A} \subset 2^X$ be a subset of 2^X closed under arbitrary increasing unions and let ϕ: $\mathcal{A} \to \mathcal{A}$ be an expanding map, i.e., $A \subseteq \phi[A]$ for all $A \in \mathcal{A}$. Define maps ϕ^α, $\alpha \in \mathrm{Ord}$, as follows: put $\phi^{(0)}(A) = A$ for all $A \subset X$, let $\phi^{(\alpha+1)}(A) = \phi(\phi^{(\alpha)}(A))$ and let $\phi^{(\beta)}(A) = \bigcup\{\phi^{(\alpha)}(A)\quad \alpha < \beta\}$ for β a limit ordinal. There is (obviously) $\tilde{\alpha}$ with $\phi^{(\tilde{\alpha}+1)} = \phi^{(\tilde{\alpha})}$; we call for each $A \subset X$ the set $\phi^{(\tilde{\alpha})}[A]$ the *closure of A under ϕ*.

Now let us state the main theorem of our paper.

THEOREM 3.1. *Let $(\mu_n X_n\colon n \in N)$ be a countable family of $\mathcal{C}$-scattered supercomplete uniform spaces. Then $\prod_N \mu_i X_i$ is supercomplete.*

Proof. Let $\boldsymbol{T} = \boldsymbol{T}(\prod_N X_i)$ and note that we can assume that the spaces $u_i X_i$ are (non-compact) fine uniform spaces, since (by [14])

$$\lambda \prod_N \mu_i X_i = \lambda \prod_N \lambda\mu_i X_i = \lambda \prod_N \mathcal{F} X_i$$

when the spaces $\mu_i X_i$ are supercomplete. To show that $\prod_N \mu_i X_i$ is supercomplete, we shall prove that for any given open cover $\mathcal{V}$ of $\prod_N X_i$, there is a well-founded tree $\mathcal{T} \in \boldsymbol{T}$, with the following properties:

(1) Root$(\mathcal{T}) = \prod_N X_i$;

(2) for each $P \in T \setminus \text{End}(\mathcal{T})$, the elements of $S(P)$ form a uniform cover of the subset P of $\prod_N X_i$;

(3) the elements $P \in \text{End}(\mathcal{T})$ refine the cover $\mathcal{V}^{<\omega}$ (consisting of all the finite unions of numbers of $\mathcal{V}$). We can (and shall) assume that $\mathcal{V}$ consists of basic open sets. (Notice that every open cover of a Tychonoff space X is in $\mathcal{F}(X)$ iff for every open cover $\mathcal{V}$ of X, $\mathcal{V}^{<\omega}$ is in $\mathcal{F}(X)$, cf. [16].)

Next we shall define a map $E\colon \boldsymbol{T} \to \boldsymbol{T}$ as follows. Let $\mathcal{T} \in \boldsymbol{T}$ and let $P \in \text{End}(\mathcal{T})$. Let us first define a tree $E(\mathcal{T}, P)$. Recall that $\text{seg}(P) = \max\{i \in N\colon \pi_i[P] \neq X_i\}$. We have to consider 4 cases.

Case 1. There is $i \leqslant \text{seg}(P)$ such that $\text{length}_{\mathscr{C}}(\pi_i[P])$ is a limit ordinal. Let $E(\mathcal{T}, P) = \text{red}_i(\mathcal{T}, P)$.

Case 2. Otherwise, if P is covered by an element of $\mathcal{V}^{<\omega}$, let $E(\mathcal{T}, P) = \mathcal{T}$.

Case 3. Otherwise, if

$$Y = \bigcap_{i \leqslant \text{seg}(P)} \pi_i^{-1}[\text{top}_{\mathscr{C}}(\pi_i[P])]$$

is not covered by finitely many elements from $\mathcal{V}$, let $E(\mathcal{T}, P) = \text{red}_{\text{seg}(P)+1}(\mathcal{T}, P)$.

