On supercomplete uniform spaces IV: Countable products

by

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Abstract. We show that the product of countably many supercomplete C-scattered spaces is supercomplete. The result implies similar but weaker theorems of [1], [17] and [4].

1. Introduction. It is well known that the product of paracompact spaces is in general not paracompact. It was proved by Z. Frolik in [5] that a countable product of locally compact paracompact spaces is paracompact. The same is true for the larger class of paracompact p-spaces of Arkhangel’skii [2]. Recently a weaker structural condition of being scattered or C-scattered has been used by K. Alster [13], M. E. Rudin and S. Watson [17], and by L. M. Friedler, H. W. Martin and S. W. Williams in [4], to obtain similar results. We prove in this paper a natural extension of their results by showing that a countable product of supercomplete C-scattered spaces is supercomplete. The notion of supercompleteness was defined by J. R. Isbell in [13]; by his result — we can take it as a definition — a uniform space μX is supercomplete iff X is topologically paracompact and the Ginsburg–Isbell locally fine coreflection (μX) fix is the fine uniformity of |μX| of X. By using the concept of metric-fine coreflections, we show at the end of the paper that a countable product of C-scattered paracompact spaces is paracompact.

Our proof uses a simple recursive technique based on well-founded (or Noetherian) trees, applied e.g. in [11], [12], [15] in the context of uniform spaces.

2. Preliminaries. This section consists of preliminary definitions. We refer the reader to [14] for basic information on uniform spaces. For the definition of the Ginsburg–Isbell locally fine coreflection λ, the reader is referred to the first three papers [3], [9], [10] in our study on supercomplete spaces. A well-founded tree is a partially ordered set T = (T, ≤) with a unique minimal element Root(T) such that every branch, i.e., a maximal linearly ordered subset, of T is finite. We denote by End(T) the set of all maximal elements of T. Given p ∈ T, the set of all immediate predecessors of p is denoted by S(p). Thus, S(p) = {q ∈ T: q > p and q > r > p for some r ∈ T}. Furthermore,

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for every $p \in \text{End}(\mathcal{F})$ there is a unique branch $\{p_0, \ldots, p_n\}$ such that $p_0 = \text{Root}(\mathcal{F})$, $p_n = p$, and $p_{i+1} \in s(p_i)$ for $0 \leq i < n$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two well-founded trees. Given $p \in \text{End}(\mathcal{F}_1)$, the symbol $\mathcal{F}_1 \vee_{p} \mathcal{F}_2$ denotes the tree obtained from $\mathcal{F}_1$ by "hanging" $\mathcal{F}_2$ below $p$, i.e., it is the tree $\mathcal{F} = (T, \leq)$ defined by

$$T = T_1 \cup (T_2 \backslash \text{Root} (\mathcal{F}_2)))$$

with the partial order satisfying $r \leq s$ iff either

(1) $q, r \in T_1$ and $q \leq s$, $r = 1, 2$, or

(2) $q \in T_1$, $r \in T_2$, and $r \leq s$.

Obviously $\mathcal{F}_1 \vee_{p} \mathcal{F}_2$ is well-founded.

There is a strict connection between well-founded trees and locally fine correlections.

Indeed, if $\mu X$ is a uniform space and $\mathcal{U} \in \Lambda_\mu$, then there is a well-founded tree $\mathcal{F}$ and a natural map $\phi: T \to 2^X$ satisfying the following properties:

(1) $\phi(\text{Root}(\mathcal{F})) = X$;

(2) $\phi[S(p)]$ is a uniform cover of $\phi(p)$ for each $p \in \text{End}(\mathcal{F})$;

(3) the sets $\cap \phi(p): p \in B(\mathcal{F})$, the collection of all branches of $\mathcal{F}$,

refine the cover $\mathcal{U}$.

Let $K$ be a closed-hereditary class of topological spaces. A space $X$ is called $K$-scattered ([19]) if every nonempty closed subset of $X$ has a point with a $K$-neighbourhood.

A $K$-exhaustion of a $K$-scattered space $X$ is a sequence $(S_n)_{n \in \mathbb{N}}$ of open subsets of $X$ such that (i) if $\gamma < \alpha$, then $S_\gamma \subseteq S_\alpha \subseteq S_\alpha \setminus \{s_\gamma: s_\gamma < \alpha\}$ is in $K$ for all $\gamma < \beta$ and (ii) $\bigcup S_\alpha = X$. The least such $\alpha$ in $K$ for $X$ will be called the $K$-length of $X$, written $\text{length}_K(X)$. The existence of $K$-exhaustions can be proved by induction. Indeed, given a $K$-scattered space $X$, let $S_\alpha = \emptyset$. Assuming that $(S_n)$ has been defined for $\gamma < \alpha$, let $Y_\alpha = X \setminus \bigcup\{S_\gamma: \gamma < \alpha\}$. In case $Y_\alpha \neq \emptyset$, $(S_n)$ is an exhaustion of $X$; otherwise the closed subspace $Y_\gamma$ contains a point with an (in $X$) open neighbourhood $U$ such that $U \cap Y_\alpha$ is in $K$; we put $S_\alpha = \bigcup\{S_\gamma: \gamma < \alpha\} \cup U$.

