On the Auslander–Reiten valued quiver of right peak rings

by

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Abstract. Let $R$ be a right peak artinian ring (1.1). We prove that under suitable assumptions (3.7) on $R$ there exists a preprojective component of the Auslander–Reiten valued quiver $\Gamma_{\text{val}}(R)$ of the category $\text{mod}_{\text{sa}}(R)$ of finitely generated socle projective $R$-modules. If $R$ admits a splitting poset decomposition (5.0) a splitting structure of the category $\text{mod}_{\text{sa}}(R)$ is described in (5.12).

1. Introduction. We recall from [28] that a semiperfect ring $R$ is a right peak ring if $R$ is a generalized matrix ring of the form

$$
R = \begin{bmatrix}
F_1 & M_2 & \ldots & M_n & M_n
\end{bmatrix}
$$

such that $\text{soc}(R)$ is an essential right ideal in $R$ isomorphic to a direct sum of finitely many copies of $P_n$ (called the right peak of $R$). Here $F_1, \ldots, F_n$ are local rings, $F = F_n$ is a division ring, $M_j$ is an $F_j\otimes F_j$-bimodule and the multiplication in $R$ is given by $F_j\otimes F_j$-bimodule maps $c_{ij}: M_j\otimes M_j \to M_i$ satisfying the natural associativity conditions. We denote by $P_1, \ldots, P_n, P_n$ the right indecomposable row ideals of $R$. Throughout we suppose that $R$ is basic, i.e. $R/\text{J}(R)$ is a product of division rings, where $\text{J}(R)$ is the Jacobson radical of $R$. We denote by $\text{mod}_{\text{sa}}(R)$ the category of finitely generated socle projective right $R$-modules. The ring $R$ is called sp-representation-finite if the number of isomorphism classes of indecomposable modules in $\text{mod}_{\text{sa}}(R)$ is finite.

We use the terminology and notation introduced in [28, 31].

Let us recall that if $I$ is a finite posed and $I^* = I \cup \{\ast\}$, where $i \prec \ast$ for all $i \in I$, then given a division ring $F$ the incidence algebra $FI^*$ is a right peak ring and $\text{mod}_{\text{sa}}(FI^*)$ is

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equivalent to the category $I$-sp of $I$-spaces over $F$ [28, 30]. On the other hand, if $A$ is an $R$-order in a simple algebra $C$ over the field of fractions of $R$ then

$$A_{C} = \begin{bmatrix} A & C \\ 0 & C \end{bmatrix}$$

is a noetherian right peak ring and the category $latt(A)$ of right $A$-lattices [26] is equivalent to the factor category $mod_{u}(A_{C})/P_{C} = \text{adj}(A_{C})$ [31]. Let us also recall from [28] that if $K_{R}$ is a vector space category and $K_{R}$ is the right peak ring of $K_{R}$ then there is a full dense functor $H: \mathcal{V}(K_{R}) \to \text{mod}_{u}(R_{K})$ preserving the representation type, where $\mathcal{V}(K_{R})$ is the factor space category of $K_{R}$. This functor is frequently applied in the representation theory of algebras [25, 29].

One of the main aims of this paper is to show (Section 3) that under suitable assumptions on $R$ there is a preprojective component $\varphi_{Q}(R)$ in the Auslander--Reiten valued quiver $\Gamma_{u}(R)$ of $\text{mod}_{u}(R)$ (see Sections 2 and 3, compare with [28; Remark 7]). In case $R$ is sp-representation-finite this will provide a simple algorithm presented in Section 4 for constructing all indecomposable modules in $\text{mod}_{u}(R)$ in a way described in [14] for mod($R$).

In Section 5 we show that if $R$ admits a splitting poset decomposition (5.0) of its valued poset $(I_{R}, d)$ then $\Gamma_{u}(R)$ can be glued from $\Gamma_{u}(A)$ and $\Gamma_{u}(B)$ along a single linear section, where $A$ and $B$ are right peak rings derived from $R$ (5.3).

We say that $R$ is a $PI$-ring if $R$ satisfies a polynomial identity, which in the case of $R$artinian means that the division rings $F_{1}, F_{2}, \ldots, F_{n}$ are finite-dimensional over their centers. In case $R$ is an artinian invariant of $R$ its value scheme $(I_{R}, d)$ [28, 29], where $I_{R} = \{i, \ldots, n, s = n+1\}$ and there is an arrow

$$i \rightarrow j$$

iff $d_{ij} = \text{length}(M_{j})_{i}, d_{ji} = \text{length}(M_{i})_{j}$ are nonzero. If the bimodules $M_{s}, \ldots, M_{n}$ are simple and $R$ is schurian (i.e., $F_{1}, \ldots, F_{n}$ are division rings) then $(I_{R}, d)$ is a valued poset [28; Prop. 2.3], with respect to the relation $i < j \iff \dim M_{j} \neq 0$.

In [20, 21] sp-representation-finite schurian $PI$-rings $R$ are characterized in terms of $(I_{R}, d)$ and it is proved that every indecomposable module $X = (X_{R}, \rho_{R})$ in $\text{mod}_{u}(R)$ over such a ring $R$ is uniquely determined by its dimension vector

$$\dim X = (x_{1}, \ldots, x_{n})$$

where $X_{i} = X_{i}(e_{i})$ and $x_{s} = \dim(X_{i})_{s}$ for $1 \leq i \leq n$. The algorithm presented in Section 3 based on the construction (3.1) of $\varphi_{Q}(R)$ allows us to calculate $\dim X$ in terms of the matrices $(d_{ij}), (d_{ij})$ for every indecomposable $X$ in $\text{mod}_{u}(R)$, where $R$ is as above.

2. Preliminaries and notation. Let us recall that $\text{mod}_{u}(R)$ is closed under direct sums, summands, kernels, extensions and has enough projectives because $F_{1}, \ldots, F_{n}$ are in $\text{mod}_{u}(R)$.

If $e = e_{1} + \ldots + e_{s}$, $A = eRe$, $M = eR(1-e)$ then

$$R_{A} \cong \begin{bmatrix} A & M \\ 0 & F \end{bmatrix}$$

and the ring [28]

$$R^{e} = \begin{bmatrix} F & M^{e} \\ 0 & A \end{bmatrix}, \quad M^{e} = \text{Hom}_{R}(M_{p}, F)$$

is a left peak $PI$-ring. If $R$ is an artinian $PI$-ring then there is a reflection duality [28; Corollary 2.7], [33; 2.6]

$$(0.0) \quad D^{e} = DV: \text{mod}_{u}(R) \to \text{mod}_{u}(R^{e})$$

where $R^{e} = (R^{e})^{op}$ and $R^{op}$ is the ring Morita dual to $R^{e}$.

It follows that $\text{mod}_{u}(R)$ has enough sp-injective modules [28; Cor. 2.7]. Let us recall from [28] that $Q$ in $\text{mod}_{u}(R)$ is sp-injective if $Q$ is injective with respect to monomorphisms $q: X \to X$ in $\text{mod}_{u}(R)$ such that $\text{Coker} \theta$ is also in $\text{mod}_{u}(R)$.

We shall need the following result.

Lema 2.1. Let $R$ be an artinian schurian upper triangular right peak $PI$-ring of the form (1.1) and let $M_{R} = \text{Hom}_{R}(M_{p}, F_{i})$, where $i, j, \ldots, n$. Then

(a) $T = R^{e}$ is an artinian schurian right peak $PI$-ring and the valued posed $(I_{R}, J)$ of $T$ is obtained from $(I_{R}, d)$ by reversing direction of all arrows between elements in $I_{R} = I_{R} = \{i, \ldots, n\}$.

(b) The injective envelope $E^{j}$ of top($P_{i}$) in $\text{mod}_{u}(R)$ has the form

$$E^{j} = (\{M_{1}, \ldots, M_{n}\}, F_{i}, 0, \ldots, 0, \theta_{i})$$

where $\theta_{i}(M_{j}) \otimes M_{p} \to M_{j}$ is such that its $F_{i}$-dual corresponds via the isomorphism

$$\text{Hom}_{R}(M_{j}, M_{k}) \cong \text{Hom}_{R}(M_{k}, M_{j})$$

to the map $\theta_{i}$ adjoint to $\theta_{ij}$. Moreover,

$$\dim E^{j} = (d_{ij}, d_{ji}, \ldots, d_{ji}, 1, 0, \ldots, 0).$$

(c) The modules

$$Q^{(j)} = E(P_{i}), \quad Q^{(j)} = \varphi^{-1} E^{j}, \quad i, j, \ldots, n$$

form a complete list of isoclasses of indecomposable sp-injective modules in $\text{mod}_{u}(R)$ and

$$\dim Q^{(j)} = (d_{j}, d_{j}, \ldots, 0, \ldots, 0).$$

where $d_{ij} = d_{ji} = d_{ji}$ and $d_{ii} = d_{ii}$ for $1 \leq i \leq n$.