Case 4. Otherwise, Y is covered by finitely many elements from $\mathcal{V}$, and we can find, for all $i \leqslant \text{seg}(P)$, open subsets U_i, W_i of X_i such that

$$\text{top}_{\mathscr{C}}(\pi_i[P]) \subset U_i \subset \bar{U}_i \subset W_i$$

and $\bigcap\{\pi_i^{-1}[\bar{W}_i]\colon i \leqslant \text{seg}(P)\}$ is covered by an element of $\mathcal{V}^{<\omega}$. (This easily follows from our requirement that the elements of $\mathcal{V}$ be basic open sets.) Let $\mathcal{F}$ be the set of all $i \leqslant \text{seg}(P)$ with $\pi_i[P]$ non-compact. In case $\mathcal{F} \neq 0$, let $E(\mathcal{T}, P)$ be obtained from $\mathcal{T}$ by adding the elements $P \setminus \pi_j^{-1}[U_j]$ $(j \in \mathcal{F})$ and $\bigcap\{\pi_i^{-1}[\bar{W}_i]\colon i \leqslant \text{seg}(P)\} \cap P$ below P (in the obvious sense defined in Section 2); otherwise, simply let $E(\mathcal{T}, P) = \text{red}_{\text{seg}(P)+1}(\mathcal{T}, P)$. (Notice that if $\pi_i[P]$ is compact, then $\text{top}_{\mathscr{C}}(\pi_i[P]) = \pi_i[P]$.)

Finally, having thus defined the trees $E(\mathcal{T}, P)$ for all $P \in \text{End}(\mathcal{T})$, put

$$E(\mathcal{T}) = \bigcup_{P \in \text{End}(\mathcal{T})} E(\mathcal{T}, P).$$

The promised tree is obtained quickly from the map E. Let $\mathcal{T}_0$ be the tree consisting of one element, $\prod_N X_i$, and let $\mathcal{T}$ be the closure of $\mathcal{T}_0$ under the map E. (Obviously, the map E constructed above is expanding; $\mathcal{T}$ is a fixed point of E.)

To show that $\text{End}(\mathcal{T})$ is a cover of $\prod_N X_i$ refining $\mathcal{V}^{<\omega}$, it is enough to prove that $\mathcal{T}$ is well founded. To see this, suppose that $\mathcal{T}$ contains an infinite branch. Hence, there is a sequence $(P_n\colon n \in N)$ of elements $P_n \in T$ such that for each $n \in N$, $P_{n+1} \in S(P_n)$ and $P_{n+1} \subset P_n$. We claim that there is a sequence $(n_i\colon i \in N)$ such that $\pi_i[P_k]$ is compact for $k = n_i$. It then follows from Lemma 2.1 that some P_r is covered by an element of $\mathcal{V}^{<\omega}$, implying that Case 2 is applied at some P_r, stopping the branch, giving the desired contradiction. Thus, assume that there is j such that $\pi_j[P_n]$ is non-compact for all n. Then every application of Case 4 to P_n reduces $\mathscr{C}$-length: $\text{length}_{\mathscr{C}}(\pi_i[P_{n+1}]) < \text{length}_{\mathscr{C}}(\pi_i[P_n])$ for some $i \leqslant j$. Since there are no infinite decreasing sequences of ordinals, Case 4 is applied at most finitely many times. Similarly, Case 1 is applied at most finitely many times with respect to any coordinate $i \in N$, and there is an infinite subset $\{n_i\colon i \in N\}$ of N such that $n_i < n_{i+1}$ for all $i \in N$, Case 3 is applied to P_{n_i}, $\text{seg}(P_{n_i}) \geqslant i$, and $\pi_i[P_{n_i}] = \pi_i[P_k]$ for all $k \geqslant n_i$. Consequently, $\text{top}_{\mathscr{C}}(\pi_i[P_{n_i}]) = \text{top}_{\mathscr{C}}(\pi_i[P_{n_k}])$ whenever $i \leqslant k$. Define

$$P'_k = \bigcap\{\pi_i^{-1}[\text{top}_{\mathscr{C}}(\pi_i[P_{n_k}])]\colon i \leqslant k\}.$$