In the sequel we consider only the class $K = \mathcal{U}$ of compact spaces, although our arguments can be extended to some other classes. (For $\mathcal{U}$-scattered paracompact spaces, see [185].

Let $(X_n: n \in \mathbb{N})$ be a countable family of topological spaces. Then $T(\prod X_n)$ denotes the set of all trees $\mathcal{F}$ with the following property:

(*) the elements of $\mathcal{F}$ are subsets of $\prod X_n$ of the form $\prod Z_n$ where each $Z_n$ is closed, and where the set $\{n \in \mathbb{N}: Z_n \neq X_n\}$ is an initial segment of $\mathbb{N}$. The partial order of $\mathcal{F}$ is the set inclusion order.

For every element $p$ of such a tree $\mathcal{F}$, define $\text{seg}(p) = \{n \in \mathbb{N}: \exists A(p) \neq X_n\}$, where $\text{seg}(p)$ is the standard projection.

Now assume that the spaces $X_n$ are $\mathcal{U}$-scattered and paracompact. Let $\mathcal{F} \in T(\prod X_n)$. For each $n \in \mathbb{N}$, the $j$-reduct of $\mathcal{F}$ at $p \in \text{End}(\mathcal{F})$, where $j \leq \text{seg}(p) + 1$, written $\text{red}_j(\mathcal{F}, p)$, is defined as follows. Put $a = \text{length}_\mathcal{U}(n, \mathcal{F})$. If $a$ is a successor ordinal, and $j < \text{seg}(p)$, let $\text{red}_j(\mathcal{F}, p) = \mathcal{F}$; if $j = \text{seg}(p) + 1$, then let $\text{red}_j(\mathcal{F}, p)$ be a normal closed cover of $X_n$ such that $X_j \neq X_n$ for $0 \leq i < 1$, and $1$, and let $\text{red}_j(\mathcal{F}, p)$ be obtained from $\mathcal{F}$ by adding the sets $\pi^{-1}_j(W_j) \cap P$ below $p$, i.e., let

$$\text{red}_j(\mathcal{F}, p) = \mathcal{F} \cup_{p} \mathcal{F}_j$$

where $\mathcal{F}_j$ is the member of $T(\prod X_n)$ consisting of $P(= \text{Root}(\mathcal{F}))$ and the sets $\pi^{-1}_j(W_j) \cap P$, $j = 0, 1$. Otherwise, when $a$ is a limit ordinal, $\pi[A]$ has an open cover by subsets $S_\beta$ with $\beta < a$ with respect to the fine uniformity of $X_n$, this cover is uniform and can be refined by a uniform closed cover $\mathcal{U}$. Let $\text{red}(\mathcal{F}, p)$ be obtained by adding all the elements of $\pi^{-1}_j(W_j) \cap P$ below $P$ in $\mathcal{F}$. (Notice that $\text{length}_\mathcal{U}(W) < a$ for all $W \in \mathcal{U}$.)

Let $X$ be $\mathcal{U}$-scattered. If $\text{length}_\mathcal{U}(X)$ is a successor ordinal, then there is a compact subset of $X$, denoted by $\text{top}(X)$, such that if $U$ is any neighbourhood of $\text{top}(X)$ in $X$, then $\text{length}_\mathcal{U}(X) < U < \text{length}_\mathcal{U}(X)$. In case $\text{length}_\mathcal{U}(X)$ is a limit ordinal, we simply define $\text{top}(X) = \emptyset$. (Notice that the functions $\text{red}$ and $\text{top}$ are defined by using the axiom of choice.)

We conclude this preliminary section by a simple lemma.

**Lemma 2.1.** Let $(P_n: n \in \mathbb{N})$ be a decreasing family of subsets of $\prod X_n$ such that for each $n \in \mathbb{N}$ there is $n_0$ such that $\pi[A]_n$ is compact. Given an open cover $\mathcal{V}$ of $\prod X_n$, there is $n_0$ such that $P_n$ is covered by finitely many elements from $\mathcal{V}$.

**Proof.** Indeed,

$$P = \prod_{n} \pi_{i}(P_{n})$$

is a product of compact sets. Thus, there is a finite $\mathcal{F} \subseteq \mathcal{V}$ such that $P \subseteq \bigcup \{V: V \in \mathcal{F}\}$. Write

$$P_{n_0} = \prod_{k=0}^{n_0} \pi_k(P_{n}) \times \prod_{k=n_0+1}^{\mathbb{N}} X_k.$$

Then $P = \bigcup \{P_{n_0}: n \in \mathbb{N}\}$. Hence, there is $n$ such that already $P_{n_0}$ is covered by $\mathcal{F}$. Choose $n$ with $P_{n_0} \subseteq P_{n}$.