Proof. (a) follows from [28; Proposition 2.5] and Proposition 2.4 below. (b) and the first part of (c) follow immediately from [28; Propositions 2.5 and 2.6]. For (2.3) first compute the forms $(2.2)$ by applying the function $\varphi^{-1}$ to $E^{j}$ in the category of finitely generated top injective right $R^{e}$-modules (denoted by $\text{mod}_{u}(R^{e})$) and then compute the corresponding dimensions keeping in mind the following result proved in [13].

Proposition 2.4. If $F_{i}$ and $F_{j}$ are division rings finite-dimensional over their centers and $N_{j}$ is a finite-dimensional $F_{i}$-$F_{j}$-bimodule then

$$\dim_{R}(N_{j}) = \dim_{N_{j}}(N_{j})$$

and

$$\dim_{N_{j}}(N_{j}) = \dim_{R}(N_{j}).$$
In the study of $\text{mod}_R(R)$ it is very convenient to use almost split sequences and irreducible maps. 

We recall from [3, 4] that a homomorphism $f: X \to Y$ in $\text{mod}_R(R)$ is irreducible if $f$ is neither a split monomorphism nor a split epimorphism and in any factorization $f = hg$ in $\text{mod}_R(R)$ either $g$ is a split monomorphism or $h$ is a split epimorphism. It is easy to check that for $X, Y$ indecomposable over an artinian ring $R$ the map $f: X \to Y$ is irreducible if and only if $f \in \text{End}(X, Y) - \text{PJB}(X, Y)$, where $J(X, Y)$ consists of all $f \in \text{Hom}_R(X, Y)$ such that $\text{id}_X - gf$ is invertible for all $g \in \text{End}_R(Y, X)$ and $\text{PJB}(X, Y)$ consists of $f = f'f'' \in J(X, Y)$ with $f' \in J(Z, Y), f'' \in J(X, Z)$ for some $Z$. We note that (see [23; 2.5])

$$\text{Irr}(X, Y) = J(X, Y)/J^2(X, Y)$$

is an $F(Y)-F(X)$-bimodule where $F(Z) = \text{End}(Z)/\text{End}(Z)$. Following Auslander [2] and Ringel [24] we define the Auslander–Reiten valued quiver $(\Gamma_{\text{mod}}(R), \delta)$ of $\text{mod}_R(R)$ as the set of isoclasses $[X]$ of indecomposable modules $X$ in $\text{mod}_R(R)$ connected by oriented valued arrows

$$[X] \xrightarrow{\delta_{XY}} [Y]$$

when $X \neq Y$ and the dimensions

$$d_{XY} = \text{dim} \text{Irr}(X, Y)_{\text{proj}}, \quad d_{XY} = \dim \text{Irr}(X, Y)_{\text{soc}}$$

are nonzero (see also [18, 19, 23]). This means that there are irreducible maps

$$\text{mod}_R(R)$$

in $\text{mod}_R(R)$. We note that $(\Gamma_{\text{mod}}(R), \delta)$ has a unique maximal element $[E(P_n)]$ and a unique minimal element $[P_0]$.

We recall from [3, 4] that an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\text{mod}_R(R)$ is said to be almost split if it does not split, $X$, $Y$ are indecomposable and in addition it has the following equivalent properties:

(a) $f$ is left almost split in the sense that given any map $h: X \to V$ in $\text{mod}_R(R)$ which is not a split monomorphism, there is a map $i: V \to Y$ such that $h' = h$. 

(b) $g$ is right almost split in the sense that given any $h: U \to Z$ in $\text{mod}_R(R)$ which is not a split epimorphism, there is a map $i: U \to Y$ such that $gi = h$.

Note that left almost split maps as well as right almost split maps are irreducible.

The reader is referred to [4, 25] for elementary facts about almost split sequences and irreducible maps in $\text{mod}_R(R)$.

Let us recall that $P, f: X \to P$ is irreducible and right almost split. Moreover, if $R$ is an artinian $PI$-ring and $Q^{\delta} = P^{\delta} \otimes E(R)$ is the indecomposable $sp$-injective module (2.2) then the induced map

$$\lambda_j: Q^{\delta} \to P^{\delta} \otimes \text{soc}(E(R))$$

is left almost split in $\text{mod}_R(R)$ because $E(R) \to E(R) \otimes \text{soc}(E(R))$ is left almost split in $\text{mod}_R(R)$.

Note also that if $P \to X$ is irreducible in $\text{mod}_R(R)$ and $X$ is indecomposable then $X$ is projective (see [27; Lemma 1.3.2]).

We say that $R$ has almost split sp-sequences if every indecomposable non-sp-injective socle projective module $X$ admits a right almost split sequence

$$0 \to X \to Y \xrightarrow{\delta} X \to 0$$

in $\text{mod}_R(R)$ and any non-projective indecomposable socle projective module $Z$ admits a left almost split sequence

$$0 \to AX \xrightarrow{\delta} X \to Z \to 0$$

in $\text{mod}_R(R)$. The terms are determined uniquely up to isomorphism.

We know from [4, 5, 26] that any right peak artin algebra $R$ has almost split sp-sequences (see also [28, Corollary 3.7]). Although this is not true for arbitrary artinian $PI$-rings (see [23; 2.5]) it is true for sp-representation-finite right peak ones ([21; Prop. 6.3]). It is also known that if $R \approx F/I^* \text{ or } R \approx A_2$ (defined in the Introduction) then $R$ has almost split sp-sequences [25, 26, 30].

3. A preprojective component in the valued translation quiver $\Gamma_{\text{mod}}(R)$. Throughout this section we suppose that $R$ is an artinian right peak ring having almost split sp-sequences. In particular we can take for $R$ a right peak artin algebra, or an incidence ring $F/I^*$ of a finite poset $I^*$, or an sp-representation-finite right peak $PI$-ring.

Following Auslander [2] a full valued subquiver $\mathcal{Q}$ of $(\Gamma_{\text{mod}}(R), \delta)$ is called a connected component if $\mathcal{Q}$ is closed under taking neighbours and for any $[X], [Y]$ in $\mathcal{Q}$ there is a sequence $[X] = [X_0], [X_1], \ldots, [X_n] = [Y]$ in $\mathcal{Q}$ where $X_0, X_1, \ldots$ are connected by an irreducible map for $i = 0, 1, \ldots, n-1$. Following [4, 17] we call a component $\mathcal{Q}$ in $(\Gamma_{\text{mod}}(R), \delta)$ preprojective (resp. preinjective) if $\mathcal{Q}$ has no oriented cycles and any $[X]$ in $\mathcal{Q}$ has the form $[\delta^{-1} P]$ (resp. $[\delta P]$) for some $P \geq 0$ and some indecomposable projective module $P$ (resp. sp-injective module $Q$). The module $N$ in $\text{mod}_R(R)$ is said to be hereditary projective (resp. hereditary sp-injective) if every submodule of $N$ is projective (resp. every socle projective indecomposable module $Y$ such that $\text{Hom}_R(N, Y) \neq 0$ is sp-injective).

Given a right peak ring $R$ as above we put (cf. [6, 7, 8])

$$\mathcal{P}_{\text{mod}}(R) = \bigcup_{i = 0}^n \mathcal{P}_i, \quad \mathcal{Q}_{\text{mod}}(R) = \bigcup_{i = 0}^n \mathcal{Q}_i,$$

where $\mathcal{P}_i = \{ [P] \in \Gamma_{\text{mod}}(R) \mid P$ is hereditary projective$\}$ and if $\mathcal{P}_i$ is defined we put $\mathcal{P}_i[j] = \delta^{-j} \mathcal{P}_i \cup \mathcal{Q}_i[j+1]$, where $[P] \in \mathcal{P}_i[j]$ iff there is a chain of irreducible maps

$$P_0 \to P_1 \to \ldots \to P_i \to P_{i+1} = P$$

with $[P_0] \in \mathcal{P}_i[j]$ and $[P_{i+1}]$ is indecomposable projective. We note that $[P_0] \in \mathcal{P}_i[j]$ and that $\mathcal{P}_i$ is finite for all $i$. The sets $\mathcal{P}_i[j], j \geq 0$, are defined dually starting from $E(P_0)$ and hereditary sp-injective modules using $\delta$.