Then $(P'_k\colon k \in N)$ satisfies the conditions of Lemma 2.1, and hence there is P'_r covered by an element of $\mathcal{V}^{<\omega}$. This implies that Case 4 is applied to P_{n_r}: a contradiction. Hence, $\mathcal{T}$ is well founded. This completes the proof of Theorem 3.1. ∎

COROLLARY 3.2 ([4]). *The product of a countable family of $\mathscr{C}$-scattered paracompact spaces is paracompact.*

COROLLARY 3.3 ([1]). *The product of a countable family of $\mathscr{C}$-scattered Lindelöf spaces is Lindelöf.*

Proof. A Tychonoff space X is Lindelöf iff the uniform space cX – where $c(X)$ denotes the uniformity generated by all countable cozero-covers – is supercomplete ([3]). Thus, if $(X_i\colon n \in N)$ is a countable family of $\mathscr{C}$-scattered Lindelöf spaces, then by 3.1 the product $\prod_N cX_i$ is supercomplete, and clearly $c(\prod_N X_i)$ is finer than $\prod_N c(X_i)$. It follows that

$$\mathcal{F}(\prod_N X_i) \subseteq \lambda \prod_N c(X_i) \subseteq \lambda c(\prod_N X_i) \subseteq \mathcal{F}(\prod_N X_i)$$

implying that $\prod_N X_i$ is Lindelöf. ∎

Notice that a Tychonoff space X is ultraparacompact iff the uniformity $\mathscr{CL}(X)$ generated by all clopen disjoint covers is fine, in fact, iff $\mathscr{CL}(X)$ is a supercomplete uniformity, since the associated trees (cf. Section 2) are well founded. Thus, we obtain additionally

COROLLARY 3.4. *The product of a countable family of ultraparacompact $\mathscr{C}$-scattered spaces is ultraparacompact.*

Remark. A space X is called *σ-$\mathscr{C}$-scattered* if it is a countable union of closed $\mathscr{C}$-scattered subspaces. A cover $\mathcal{V}$ of a uniform space μX is called *σ-uniform* if there is a countable collection $(F_n\colon n \in N)$ of closed subspaces of X such that $X = \bigcup\{F_n\colon n \in N\}$ and for each n, $\mathcal{V} \restriction F_n$ is a uniform cover of the subspace F_n. Notice that if we replace in the proof of 3.1 $\mathscr{C}$-scattered by σ-$\mathscr{C}$-scattered, then – by using virtually the same proof but in applying the maps red_i to points P with a new coordinate i, we split P into countably many closed $\mathscr{C}$-scattered parts – we obtain a well-founded tree $\mathcal{T}$ such that for every $P \in T \setminus \text{End}(\mathcal{T})$, $S(P)$ is a σ-uniform cover of P. Note that the uniformity generated by all σ-uniform open covers of μX is the metric-fine coreflection $m\mu$ (see [6]). (Recall here that every uniform cover has a σ-uniformly discrete refinement, hence so does every σ-uniform cover.) By the modified proof of 3.1, $\lambda m \prod_N \mu_i$ contains every open cover of $\prod_N X_i$ which implies the following

THEOREM 3.5. *Let* $(\mu_i X_i\colon i \in N)$ *be a countable family of* σ-$\mathscr{C}$*-scattered supercomplete spaces. Then* $m(\prod_N \mu_i X_i)$ *is supercomplete.*

COROLLARY 3.6. *A countable product of* σ-$\mathscr{C}$*-scattered paracompact spaces is paracompact.*