**3. The result.** In this section we prove that the product of a countable family of $\mathcal{U}$-scattered supercomplete spaces is supercomplete. In the proof we use well-founded trees and $\mathcal{U}$-exhaustions, defined in Section 2, together with the following principle. Let $X$ be a set, let $\alpha < 2^X$ be a subset of $2^X$ closed under arbitrary increasing unions and let $\phi: \alpha \to \alpha$ be an expanding map, i.e., $A \subseteq \phi[A]$ for all $A \in \alpha$. Define maps $\phi^*, \phi^* \in \text{Ord}$, as follows: put $\phi^*(A) = A$ for all $A \subseteq X$, let $\phi^*(A) = \bigcup \{\phi^*(B): B \subseteq X\}$ for $\beta$ a limit ordinal. There is (obviously) $\phi^* \in \text{Ord}$ with $\phi^*(A) = \phi^*(B)$; we call for each $A \subseteq X$ the set $\text{closure}(A)$ under $\phi$.

Now let us state the main theorem of our paper.

**Theorem 3.1.** Let $(\mu X_n: n \in \mathbb{N})$ be a countable family of $\mathcal{U}$-scattered supercomplete uniform spaces. Then $\prod_{n} \mu X_n$ is supercomplete.

**Proof.** Let $T = T(\prod_{n} X_n)$ and note that we can assume that the spaces $\mu X_n$ are (non-compact) fine uniform spaces, since (by [14])

$$\lambda \prod_{n} X_n = \lambda \prod_{n} \mu X_n = \lambda \prod_{n} X_n$$

when the spaces $\mu X_n$ are supercomplete. To show that $\prod_{n} \mu X_n$ is supercomplete, we shall prove that for any given open cover $\mathcal{V}$ of $\prod_{n} X_n$, there is a well-founded tree $\mathcal{F} \in T$, with the following properties:
(1) Root(\(\mathcal{F}\)) = \bigcap X_i;

(2) for each \(P \in \text{End}(\mathcal{F})\), the elements of \(S(P)\) form a uniform cover of the subset \(P\) of \(\bigcap X_i\);

(3) the elements \(P \in \text{End}(\mathcal{F})\) refine the cover \(\gamma^{<\omega}\) (consisting of all the finite unions of numbers of \(\gamma\)). We can (and shall) assume that \(\gamma\) consists of basic open sets. (Notice that every open cover of a Tychonoff space \(X\) is in \(\mathcal{F}(X)\) iff for every open cover \(\gamma\) of \(X\), \(\gamma^{<\omega}\) is in \(\mathcal{F}(X)\), cf. [16]).

Next we shall define a map \(E: T \to T\) as follows. Let \(\mathcal{F} \in T\) and let \(P \in \text{End}(\mathcal{F})\). Let us first define a tree \(E(\mathcal{F}, P)\). Recall that \(\text{seg}(P) = \max\{i \in \mathbb{N} : P_i \neq X_i\}\). We have to consider 4 cases.

Case 1. There is \(i \leq \text{seg}(P)\) such that \(\text{length}_\gamma(P_i)\) is an ordinal. Let \(E(\mathcal{F}, P) = \text{red}_\gamma(\mathcal{F}, P)\).

Case 2. Otherwise, if \(P\) is covered by an element of \(\gamma^{<\omega}\), let \(E(\mathcal{F}, P) = \mathcal{F}\).

Case 3. Otherwise, if \(\gamma\) is not covered by finitely many elements from \(\gamma\), let \(E(\mathcal{F}, P) = \text{red}_\gamma(\mathcal{F}, P)\).

Case 4. Otherwise, \(\gamma\) is covered by finitely many elements from \(\gamma\), and we can find, for all \(i < \text{seg}(P)\), open subsets \(U_i\) of \(X_i\) such that

\[
\text{top}_\gamma(P_i) \subseteq U_i \subseteq \mathcal{F}
\]

and

\[
\bigcap \{\pi^{-1}_i(\mathcal{F}) : i < \text{seg}(P)\}\n\]

is covered by an element of \(\gamma^{<\omega}\). (This easily follows from our requirement that the elements of \(\gamma\) be basic open sets.) Let \(E\) be the set of all \(i < \text{seg}(P)\) with \(\pi_i(P)\) non-compact. In case \(\mathcal{F} \neq 0\), let \(E(\mathcal{F}, P)\) be obtained from \(\mathcal{F}\) by adding the elements \(P_i^{-1}(U) : i \in \mathcal{F}\) and \(\pi_i^{-1}(\mathcal{F}) : i < \text{seg}(P)\) above \(P\) (in the obvious sense defined in Section 2); otherwise, simply let \(E(\mathcal{F}, P) = \text{red}_\gamma(\mathcal{F}, P)\). (Notice that if \(\pi_i(P)\) is compact, then \(\text{top}_\gamma(P_i) = \pi_i(P)\).)