Throughout for the sake of simplicity we shall identify $X$ with its isomorphism.
class \(\{X\}\) and we shall write \(J(P)\) instead of \(P, J(R)\). We shall write \(\Gamma_\alpha(R)\) instead of \(\{\Gamma_\alpha(R)\}, \Phi\).

We know from [24, 34] (see also [12, 18, 19]) that there are natural ring isomorphisms \(\gamma: F(Y) \to F(\delta Y)\), where \(F(Z) = \text{End}(Z)\), End (Z), and a nondegenerate bilinear form

\[
\gamma_{XY}: \text{Irr}(dY, X) \otimes_{\text{Irr}(X, Y)} \text{Irr}(X, Y) \to F(X)
\]

which induces an \(F(Y) \cdot F(X)\)-bimodule isomorphism

\[
\text{Irr}(F(Y), F(X))(\alpha, \beta) = \text{Hom}_{F(Y) \cdot F(X)}(\text{Irr}(F(Y), F(X)))
\]

for any indecomposable modules \(X, Y\) in \(\text{mod}_R(R)\), \(\alpha \neq 0\). Moreover, if \(\alpha \neq 0\) then \(\alpha\) induces an \(F(Y) \cdot F(X)\)-bimodule isomorphism

\[
\text{Irr}(F(Y), F(X)) = \text{Irr}(dX, F(Y))
\]

over ring isomorphisms \(\gamma_X, \gamma_Y\).

We shall show below that \(\Gamma_\alpha(R), \alpha, \beta, \gamma_X\) is a valued translation quiver in the sense of [15, 24, 18]; we call it the valued translation quiver of \(\text{mod}_R(R)\).

We notice that if we suppose that \(R\) is a PI-ring then \(F(X)\) and \(F(Y)\) are division PI-rings and in view of Proposition 2.4 the isomorphism (3.4) yields

\[
(dX, F(Y)) = (dX, dY)
\]

Now we are able to prove one of the main results of this paper. It generalizes some of the results in [9, 11].

**Theorem 3.7.** Let \(R\) be a hereditary algebra having almost split sp-sequences. Moreover, suppose that for every \(i\) there is a indecomposable module \(T_i\) and \(i, g_i\) such that \(J(P_i) \cong \Psi_i\). Then \(\Psi_i \cong \Psi(P_i)\) is a projective component of \(\Gamma_\alpha(R)\).

If, in addition, \(R^\alpha\) has a Morita duality then the functor \(\mathcal{D}: \text{mod}_R(R) \to \text{mod}_R(R^\alpha)\) (2.10) carries \(\Psi_i\) to the preprojective component \(\Psi_i\).

**Proof.** We follow an idea of Bautista–Larrson [7]. We proceed in several steps.

1. \(\mathcal{P}_i \cap \mathcal{P}_j\) is empty for \(i \neq j\). If we apply induction on \(u\). Let \(x \in \mathcal{P}_u\). But \(x \in \mathcal{P}_u\) if \(u = n\). Therefore \(x \in \mathcal{P}_n\).

2. \(X_1, ..., X_n\) are indecomposable. Assume that \(j\) is minimal with respect to this property. Then some \(X_j\) is projective because otherwise \(d(x)\) is an oriented cycle of irreducible maps in \(\mathcal{P}_u\). Therefore, to our choice of \(j\). If \(x\) is projective then \(x = 0\). Since \(y = 0\) is indecomposable and therefore \(T \neq X_j, \mathcal{P}_u\); \(X_j\), it follows from (ii) that \(X_j, \mathcal{P}_u\) is projective because otherwise according to (3.4) there is an irreducible map \(dX_i = X_1, ..., X_n\), and therefore \(X_1 = T \cong X_j, \mathcal{P}_u\); a contradiction with (ii) (i). This proves that \(\mathcal{P}_u\) is a choice of proper monomorphisms between indecomposable projective modules. This contradiction finishes the proof of 3.

3. \(X \to Y\) is an irreducible map in \(\text{mod}_R(R)\) and \(X, Y\) are indecomposable with \(\mathcal{P}_{1, 2} \cap \mathcal{P}_3, \mathcal{P}_4\). For, if \(X\) is projective then \(\mathcal{P}_3, \mathcal{P}_4\) and therefore there is a chain \(\mathcal{P}_3, \mathcal{P}_4\). Since \(\mathcal{P}_3, \mathcal{P}_4\) is indecomposable, \(X \not\cong T \neq P_i \in \mathcal{P}_u\) as required. If \(X\) is projective then \(\mathcal{P}_3, \mathcal{P}_4\) and \(\mathcal{P}_4\) is an irreducible map \(dX_i = X_1, ..., X_n\). Since \(\mathcal{P}_4\) is not projective then \(\mathcal{P}_4\) is not projective and \(X \cong T \neq P_i \in \mathcal{P}_u\). Thus 4 is proved.

5. \(X_j, \mathcal{P}_u\) is a preprojective component. It is easy to conclude from 1, 2, 3 and 4 that \(\mathcal{P}_u\) has no oriented cycles. Further, if \(\mathcal{P}_u\) is not projective then the existence of a left almost split sp-sequences ending with \(Y\) yields the existence of irreducible maps \(dX_i = Y, Y\) indecomposable and in view of 2 and 4, \(\mathcal{P}_u\) for some \(i = j\).

\[ J \subseteq \text{Fundamenta Mathematicae} 163(2) \]
Continuing this way we will find $\tau \in \mathbb{N}$ such that $A' \mathcal{Y}$ is projective because otherwise we get a contradiction with $2'$. The remaining part of the proposition easily follows from the duality (2.0).

Note that Theorem 3.7 applies to $\mathcal{P}_1$-rings satisfying the conditions (i), (ii) below because of the following result.

**Proposition 3.8.** Let $R$ be a schurian basic artinian right peak $\mathcal{P}_1$-ring (1.1) such that
(i) $d_{ij} d_{jk} \leq 3$ for $j = 1, \ldots, n$;
(ii) the valued poset $(\mathcal{V}, \mathcal{A})$ does not contain as an upper valued subposet one of the posets

\[ G_1: \circ \rightarrow (\nexists \circ) \circ, \quad G_2: \circ \rightarrow (\nexists \circ) \circ. \]

Then $R$ is isomorphic to an upper triangular form (1.1) and for every $P_i = q_i R, i = 1, \ldots, n$, there is an indecomposable module $T_i = T(P_i)$ such that $F(T_i) = \text{End}(T_i)$ is a division ring,

\[ J(P_i) = T_i^n \]

with $q_i = \text{GCD}(d_{ii} \ldots d_{ii})$, and

\[ \text{dim}_R \text{Hom}_R(T_i, P_i), \text{dim}_R \text{Hom}_R(T_i, P_i)_{(q_i T_i)} = \begin{cases} (d_{ij} d_{jk}) & \text{if } d_{ij} = d_{ij} d_{ik} \text{ for all } k > j, \\ (1, 1) & \text{otherwise}. \end{cases} \]

For every $Q^0, i = 1, \ldots, n$, there exists an indecomposable module $\tilde{T}_i$ such that $F(\tilde{T}_i)$ is a division ring and

\[ \varphi^{-1}(k^0/\text{soc } E(0)) \cong \tilde{T}_i, \]

where $\varphi_1 = \text{GCD}(d_{ii} d_{ij} - d_{ii}, \ldots, d_{ii} d_{ij} - d_{ij} - 1, d_{ij} d_{ij} - d_{ij} - 1, \ldots, d_{ii} d_{ij} - d_{ij} - 1, d_{ii} d_{ij} - d_{ij} - 1)$ (see 2.6),

\[ \text{dim}_R \text{Hom}_R(Q^0, \tilde{T}_i), \text{dim}_R \text{Hom}_R(Q^0, \tilde{T}_i)_{(q_i T_i)} = \begin{cases} (d_{ij} d_{ik}) & \text{if } d_{ij} = d_{ij} d_{ik} \text{ for all } i < j, \\ (1, 1) & \text{otherwise}. \end{cases} \]

**Proof.** The first statement follows from [21, Proposition 7.1] and its proof. The second follows from the first in view of Lemma 2.1 (a) and the duality (2.0) which carries the map (2.6) to the map $J(P_i) \rightarrow P_i$ in $\text{mod}_{\mathcal{P}_1}(R)$. By Theorem 3.7 and Proposition 3.8 we get

**Corollary 3.9.** If $R$ is a schurian artinian right peak $\mathcal{P}_1$-ring (1.1) satisfying conditions (i), (ii) above and having almost split sp-sequences then $\mathcal{P}_1(R)$ is a preprojective component of $\Gamma_{\mathcal{P}_1}(R)$. In particular, this happens if $R$ is an sp-representation-finite schurian $\mathcal{P}_1$-ring [21; 2.10].