References

[1] K. Alster, *A class of spaces whose Cartesian product with every hereditarily Lindelöf space is Lindelöf*, Fund. Math. 114 (1981), 173–181.
[2] A. Arkhangel'skii, *On a class of spaces containing all metric and all locally bicompact spaces*, Soviet Math. Dokl. 4 (1963), 1051–1055.
[3] H. Corson, *Determination of paracompactness by uniformities*, Amer. J. Math. 80 (1958), 185–190.
[4] L. M. Friedler, H. W. Martin and S. W. Williams, *Paracompact C-scattered spaces*, Pacific J. Math. 129 (1987), 277–296.
[5] Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. Math. 8 (1960), 747–750.
[6] — *A note on metric-fine spaces*, Proc. Amer. Math. Soc. 46 (1974), 111–119.
[7] S. Ginsburg and J. R. Isbell, *Some operators on uniform spaces*, Trans. Amer. Math. Soc. 93 (1959), 145–168.
[8] A. Hohti, *On supercomplete uniform spaces*, Proc. Amer. Math. Soc. (3) 87 (1983), 557–560.
[9] — *On supercomplete uniform spaces* II, Czechoslovak Math. J. 37 (1987), 376–385.
[10] — *On supercomplete uniform spaces* III, Proc. Amer. Math. Soc. 97 (1985), 339–342.
[11] A. Hohti and J. Pelant, *On complexity of metric spaces*, Fund. Math. 25 (1985), 133–142.
[12] M. Hušek and J. Pelant, *Extensions and restrictions in products of metric spaces*, Topology Appl. 25 (1987), 245–252.
[13] J. Isbell, *Supercomplete spaces*, Pacific J. Math. 12 (1962), 287–290.
[14] — *Uniform spaces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R. I., 1964.
[15] J. Pelant, *Locally fine uniformities and normal covers*, Czechoslovak Math. J. 37 (112) (1987), 181–187.
[16] M. D. Rice, *A note on uniform paracompactness*, Proc. Amer. Math. Soc. 62 (1977), 359–362.
[17] M. E. Rudin and S. Watson. *Countable products of scattered paracompact spaces*, ibid. 89 (1983), 551–552.
[18] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math. 73 (1971), 59–74.
[19] — *Spaces defined by topological games*, ibid. 88 (1975), 193–223.

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On supercomplete uniform spaces V: Tamano's product problem

by

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Abstract. In this paper we solve the analogue of Tamano's problem [8] for supercomplete spaces. We show that a supercomplete space μX has the property that its product with every supercomplete space is again supercomplete if, and only if, X is C-scattered [19].

1. Introduction. This is the last member in our series of papers [4]–[7] on supercomplete uniform spaces. These spaces were introduced and characterized by J. R. Isbell in [11]. By definition, μX is *supercomplete* if the uniform hyperspace $H(\mu X)$, equipped with the Hausdorff uniformity, is a complete uniform space. By [11], supercompleteness is a uniform form of paracompactness: μX is supercomplete iff (1) X is (topologically) paracompact and (2) the Ginsberg–Isbell locally fine coreflection $\lambda\mu$ [3], [11] is the fine uniformity of X. (In this case, every open cover of X can be analyzed combinatorially by using uniform covers as a starting point.) This notion has also been studied in the context of linear spaces and closed graph theorems [2], [15]; [10] gives an application to homogeneous spaces. Several results concerning product spaces and supercompleteness have been obtained in [4]–[7] and [8]; closely related questions on uncountable products are dealt with in [17].

In [18], H. Tamano asked for a characterization of paracompact spaces the product of which with every paracompact space is paracompact. While it is known [16] that in the class of p-spaces of Arkhangel'skii [1], such paracompact spaces are σ-locally compact, the general problem has proved to be difficult. In this paper we solve the analogous question for supercomplete spaces, with a relatively simple proof.

2. Preliminaries. The basic reference to uniform spaces is [12]. For a completely regular space X, $\mathscr{F}(X)$ denotes the fine uniformity of X, consisting of all the normal covers of X, and βX denotes the Čech–Stone compactification of X. The basic properties of the Čech–Stone compactification can be found e.g. in [20]. We repeat here the definition of (slowed-down) Ginsburg–Isbell derivatives (see [9]) of uniformities. Let $\mathscr{C}(X) \subseteq P(P(X))$ denote the collection of all covers of X. Then $\mathscr{C}(X)$ is ordered by the relation $\prec$ of refinement. Let μ, ν be filters in $\mathscr{C}(X)$ with respect to $\prec$. The symbol ν/μ

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