Finally, having thus defined the trees \(E(\mathcal{F}, P)\) for all \(P \in \text{End}(\mathcal{F})\), put

\[
E(\mathcal{F}) = \bigcup_{P \in \text{End}(\mathcal{F})} E(\mathcal{F}, P).
\]

The promised tree is obtained quickly from the map \(E\). Let \(\mathcal{F}\) be the tree consisting of one element, \(\bigcap X_i\), and let \(\mathcal{F}\) be the closure of \(\mathcal{F}\) under the map \(E\). (Obviously, the map \(E\) constructed above is expanding; \(\mathcal{F}\) is a fixed point of \(E\)).

To show that \(\text{End}(\mathcal{F})\) is a cover of \(\bigcap X_i\), it is enough to prove that \(\mathcal{F}\) is well-founded. To see this, suppose that \(\mathcal{F}\) contains an infinite branch. Hence, there is a sequence \((P_n : n \in \mathbb{N})\) of elements \(P_n \in \mathcal{F}\) such that for each \(n \in \mathbb{N}\), \(P_n \subseteq S(P_{n+1})\). We claim that there is a sequence \((n_r : r \in \mathbb{N})\) such that \(\pi_r(P_{n_r})\) is compact for \(r = n\). It then follows from Lemma 2.1 that some \(P_n\) is covered by an element of \(\gamma^{<\omega}\), implying that Case 2 is applied at some \(P_n\), stopping the branch, giving the desired contradiction. Thus, assume that there is \(\mathcal{F}\) such that \(\pi_r(P_{n_r})\) is non-compact for all \(n\). Then every application of Case 4 to \(P_n\) reduces \(\mathcal{F}\)-length.
Theorem 3.5. Let \( \{\mu_k X_k \mid k \in \mathbb{N}\} \) be a countable family of \( \sigma\delta \)-scattered supercomplete spaces. Then \( m(\bigcap_k \mu_k X_k) \) is supercomplete.

Corollary 3.6. A countable product of \( \sigma\delta \)-scattered paracompact spaces is paracompact.

References


On supercomplete uniform spaces V: Tamano's product problem

by

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Abstract. In this paper we solve the analogue of Tamano's problem [8] for supercomplete spaces. We show that a supercomplete space \( \mu X \) has the property that its product with every supercomplete space is again supercomplete if, and only if, \( X \) is \( C \)-scattered [19].

1. Introduction. This is the last member in our series of papers [4]-[7] on supercomplete uniform spaces. These spaces were introduced and characterized by J. R. Isbell in [11]. By definition, \( \mu X \) is supercomplete if the uniform hyperspace \( H(\mu X) \), equipped with the Hausdorff uniformity, is a complete uniform space. By [11], supercompleteness is a uniform form of paracompactness: \( \mu X \) is supercomplete iff (1) \( X \) is (topologically) paracompact and (2) the Ginsberg–Isbell locally fine corefinement \( H(\mu X) \) is the fine uniformity of \( X \). In this case, every open cover of \( X \) can be analyzed combinatorially by using uniform covers as starting point. This notion has also been studied in the context of linear spaces and closed graph theorems [2], [15]; [10] gives an application to homogeneous spaces. Several results concerning product spaces and supercompleteness have been obtained in [4]-[7] and [8]; closely related questions on uncountable products are dealt with in [17].

In [18], H. Tamano asked for a characterization of paracompact spaces the product of which with every paracompact space is paracompact. While it is known [16] that in the class of \( \gamma \)-spaces of Arkhangel'skii [2], such paracompact spaces are \( \alpha \)-locally compact, the general problem has proved to be difficult. In this paper we solve the analogous question for supercomplete spaces, with a relatively simple proof.

2. Preliminaries. The basic reference to uniform spaces is [12]. For a completely regular space \( X \), \( \mathcal{U}(X) \) denotes the fine uniformity of \( X \), consisting of all the normal covers of \( X \), and \( A X \) denotes the Čech–Stone compactification of \( X \). The basic properties of the Čech–Stone compactification can be found e.g. in [20]. We repeat here the definition of (slowed-down) Ginsburg–Isbell derivatives (see [9]) of uniformities. Let \( \mathcal{U}(X) \) denote the collection of all covers of \( X \). Then \( \mathcal{U}(X) \) is ordered by the relation \( \prec \) of refinement. Let \( \mu, \nu \) be filters in \( \mathcal{U}(X) \) with respect to \( \prec \). The symbol \( \nu/\mu \)