From Theorem 3.7 and its proof immediately follows

**Corollary 3.10.** Let $R$ be as in Theorem 3.7. Then

(a) $X \in \mathcal{P}_1(R)$ if and only if $X$ is sp-preprojective in the sense that the number of nonisomorphic indecomposable modules $Z$ in $\text{mod}_{\mathcal{P}_1}(R)$ with $\text{Hom}_{\mathcal{P}_1}(Z, X) \neq 0$ is finite.

(b) $Y$ belongs to a preinjective component of $R$ if and only if $Y$ is sp-preinjective in the sense that the number of nonisomorphic indecomposable modules $Z$ in $\text{mod}_{\mathcal{P}_1}(R)$ with $\text{Hom}_{\mathcal{P}_1}(Z, Y) \neq 0$ is finite.

Applying well-known arguments of Auslander [2] we get

**Corollary 3.11.** Let $R$ be as in Theorem 3.7. Then the following conditions are equivalent:

(a) $R$ is sp-representation-finite.
(b) $(\Gamma_{\mathcal{P}_1}(R), \mathcal{A}) = \mathcal{P}_1(R).
(c) Every module in $\text{mod}_{\mathcal{P}_1}(R)$ is sp-preinjective.
(d) $\mathcal{P}_1(R)$ is finite.
(e) $E(P_{\infty})$ is sp-preprojective.

**Remark 3.12.** It follows from the corollaries above that if $R$ is sp-representation-finite having the duality (2.0) then $\Gamma_{\mathcal{P}_1}(R)$ has the following shape [like in the hereditary case [25]]:

\[ \mathcal{P}_1(R) \quad \mathcal{P}_1(R) \quad \mathcal{P}_1(R) \quad \mathcal{P}_1(R) \quad \mathcal{P}_1(R) \]

where $\mathcal{P}_1(R)$ is a disjoint union of components and there are no maps from modules in $\mathcal{P}_1(R)$ (resp. in $\mathcal{P}_1(R)$) to modules in $\mathcal{P}_1(R) \cup \mathcal{P}_1(R)$ (resp. in $\mathcal{P}_1(R)$). In case $R$ is sp-representation-finite $\mathcal{P}_1(R)$ is empty and $\mathcal{P}_1(R) = \mathcal{P}_1(R)$.

Now we are able to prove an sp-counterpart of a result of Bautista–Larrion–Salmeron [7, 8] (see also Ringel [25]).

**Proposition 3.13.** Let $R$ be an artinian schurian basic right peak $\mathcal{P}_1$-ring having almost split sp-sequences and such that $d_{ij} d_{jk} \leq 3$ for all $j \in \mathcal{X}_i$ and $(\mathcal{V}, \mathcal{A})$ does not contain upper subposets $G_2$ and $G_3$. Then $\mathcal{P}_1(R)$ is a preprojective component of $\Gamma_{\mathcal{P}_1}(R)$ and has the following properties:

(a) $\text{Ext}(X, X) = 0$ and $\text{End}(X)$ is a division ring for all $X$ in $\mathcal{P}_1(R)$.
(b) For any arrow

\[ X \xrightarrow{(\varphi_{\mathcal{X}_i})} Y \]

in $\mathcal{P}_1(R)$ there exists an indecomposable projective module $P$ with $J(P) \cong \mathcal{T}_i, T$ indecomposable, such that $(d_{ij}, d_{jk})$ or $(d_{ij}, d_{jk})$ is equal to $(d_{ij}, d_{jk})$ and $d_{ij} d_{jk} \leq 3$.
(c) $(\mathcal{P}_1(R), \mathcal{A}, \mathcal{S})$ is a symmetrically valued translation quiver (i.e. it has no valued loops $\varphi_0$ and no minimal arrows $\varphi_0$ with $d \geq 2$ [10, 24, 18]) with trivial fundamental group in the sense of Baobang–Gabriel [10, 15] (see also [24, 19]).

(d) $F(X) = \text{End}(X)$ is a division ring for every $X$ in $\mathcal{P}_1(R)$ and $\text{Hom}_{\mathcal{P}_1}(X, Y)$ is finite-dimensional over $F(X)$ as well as over $F(Y)$ for every $X, Y$ in $\mathcal{P}_1(R)$.
Proof. We know from Corollary 3.9 that \( P_{\alpha} = \mathcal{P}_{\alpha}(R) \) is a preprojective component in \( \Gamma_{\alpha}(R) \).

(a) If \( f \in \text{End}(X) \) is non-invertible then \( f \) is a sum of compositions of irreducible maps in \( \text{mod}_{\alpha}(R) \) because the direct sum of all modules \( Y \) in \( \mathcal{P}_{\alpha} \) with \( \text{Hom}_{\alpha}(X, Y) \neq 0 \) is finite and therefore has semi-primary endomorphism ring. Hence there is a cycle in \( \mathcal{P}_{\alpha} \), a contradiction. If \( \text{Ext}^1_{\alpha}(X, X) \neq 0 \) then by arguments used above there is a cycle in \( \mathcal{P}_{\alpha} \) and we again get a contradiction.

(b) Since \( X, Y \) are in \( \mathcal{P}_{\alpha} \) there is \( t \geq 0 \) such that the modules \( d^t X, d^t Y \) are both non-zero and one of them is indecomposable projective. First suppose that \( d^t Y \) is projective. Since we know from Proposition 3.8 that \( J(P) \cong T^g \), where \( g \geq 0 \) and \( T \) is indecomposable, (b) follows from (3.5). Next suppose that \( d^t Y \) is not projective and \( d^t X = P \) is projective. Again, by Proposition 3.8, \( J(P) \cong T^g \) where \( g \geq 0 \) and \( T \) is indecomposable. Since according to (3.4) and (3.5) there is an irreducible map \( d^{g+1} Y \rightarrow d^t X \) we have \( d^{g+1} Y \cong T \). Now (b) follows from (3.4)-(3.6).

(c) The first assertion follows from (b) and Proposition 3.8. The second immediately follows from the definition of the fundamental group because \( \mathcal{P}_{\alpha} \) has no oriented cycles and \( J(P) \cong T^g \) for some \( g \geq 0 \) and \( T \) indecomposable.

(d) If \( X \cong Y \) are in \( \mathcal{P}_{\alpha} \), then by the arguments used in the proof of (a) any \( f \in \text{Hom}_{\alpha}(X, Y) \) is a sum of compositions of irreducible maps. It follows that for any \( t \geq 1 \) there is an \( F(Y) \cdot F(X) \)-bimodule epimorphism

\[
\zeta_t : \bigoplus_{i=1}^{m} \text{Irr}(X, X_i) \otimes \text{F}_{\alpha}(X, X_i) \otimes \text{Irr}(Y, Y_j) \rightarrow f^t(X, Y)
\]

where the sum is finite and the \( X_i \) are in \( \mathcal{P}_{\alpha} \). Since by Corollary 3.10, \( f^t(X, Y) = 0 \) for some \( m, \) in view of \( \zeta_t, \) (d) follows from (b) by an easy induction on \( t \). This completes the proof.

Corollary 3.14. Let \( R \) be a basicartinian \( \alpha \)-representation-finite right peak \( PI \)-ring.

Then the following conditions are equivalent:

(a) \( (\Gamma_{\alpha}(R), d, \alpha) \) is a symmetrizable valued translation quiver and \( R \) is simply connected in the sense that the fundamental group of \( (\Gamma_{\alpha}(R), d, \alpha) \) is trivial.

(b) \( (\Gamma_{\alpha}(R), d) \) has no oriented cycle.

(c) \text{End}(X) \) is a division ring for every indecomposable module \( X \) in \( \text{mod}_{\alpha}(R) \).

(d) \( R \) is schurian.

Proof. (a) \( \Rightarrow \) (b) and (c) \( \Rightarrow \) (d) are obvious.

(b) \( \Rightarrow \) (d). Apply arguments in the proof of Proposition 3.13(a).

(a) \( \Rightarrow \) (c). We know from [21; Proposition 6.3] that \( R \) has almost split \( \alpha \)-sequences. Then by Corollary 3.9, \( \mathcal{P}_{\alpha}(R) \) is a preprojective component and \( (\Gamma_{\alpha}(R), d) = \mathcal{P}_{\alpha}(R) \) according to Corollary 3.11. Consequently (a) and (c) follow from Proposition 3.12 and the proof is complete.

