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On hereditarily decomposable hereditarily equivalent non-metric continua

by

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Abstract. We give a method for constructing hereditarily equivalent Hausdorff arcs. The method yields four topologically distinct examples, two of which are known and two new. An example is given of a hereditarily decomposable, hereditarily equivalent continuum which is not an arc. This example shows that theorems of Henderson, Mahavier and Thomas, and Oversteegen and Tymchatyn will not generalize to the non-metric setting.

§ 1. Introduction. A continuum is a compact connected Hausdorff space. A continuum is said to be *decomposable* if it is the union of two of its proper subcontinua and *hereditarily decomposable* if each of its non-degenerate subcontinua is decomposable. A continuum is said to be *hereditarily equivalent* if it is homeomorphic to each of its non-degenerate subcontinua. Clearly decomposable hereditarily equivalent continua are hereditarily decomposable. In 1921 Mazurkiewicz [Maz] raised a question which has still not been fully answered, namely whether there are any hereditarily equivalent metric continua other than the arc [0, 1]. Henderson [H] has shown that the arc is the only decomposable example. Moise [Moi] has shown that the pseudo-arc is hereditarily equivalent, thus answering Mazurkiewicz' original question. However, the question whether there are any examples beyond these two remains open and appears to be very difficult.

In this paper we investigate non-metric decomposable hereditarily equivalent continua. All of the examples we have been able to find of such continua in the literature are Hausdorff arcs. These examples are due to Arens [Are1] and Babcock [B], all described in [B], and a new example due to Hart and van Mill [H–V]. A recent paper of Ward [W] contains a lengthy and interesting discussion of general Hausdorff arcs, including some results on homogeneity closely related to Babcock's. In § 2 below we give a method for constructing hereditarily equivalent Hausdorff arcs which yields the arc [0, 1], Aren's original example and two new examples. An extension of the method, to be described in a future paper, will produce all of the Arens–Babcock examples and many others. Other new examples may well be possible from our method described in this paper (see Theorem 2.7 and Question 2.10). In § 3 we give an example of a decomposable

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hereditarily equivalent continuum which is not an arc. The example shows that Henderson's theorem cited above and a related structure theorem of Mahavier [Ma] and Thomas [T] for hereditarily decomposable chainable continua will not generalize to the non-metric setting. We wish to thank Profs. E. D. Tymchatyn and L. E. Ward, Jr. for drawing our attention to the papers of Arens and Hart–van Mill respectively, and Prof. G. R. Gordh for several helpful conversations. We also wish to thank the referee, whose suggestions substantially improved our presentation.

§ 2. Hereditarily equivalent Hausdorff arcs.

DEFINITION 2.1. A *Hausdorff arc* is a continuum admitting a total order such that its topology is the order topology.

It is well known that a continuum is a Hausdorff arc if and only if it has exactly two non-separating points. Hereditarily equivalent Hausdorff arcs are all first countable and have cardinality $\mathfrak{c}$ (see [Tr] and [B]). We define interval notation in the usual way.

LEMMA 2.2. *Let $X = [a, b]$ be a hereditarily equivalent Hausdorff arc. Let $p \in (a, b]$. Then there is an order preserving homeomorphism of $[a, b]$ to $[a, p]$.*

Proof. Let $p \in (a, b]$. Then there is a homeomorphism h': $[a, b] \to [a, p]$. If h' fails to preserve order, then $h'(a) = p$ and $h'(b) = a$. Moreover, there must be a fixed point x_0 for h', so that h' carries the interval $[x_0, b]$ onto $[a, x_0]$ and the interval $[a, x_0]$ onto the interval $[x_0, p]$. Define h: $[a, b] \to [a, p]$ by setting $h(x) = x$ for $x \in [a, x_0]$ and $h(x) = h'(h'(x))$ for $x \in [x_0, b]$. It is easy to check that h is order preserving. ■

Lemma 2.2 implies that every hereditarily equivalent Hausdorff arc admits order preserving homeomorphisms onto each of its non-degenerate subarcs. Hart and van Mill [H–V] have given an example which admits no order reversing homeomorphisms.

We now proceed to our construction of several distinct hereditarily equivalent Hausdorff arcs. Let X be a hereditarily equivalent Hausdorff arc.

DEFINITION 2.3. A set $A \subset X$ is *homogeneously embedded* provided that for each $p < q$ and $r < s$ in X there exists a homeomorphism h: $[p, q] \to [r, s]$ such that

$$h[(p, q) \cap A] = (r, s) \cap A.$$

DEFINITION 2.4. Let X be a T_2-arc and $A \subset X$. We define a countable inverse system (X_n, f_m^n) as follows: each X_n is a T_2-arc, each map f_n^{n+1}: $X_{n+1} \to X_n$ is monotone. Put $A_0 = A$ and $X_0 = X$. f_0^1: $X_1 \to X_0$ is a monotone map such that

$$(f_0^1)^{-1}(x) = \begin{cases} \{\text{point}\} & \text{if } x \notin A_0, \\ B_x^1 \approx X & \text{if } x \in A_0, \end{cases}$$

where B_x^1 is a homeomorphic copy of X for each $x \in A_0$ and $\approx$ denotes an order isomorphism.