4. An algorithm for describing \( \mathcal{P}_{\alpha}(R) \). Suppose that \( R \) is a basic schurian artinian right peak \( PI \)-ring of the form (1.1) which is upper triangular and the valued poset \( (I, d) \) has the properties (i) and (ii) of Proposition 3.8. Then according to Corollary 3.9, \( \mathcal{P}_{\alpha}(R) \) is a preprojective component and applying arguments of Happel [16] similarly to [21; Section 8] one can show that every indecomposable \( X \) in \( \mathcal{P}_{\alpha}(R) \) is uniquely determined by its dimension vector \( \dim X \) provided \( R \) is a finite-dimensional algebra over a field.

Following the method initiated by Bautista, Brenner and Ringel (see [14]) we can describe \( \mathcal{P}_{\alpha}(R) \) in terms of \( \dim X, X \in \mathcal{P}_{\alpha}(R) \) as follows:

(a) Given \( R \) of the form (1.1) which is upper triangular we calculate the numbers \( d_{ij}, d_{ji}, i, j = 1, \ldots, n \), \( * \) (1.2) and we write two sets of vectors in \( \mathbb{Z}^n \):

\[
\mathcal{P} = \{p_1, \ldots, p_n, p_{n+1} = p_n\}, \quad \mathcal{Q} = \{q_1, \ldots, q_n\}
\]

where

\[
p_i = \dim P_i = (0, \ldots, 1, d_{ji}, \ldots, d_{ji}),
\]

\[
d_{ii} = d_{ii} - d_{ij} = d_{ii} - d_{ij} = d_{ii}
\]

with \( d_{ii} = d_{ii}, d_{ij} = d_{ij} \) and \( d_{ii} = d_{ii} \) for all \( i \geq 1 \) (see 2.3).

(b) Consider the set \( \mathcal{F} = \{t_1, \ldots, t_n\} \), where

\[
t_i = \text{g.c.d.} \left( d_{ij}, d_{ji}, \ldots, d_{ji} \right)
\]

and \( g_j = \text{GCD}(d_{ij}, \ldots, d_{ij}) \). Note that according to Proposition 3.8 we have \( J(P) \cong T(P)^{P_j} \), \( P_j = \dim T(P) \), and the valued arrow from \( T(P) \) to \( P_j \) in \( (\Gamma_{n}(R), d) \) has the form

\[
T(P) \rightarrow_T P_j \quad \text{if} \quad d_k = d_{kk} = d_{kk} \quad \text{for all} \quad k > j.
\]

(c) We define inductively sets \( l_0, l_1, \ldots, l_n \) of vectors in \( \mathbb{Z}^n \) and valued arrows between vectors of \( l_i \) and \( l_{i+1} \) as follows:

(i) We mark vectors of \( l_j \) as pairwise different points on the line \( x_j = j \) in the plane \( \mathbb{R}^2 \).

(ii) Let \( l_i = \{p_k\} \) and define \( l_i \) as the set of vectors \( p_i \in \mathcal{P} \) such that \( t(p_i) = p_k \). For any such \( i \) we connect \( p_k \) with \( p_i \) by the valued arrow

\[
\frac{\text{g.c.d.}(d_{ii}, d_{ii})}{\text{g.c.d.}(d_{ii}, d_{ii})} \rightarrow p_i.
\]

(iii) Given \( x \in l_{i-1} - \mathbb{Z}^n \) we form a mesh...
where \( x_1, \ldots, x_i \) are all vectors in \( l_k \) connected with \( x_i \). \( d_j = d'_x, d_j = d_{xx}, \)
\[ A^-(x) = \sum_{j=1}^{d} x_j d'_j - x. \]
We put
\[ l_{k+1} = \{ A^-(x) \mid x \in l_k \cup \beta \} \cup \{ \sum_{j \in T} \} . \]

We connect \( x_1, \ldots, x_{d} \) with \( A^-(x) \) as is marked on the mesh above and we connect \( t_j \) with \( t_{j'} \) by the valued arrow (4.2) with \( T(P_j), P_j, t_j, t_{j}, p_j, p_{j} \) interchanged.

Note that if \( x = \dim X \in \beta \cup \alpha \) and \( X_1, \ldots, X_i \) are all indecomposable modules in \( \text{mod}_{R_p}(R) \) connected by irreducible maps starting from \( X_i \), then taking \( x_j = \dim X_j \) we get \( A^-(x) = \dim A^-(X) \).

This together with the remark in (ii) shows that \( \varphi_{R_p}(R) = \bigcup_{j=1}^{\infty} L_j \) if we identify \( X \) with \( \dim X \).

(4.3) Conclusion. The procedure will stop at step \( m \) if \( \varphi_{R_p}(R) = \dim E \in l_m \). In this case \( R \) is sp-representation-finite and \( \{ \Gamma_{R_i}(R), d \} = I_{\infty} \cup \ldots \cup I_{\infty} \) (Corollary 3.11). Thus we get a simple procedure for describing \( \varphi_{R_p}(R) \) (and dually for describing the preinjective component \( A_{R_p}(R) \)) which is the Auslander–Reiten quiver provided \( R \) is sp-representation-finite.

**Example 4.4.** Let \( R \) denote the real numbers and \( C \) the complex numbers. Consider the \( 7 \times 7 \) matrix ring

\[
T = \begin{bmatrix}
R & R & R & R & C & 0 & C \\
R & R & R & C & 0 & C & R \\
C & 0 & C & 0 & R & C & R \\
0 & C & R & C & 0 & C & R
\end{bmatrix}
\]

where the maps \( c_{ij} \) are given by the multiplication in \( C \). Then \( T \) is a right peak \( R \)-algebra which is sp-representation-finite and

\[
1 \rightarrow 2 \uplus 3 \\
\downarrow \downarrow \downarrow \downarrow \\
(1, 2, 3)
\]

Applying the algorithm one can construct \( \{ \Gamma_{R_p}(R), d \} \) presented in Figure 1 (4).

Other examples can be found in [21; Appendix] and in [20].

**5. The Auslander–Reiten quiver \( \Gamma_{R_p}(R) \) of a splitting ring.** Suppose that \( R \) is an artinian right peak \( PL \)-ring of the form (1.1) which is upper triangular and has almost split sp-sequences. In particular, \( R \) has the reflection duality (2.0). Moreover, we suppose that \((1, 2, 3)\) is a valued posed with respect to the relation \( i < j \iff M_j \neq 0 \). Following [21;
Definition 4.5] We say that $(I, d)$ has a splitting decomposition if there is a disjoint union poset decomposition

$$I_j = I' + C + I'$$

such that $I', I'$ are not empty and

(i) $I' < I'$ for all $I' \in I$, $I' \in I'$ and there are no relations $e < I' \in I$, $e \in C, I' \in I$,

(ii) $(C, d)$ with $C = C \cup \{e\}$ is a homogeneous chain

$$(C, d): e_1 \rightarrow e_2 \rightarrow \ldots \rightarrow e_n \rightarrow e_{n+1} = e$$

$d_{e_{j+1}, e_j} = d_{e_{j+1}} = 1$.

(iii) $d_{I', I'}$ for all $I' \in I$, $I' \in I'$.

In this case $(I, d)$ has the form

$$(5.1)$$

Throughout suppose that $(I, d)$ admits a splitting poset decomposition (5.0) and put

$$e = e_1 + e(C + I'), \quad \eta = e_1 + e(C + I'), \quad \eta' = e(C + I'), \quad \eta_{I'} = e(C + I'),$$

$J' = J \cup \{e\}$ where $e(J) = \sum c_j$ for $J \subseteq I$. Moreover, we put

$$A = eR, \quad B = \eta R.$$  

Note that $A$ and $B$ are the right peak rings $R_{I', e}$ and $R_{C + I', e}$ of $R$ by omitting all rows and columns with indices in $I'$ and $I$, respectively. Following [28, 29, 31] consider the functors (see [31])

$$(5.4)$$

$$\text{mod}_D(R) \cong \text{mod}_D(R) \cong \text{mod}_R(B),$$

where $r_x = r_{I', e} = r_{C + I', e} = r_{C + I'}$ are the restriction functors to $(I' + C, e(C + I'))$, i.e. $r_x(X) = X e(C + I') e = (X e(C + I')) e$.

$$L(Y) = \text{Hom}_R(Re, Y), \quad T(Z) = Z \otimes_R \eta R.$$  

Following [32; 5.195] and [33] we are going to show that $(I', d)$ is a simple glueing of $(I', d)$ and $(C, d)$ by applying the functors $L$ and $T$.

Proposition 5.6. Under the assumptions and notations above we have:

(a) The functors $L$, $T$ are full, faithful and $r_x L = \text{id}$. $T$ is left adjoint to $r_x$ and $L$ is right adjoint to $r_x$.

(b) If $Y'$ is indecomposable in $\text{mod}_D(A)$ then the restriction $(L(Y') e(C + I'))$ of $L(Y')$ to $(I' + C, e(C + I'))$ is isomorphic to $E''(P_x) e = (P_x) e(C + I')$, where $h = \dim(Y') e = (P_x)$ and $E''(P_x) e = P_x e = P_x e(C + I')$ is the injective envelope of $P_x$ in $\text{mod}_D(e(C + I'))$.

(c) If $X$ is indecomposable in $\text{mod}_D(R)$ and $X e = 0$ then $r_x(X)$ is indecomposable and $L(r_x(X)) = X e(C + I')$. Every indecomposable module $X$ in $\text{mod}_D(R)$ is in one of the images $L(T, \text{id})$, $L(\text{id}, T)$.

Proof. (a) is well known (see [1]).

(b) Since $L$ is right adjoint to $r_x$ it carries injectives to injectives. Then the first part of (b) follows by the arguments in the proof of [28; Prop. 2.5(a)]. The second part of (b) and the first two statements in (c) follow immediately from the definitions.

(c) It follows from [21; Lemma 4.6] that the splitting poset decomposition (5.0) induces a corresponding ring splitting decomposition of $R$ [21; Def. 4.2] and therefore (c) follows from [21; Theorem 4.3].

(d) $L$ is left exact as a right adjoint to the exact functor $r_x$. Let $f: Y_x \rightarrow Y_x$ be surjective. In order to prove that $L(f)$ is surjective it is sufficient to show that the restrictions of $L(f)$ to $(I' + C)$ and $(C + I')$ are surjective. The first is $r_x(f) = f$ and the second is $r_x(f) = r_x(f) e = (C + I') e$. Since the induced surjection $s_0(Y) e = s_0(L(Y) e(C + I'))$ splits, according to (b) $r_x(f)$ is a splitable epimorphism and therefore $L(f)$ is surjective.

Since $Q^0$ and $\text{mod}_D(R)$ are indecomposable and $Q^0 \otimes_R \eta_R = 0 = \text{mod}_D(R)$, (c) yields $Q^0 \cong L(r_x(Q^0)) = L(\text{id})$ because one can easily show that $r_x(Q^0) = Q^0$.

The remaining statements in (d) and (c) can be proved in a similar way. The proof is complete.

In the proof of our glueing theorem we shall need the following three lemmas. We suppose that $R$ is as above.

Lemma 5.7. Let $\tilde{Q}_j$ denote the $\text{sp}$-injective $B$-module $Q^0_j$, $j = 0, 1, \ldots, m$ (see (2.2)). Then

(a) The restriction $\tilde{Q}_j e = (\tilde{Q}_j) e(C + I')$ is isomorphic to the injective envelope $E''(P_x) e = (P_x) e(C + I')$ of $P_x$ in $\text{mod}_D(e(C + I'))$.

(b) $\text{mod}_D(A) \cong \text{mod}_D(A) \cong \text{mod}_D(A)$.

(c) If $X$ is indecomposable in $\text{mod}_D(R)$ and $X e = 0$ then $r_x(X)$ is indecomposable and $L(r_x(X)) = X e(C + I')$. Every indecomposable module $X$ in $\text{mod}_D(R)$ is in one of the images $L(T, \text{id})$, $L(\text{id}, T)$.
(5.8)

(b) It is easy to see that every indecomposable summand $Z'$ of $Z$ has also the property of $Z$ in (b) and therefore $\text{Hom}_A(\mathcal{Q}_m, Z) \neq 0$. Since $\mathcal{Q}_m$ is hereditary sp-injective, $Z'$ is sp-injective and therefore there are monomorphisms $\mathcal{Q}_m \to Z \to \mathcal{Q}_0 = E_0(P_0)$. Then

(a) yields $Z' \cong Q_j$ for some $j \leq m$.

It follows from Proposition 5.6(e) and (b) that $Y_i = r_j(P_i)$ is indecomposable, $P_i \cong L_0, r_j(P_i) \cong Y_\omega \cong E_0(L_0)^h$ with $h = d_{ij}$ and the proof of (b) is complete.

(c) The first part follows from the shape of $\mathfrak{I}$ (see (5.1)) and the second one follows from (a) and the definition (3.1).

**Lemma 5.9.** Let $Y, Y'$ and $Z, Z'$ be indecomposable modules in $\text{mod}_A(A)$ and $\text{mod}_B(B)$ respectively. Then

(a) $\text{Hom}_A(LY, TZ) = 0$ if $Y' \neq 0$.

(b) The natural epimorphisms

\[
\begin{align*}
\mathcal{L}: \text{Irr}(Y, Y') & \to \text{Irr}(LY, LY), \\
\mathcal{T}: \text{Irr}(Z, Z') & \to \text{Irr}(TZ, TZ)
\end{align*}
\]

are bimodule isomorphisms over the natural ring isomorphisms $F(Y) \cong F(LY)$, $F(Y') \cong F(LY)$, $F(Z) \cong F(TZ)$, $F(Z') \cong F(TZ)$, respectively. In particular, $\mathcal{L}$ and $\mathcal{T}$ carry irreducible maps to irreducible ones.

**Proof.** (a) Since $Y' \neq 0$, by Proposition 5.6 we have $r_j(LY) \cong (LY)\phi_i \cong E_0(L_0)^h$, where $h = \text{dim}(Y_\omega)_j$, and according to Lemma 5.7, $r_j(LY)$ is a direct sum of copies of $\mathcal{Q}_{i_0}, \ldots, \mathcal{Q}_{i_m}$. Hence if we assume that there is nonzero $f \in \text{Hom}_A(LY, TZ)$ then the restriction of $f$ to $\text{soc}LY \cong r_j(LY)\phi_i$ is nonzero and therefore $\text{Hom}_A(\mathcal{Q}_i, Z) \neq 0$. It follows from Lemma 5.7 that $Z \cong \{0\}$ for some $0 \leq i \leq j$ and $TZ \cong T\phi_i \cong L\phi_i$, where $P_{\mu_j} = \mathcal{Q}_{i_j}$. Hence $\text{Hom}_A(LY, TZ) \cong \text{Hom}_A(LY, L\phi_i) \cong \text{Hom}_A(Y, P_{\mu_j}) = 0$, because $P_{\mu_j}$ is hereditary projective and $Y$ for $i = 1, \ldots, m+1$. This contradiction finishes the proof of (a).

(b) Suppose $\mathcal{T}(g) = 0$, where $g \in \text{Hom}_B(Z, Z')$, then $\mathcal{T}(g)$ has a factorization $TZ \to X \to TZ'$ in $\text{mod}_B(R)$ with $h \in \text{Irr}(Z, X)$ and in view of (a), $X$ can be chosen of the form $X = T(Z')$. Since $T$ is full and faithful, $g = \text{the } J^*(Z, Z')$. Thus $\mathcal{T}(g) = 0$ and $\mathcal{T}$ is bijective.

Suppose $\mathcal{L}(f) = 0$ where $f \in \text{Irr}(Y, Y)$. If $LY$ is not in the image of $T$ then as above we conclude that $\mathcal{L}(f) = 0$. Suppose $LY \cong Z(T)$, note that this happens if and only if $Y \cong P_{\mu_j}$ for some $j = 1, \ldots, m+1$, $\mu_{j+1} = 0$. Suppose $Y = P_{\mu_j}$ and note that $LY \cong T(\mathcal{Q}_{i_j})$ (Lemma 5.7). If $LY$ is in the image of $T$ then we are done by (5.9) for $T$. Suppose $LY \neq \text{Im}T$ and $\mathcal{L}(f) = 0$. Then $L(f)$ is a sum of composed maps $L_{\mu_j} \cong \mathcal{T}(\mathcal{Q}_{i_j}, \mathcal{Q})$, where $h \in J(\mathcal{Q}_{i_j}, X_j)$, $i_j \in J(X_j, LY)$ are both nonzero. Assume that $X_j \cong T(Z_j)$. Then $h = \text{the } L(h) = 0$ for some $h \in J(X_j, Z_j)$ and Lemma 5.8 yields $Z_j \cong \mathcal{Q}_{i_j}$, $X_j \cong \text{Im}T(Z_j) \cong \text{Im}L_{\mu_j}$ for some $i_j \leq j$. This proves that all modules $X_j$ are of the form $L(Y)$ because of Proposition 5.6(e). It follows that $f = \sum f \phi_i$ for some $f \in \text{Irr}(Y, Y)$, $f \in \text{Irr}(P_{\mu_j}, Y)$ and therefore $f = 0$. This finishes the proof.

**Lemma 5.11.** Suppose that $Y_i, Z_i$ are indecomposable and let

\[
\mathfrak{Y}: 0 \to Y \to Y' \to Y'' \to 0, \quad \mathfrak{Z}: 0 \to Z \to Z' \to Z'' \to 0
\]

be almost split exact sequences in $\text{mod}_A(A)$ and $\text{mod}_B(B)$ respectively. Then $\mathfrak{Y}$, $\mathfrak{Z}$ are almost split sequences.

**Proof.** Since $LY, L_{\mu_j}, TZ, TZ'$ are indecomposable it is sufficient to prove that $L(Y), L_{\mu_j}$ are left almost split. Let $f: LY \to X$ be a nonzero nonisomorphism and let $X$ be indecomposable in $\text{mod}_B(B)$. We know from Proposition 5.6(e) that either $X \cong LU$ or $X \cong TV$ for some $U \in \text{mod}_A(A)$, $V \in \text{mod}_B(B)$. If $X \cong LU$ there is a nonisomorphism $g: Y \to U$ such that $f = L(g)$. Since $\mathfrak{Y}$ is almost split $g$ factors through $u$ and $f$ factors through $L(u)$. Suppose $X \cong TV$. Since $f \neq 0$, in view of Lemma 5.3(a) we have $(LY)\phi_i = 0$ and therefore $LY \cong Z(T)$ for some $Z$. It follows as in the proof above that $Y \cong P_{\mu_j}$, $Z \cong \mathcal{Q}_{i_j}$, for some $j = 1, \ldots, m+1$ (see 5.8). Since $\text{Hom}_A(\mathcal{Q}_{i_j}, Y) \cong \text{Hom}_A(LY, X) = 0$, in view of Lemma 5.3 we have $Y \cong \mathcal{Q}_{i_j}$, for some $i < j$ and $X \cong TV \cong \mathcal{T}(\mathcal{Q}_{i_j}) \cong L\phi_i$. If $h: Y \to P_{\mu_j}$, such that $L(h) = f$ then $h$ factors through $u$ and $f$ factors through $L(u)$ as required. Consequently $\mathfrak{Y}$ is almost split. The proof of the remaining part is left to the reader.

Now we are able to prove the main result of this section.

**Theorem 5.12.** Let $R$ be an Artinian right peak PI-ring of the form (1.1). Suppose that $R$ is upper triangular, has almost split sequences and that $\mathfrak{Y}, \mathfrak{Z}$ is a valued poset with respect to $i < j \to M_j$. If $\mathfrak{Y}$ or $\mathfrak{Z}$ has a splitting decomposition (5.9) then

(a) The exact functors $\mathcal{L}, \mathcal{T}$ carry almost split sequences to almost split sequences and induce the bimodule isomorphisms (5.10).

(b) Every almost split sequence $X$ in $\text{mod}_A(R)$ is either of the form $\mathfrak{T}(\mathfrak{Z})$ or of the form $\mathfrak{L}(\mathfrak{Y})$, where $\mathfrak{Y}$ and $\mathfrak{Z}$ are almost split sequences. Every irreducible map $f: X \to X'$ between indecomposable modules in $\text{mod}_A(R)$ is of one of the forms $f(L(Y)), f(L(Z))$, where $Y, Z$ are irreducible.

(c) An indecomposable module $X$ in $\text{mod}_A(R)$ belongs to $\text{Im}L \cap \text{Im}T$ if and only if $X = L(P_{\mu_j}) \cong T(\mathcal{Q}_{i_j})$ for some $j = 1, \ldots, m+1$, where $\mu_{j+1} = 0$ (see 5.8).

(d) If, in addition, $R$ satisfies the assumptions of Theorem 3.7 then $\mathfrak{S}(B, A, Z)$ has the form below (in the notation of Remark 3.12) obtained from $\mathfrak{S}(B, A, Z)$ and $\mathfrak{S}(B, A, Z)$ by the identification of the final section $\mathcal{Q}_{i_0}, \mathcal{Q}_{i_1}, \ldots, \mathcal{Q}_{i_m} \cong \mathcal{Q}_{i_0}, \mathcal{Q}_{i_1}, \ldots, \mathcal{Q}_{i_m} \cong \mathcal{Q}_{i_0}, \mathcal{Q}_{i_1}, \ldots, \mathcal{Q}_{i_m}$ with the starting section $P_{\mu_0} \to P_{\mu_1} \to \cdots \to P_{\mu_m}$ of $\text{mod}_A(A)$ (see 5.8), where $A, B$ are the rings (5.3) and there are no maps from the right to the left.
Proof. (a) follows from Lemmas 5.9 and 5.11, whereas (c) follows from the proof of Lemma 5.9.

(b) We know from Proposition 5.6(e) that every indecomposable module $X$ in $\text{mod}_{\mathcal{A}}(R)$ belongs either to $\text{Im} \ T$ or to $\text{Im} \ L$.

Let $X : 0 \rightarrow X' \rightarrow X'' \rightarrow 0$ be an almost split sequence in $\text{mod}_{\mathcal{A}}(R)$ and let $X'$ be indecomposable. If $X \cong LY$ then we know from Proposition 5.6(d) that $Y$ is not sp-injective and therefore there is an almost split sequence $Y$ in $\text{mod}_{\mathcal{A}}(R)$ starting with $Y$. By Lemma 5.11, $L(Y)$ is an almost split sequence starting with $X$ and therefore, $X \cong L(Y)$.

Suppose now that $X \not\cong \text{Im} \ L$. Then $X \cong T(Z)$ for some $Z$ and we know from (e) that $X \cong T(Z)$ for $j = 0, 1, \ldots, m$. Then, in view of Proposition 5.6(e), $Z$ is not sp-injective and therefore there is an almost split sequence $Z$ in $\text{mod}_{\mathcal{A}}(R)$ starting with $Z$. It follows from Lemma 5.11 that $X \cong T(Z)$ as desired. Since the second statement in (b) follows from the first, $X)$ is proved.

Since (d) follows from (a)-(c) the theorem is proved.

Corollary 5.13. Let $I$ be a poset having a disjoint union poset decomposition $I = I + C + I'$, where $C: c_1 \rightarrow c_2 \rightarrow \ldots \rightarrow c_n$, $I' = I'$, $C$: is nonempty and the condition (b) below (5.0) is satisfied. If $T: (C + I')^s \rightarrow I$-sp is the natural embedding functor and $L: (I' + C)^s \rightarrow I$-sp is given by $L(M, M)_{\text{ext}} = c_1c_2c_3 \ldots c_n$, where $M = M$, $M = M$, $M = M$ for $i \not= 1, 2, 3, \ldots, n$, then $L, T$ are exact and have the properties (a)-(d) of Theorem 5.12 with $I$-sp, $(I' + C)^s$, $(C + I')^s$ and $\text{mod}_{\mathcal{A}}(R)$, $\text{mod}_{\mathcal{A}}(R), \text{mod}_{\mathcal{A}}(B)$ interchanged.

Proof. Apply Theorem 5.12 to $R = F[I]$. Note that $\text{mod}_{\mathcal{A}}(F[I]) \cong I$-sp, $\text{mod}_{\mathcal{A}}(R) \cong (I' + C)^s$, $\text{mod}_{\mathcal{A}}(B) \cong (C + I')^s$ and the functors $L, T$ above coincide with the functors $L, T$ in (5.4) (see [30]).

Remark 5.14. (1) Let

$$R = \begin{bmatrix} S & M_s \\ 0 & B \end{bmatrix}$$

where $S = e' s e'$, $M = e' R e'$ (see (5.2)). Then $M_s$ is a direct sum of copies of hereditary sp-injective B-modules $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ (Lemma 5.7(b)) and therefore we are in a situation similar to that in [34] (compare Theorem 5.12 with [34]; Theorem 1 and Remark 1).

(2) It would be interesting to prove a counterpart of Theorem 5.12 for a ring splitting of $R$ in the sense of [21; Definition 4.2]. Example 4.4 shows that Theorem 5.12 and Lemmas preceding it do not hold for ring splittings. In fact $I_s = I' + C + I''$ with $I' = \{1, 2, 4\}$, $C = \{3, 6\}$, $I'' = \{5\}$ induce a splitting system of functors (5.4) with $R = T$. The modules from $\text{Im} \ L$ and from $\text{Im} \ L \cap \text{Im} \ T$ in Figure 1 are underlined and wave underlined respectively.

Another illustration of the construction $(I^s, I, d, a)$ in the case of a ring splitting by successively applying (5.4) is given in Example 5.15 below.

(3) In [33] a splitting theorem is proved for directed multiplex bound quiver algebras. It plays an important role in determining coordinate supports of indecomposable socle projective modules over covering algebras of right peak algebras (see [22]).

Example 5.15. Let $C$ and $R$ be the complex and real numbers, respectively, and $c_{234}: R \otimes R \rightarrow C$ is the natural embedding, $c_{234}: C \otimes C \rightarrow C$ is such that the composed map $c_{234}: C \otimes \text{Hom}_R(C, C) \rightarrow C$ is the identity. The remaining maps $c_{23}$ are multiplications. Note that

$$\begin{cases}
(2, 3) & (2, 3) \\
(2, 1) & (1, 2) \\
(1, 2) & (3, 2) \\
(1, 2) & (3, 2)
\end{cases}$$

has no splitting decomposition; however, $I_s = I' + C + I''$ with $I' = \{1\}$, $C = \{2, 3, 4, 5\}$, $I'' = \{6\}$ induces a ring splitting with the corresponding rings $A$, $B$ as in (5.3) obtained from $R$ by omitting the first row and the first column, and the sixth row and the sixth column respectively. Note that $I_s = \{1, 2, 3, 4, 5\}$. $I_b = \{2, 3, 4, 5, 6\}$. For the rings $A$ and $B$ there exist ring splittings induced by the following decompositions:

$I_A = I' = C + I''$ and $I_b = I_a + C_b + I_a'$.

where $I_A = \{2\}$, $C_A = \{1, 2, 3, 4\}$, $I_A' = \{5\}$, $I_b = \{2\}$, $C_B = \{3, 4, 6\}$, $I_b = \{5\}$.

Let $(A^1, B^1)$ and $(A^2, B^2)$ be the corresponding splitting pairs like in (5.3). Then

$$(I_A^s, d): 2 \rightarrow 3 \rightarrow 1 \rightarrow 4 \rightarrow 5 \rightarrow 2.$$
Note that the rings $A^1$ and $B^2$ are hereditary sp-representation-infinite of type $F_{4}$.
The ring $B^1$ is hereditary representation-finite of type $F_{12}$ and the ring $A^2$ is
sp-representation-finite of type $F_{3}$ (see [21: Appendix]). The Auslander–Reiten valued
quivers of the rings above are presented in Figure 2.

\[
\begin{align*}
\Gamma_{w}(A^1): & \quad \text{Fig. 2} \\
\Gamma_{w}(B^1): & \\
\Gamma_{w}(A^2): & \\
\Gamma_{w}^{(3)}(A^2): & \\
\end{align*}
\]

Let us denote by $L$, $T$, $E$, $T^1$, $E$, $T^2$ the functors (3.4) corresponding to the pairs $(A, B)$,
$(A^1, B^1)$, $(A^2, B^2)$, respectively. One can show that $\Gamma_{w}(A)$ and $\Gamma_{w}(B)$ are obtained from
$\Gamma_{w}(A^1)$, $\Gamma_{w}(B^1)$ and $\Gamma_{w}(A^2)$, $\Gamma_{w}(B^2)$ by gluings along sections in $\text{Im} L \cap \text{Im} T^1$ and
$\text{Im} E \cap \text{Im} T^2$ presented in Figure 3. The marked sections represent the indecomposables

\[
\begin{align*}
(0, 0, 1, 0, 1, 1) & \rightarrow (0, 0, 2, 1, 2, 2) \rightarrow (1, 0, 2, 1, 2, 2) \rightarrow (0, 0, 2, 0, 1, 1), \\
(0, 1, 0, 1, 2, 1) & \rightarrow (0, 2, 1, 2, 3, 2) \rightarrow (0, 2, 1, 2, 4, 2) \rightarrow (0, 2, 0, 1, 2, 1)
\end{align*}
\]
in terms of dimension vectors \((x_1, \ldots, x_n, x_{n+1}), (x_2, \ldots, x_n, x_{n+2})\), respectively. Moreover, \(+\ast\) and \(+\bullet\) are the vectors \((1, 0, 1, 1, 1)\) and \((0, 0, 2, 1, 1, 1)\); \(+\ast\) belongs to a tube of rank 3 and \(+\bullet\) belongs to a tube of rank 2.

We mark by \(\circ\) the modules in \(\text{Im} L, \text{Im} L^1, \text{Im} L^2\), and by \(\ast\) the ones in \(T, \text{Im} L^1, \text{Im} L^2\). The modules in \(\text{Im} T^1 \cap \text{Im} L^1, \text{Im} T^2 \cap \text{Im} L^2, \text{Im} T \cap \text{Im} L\) are denoted by \(+\bullet\). The gluing respects the natural order in the quivers.

Now by simple calculations one can show that \(\Gamma^\circ_\ast(R)\) can be obtained from \(\Gamma_\ast^\circ(A)\) and \(\Gamma_\ast^\circ(B)\) by glueing along a section in \(\text{Im} L \cap \text{Im} T\) as presented in Figure 4. The indecomposable modules in this section are represented in terms of their dimension vectors \((x_1, x_2, \ldots, x_N, x_{n+1})\), i.e.: \((0, 0, 0, 1, 1, 2, 1)\) \(\rightarrow\) \((0, 0, 2, 1, 1, 2, 1)\) \(\rightarrow\) \((0, 0, 2, 1, 1, 2, 1)\). The singular point \(+\ast\) \(=\ast\bullet\) in the square belongs to a tube of rank 2, its dimension vector is \((0, 0, 2, 1, 1, 2, 1)\). Note that the triple in the square equals \((1, 0, 2, 0, 1, 2, 1)\) \(\rightarrow\) \((2, 0, 3, 1, 2, 4, 2)\) \(\rightarrow\) \((1, 0, 1, 1, 2, 1)\) and belongs to a tube of rank 3. The modules \(+\ast\), \(+\bullet\), \(+\circ\) form one \(d\)-orbit in the tube.

The boldface sections in Figure 4 should be identified.

References


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On supercomplete uniform spaces IV: Countable products

by

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Abstract. We show that the product of countably many supercomplete C-scattered spaces is supercomplete. The result implies similar but weaker theorems of [1], [17] and [4].

1. Introduction. It is well known that the product of paracompact spaces is in general not paracompact. It was proved by Z. Frolik in [5] that a countable product of locally compact paracompact spaces is paracompact. The same is true for the larger class of paracompact p-spaces of Arkhangel’ski˘ı [2]. Recently a weaker structural condition of being scattered or C-scattered has been used by K. Alster [13], M. E. Rudin and S. Watson [17], and by L. M. Friedler, H. W. Martín and S. W. Williams in [4], to obtain similar results. We prove in this paper a natural extension of their results by showing that a countable product of supercomplete C-scattered spaces is supercomplete. The notion of supercompleteness was defined by J. R. Isbell in [13]; by his result — we can take it as a definition — a uniform space (X, µ) is supercomplete if X is topologically paracompact and the Ginzburg–Isbell locally fine coreflection (µ) is the fine uniformity of (X, X) of X. By using the concept of metric-fine coreflections, we show at the end of the paper that a countable product of C-scattered paracompact spaces is paracompact.

Our proof uses a simple recursive technique based on well-founded (or Noetherian) trees, applied e.g. in [11], [12], [15] in the context of uniform spaces.

2. Preliminaries. This section consists of preliminary definitions. We refer the reader to [14] for basic information on uniform spaces. For the definition of the Ginzburg–Isbell locally fine coreflection λ of a uniform space (X, µ) the reader is referred to the first three papers [3], [9], [10] in our study on supercomplete spaces. A well-founded tree is a partially ordered set T = (T, ≤) with a unique minimal element Root(T) such that every branch, i.e. maximal linearly ordered subset, of T is finite. We denote by End(T) the set of all maximal elements of T. Given p ∈ T, the set of all immediate ≤-successors of p is denoted by S(p). Thus, S(p) = {q ∈ T : q > p and q > r > p for no r ∈ T}. Furthermore,