Each copy B_x^1, $x \in A_0$, of X contains a copy A_x^1 of A.

Denote by $A_1 = \bigcup_{x \in A_0} A_x^1 \subset X_1$. f_1^2: $X_2 \to X_1$ is a monotone map such that

$$(f_1^2)^{-1}(x) = \begin{cases} \{\text{point}\} & \text{if } x \notin A_1, \\ B_x^2 \approx X & \text{if } x \in A_1, \end{cases}$$

where B_x^2 is a homeomorphic copy of X for each $x \in A_1$.

Each copy B_x^2, $x \in A_1$, of X contains a copy A_x^2 of A.

Put $A_2 = \bigcup_{x \in A_1} A_x^2 \subset X_2$. Inductively define A_{n+1}, X_{n+1} and f_n^{n+1}: $X_{n+1} \to X_n$ as a monotone map such that

$$(f_n^{n+1})^{-1}(x) = \begin{cases} \{\text{point}\} & \text{if } x \notin A_n, \\ B_x^{n+1} \approx X & \text{if } x \in A_n. \end{cases}$$

Each copy B_x^{n+1} of X contains a copy A_x^{n+1} of A $(x \in A_n)$.

Put $A_{n+1} = \bigcup_{x \in A_n} A_x^{n+1}$.

We will denote the inverse limit of this system by $(X, A)_\omega = \varprojlim(X_n, f_n^{n+1})$. Note that $(X, A)_\omega$ is a T_2-arc.

THEOREM 2.5. *Let X be a hereditarily equivalent T_2-arc and suppose $A \subset X$ is homogeneously embedded. Then $(X, A)_\omega$ is a hereditarily equivalent T_2-arc.*

Proof. It follows immediately from the construction that $(X, A)_\omega$ is a T_2-arc. Hence it remains to be shown that $(X, A)_\omega$ is hereditarily equivalent. Suppose $X = X_0 = [a_0, b_0]$ and $X_n = [a_n, b_n]$. Note that for each $p < q$ and $r < s$ in X_0, where $\{q, s\} \subset A$, $\mathrm{Cl}(f_0^{-1}((p, q])) \approx \mathrm{Cl}(f_0^{-1}((r, s]))$ (under an order preserving homeomorphism). Moreover, we may assume that this homeomorphism maps $f_0^{-1}(q)$ homeomorphically onto $f_0^{-1}(s)$.

Let $Y \subset (X, A)_\omega$ be a nondegenerate subcontinuum and let n_1 be minimal such that $Y_{n_1} = f_{n_1}(Y)$ is non-degenerate, where f_{n_1}: $(X, A)_\omega \to X_{n_1}$ denotes the natural projection. Then Y_{n_1} is contained in a copy of $X_0 = X$. Put $Y_n = f_n(Y) = [c_n, d_n]$ and $Y = [c, d]$. Choose $m_0 \in (a_0, b_0) \cap A_0$ and $k_{n_1} \in (c_{n_1}, d_{n_1}) \cap A_{n_1}$. It suffices to show that $f_0^{-1}([a_0, n_0]) \approx f_{n_1}^{-1}([c_{n_1}, k_{n_1}]) \cap Y$ and $f_0^{-1}([m_0, b_0]) \approx f_{n_1}^{-1}([k_{n_1}, d_{n_1}]) \cap Y$ under homeomorphisms which agree on $f_0^{-1}(m_0)$. We will show that

$$f_0^{-1}([a_0, m_0]) \approx f_{n_1}^{-1}([c_{n_1}, k_{n_1}]) \cap Y.$$

Suppose first that $a_0 \notin A$ and $(f_{n_1+j}^{n_1+j+1})^{-1}(c_{n_1+j}) \cap Y_{n_1+j+1}$ is a point for each $j \geqslant 0$. Then

$$f_0^{-1}([a_0, m_0]) = \mathrm{Cl}(f_0^{-1}((a_0, m_0])) \approx \mathrm{Cl}(f_{n_1}^{-1}((c_{n_1}, k_{n_1}])) = f_{n_1}^{-1}([c_{n_1}, k_{n_1}])$$

and we are done.

Suppose next that $a_0 \in A$ and $(f_{n_1+j}^{n_1+j+1})^{-1}(c_{n_1+j}) \cap Y_{n_1+j+1}$ is a point for each $j \geqslant 0$.

Choose a sequence $k_{n_1} > r_1 > r_2 > \ldots > c_{n_1}$ such that $r_i \in A_{n_1}$ and $\lim r_i = c_{n_1}$. Then $f_{n_1}^{-1}((r_1, k_{n_1}]) \approx f_0^{-1}((a_0, m_0])$ and for each $j \geqslant 1$ $f_{n_1}^{-1}((r_{j+1}, r_j]) \underset{h_j}{\approx} f_j^{-1}((a_j, m_j])$, where $(f_{j-1}^j)^{-1}(a_{j-1}) = [a_j, m_j]$. It is easy to see that the homeomorphisms h_j define a homeomorphism of

$$f_0^{-1}([a_0, m_0]) = \mathrm{Cl}\Big(\bigcup_{j=0}^{\infty} f_j^{-1}((a_j, m_j])\Big) \quad \text{onto}$$

$$Y \cap f_{n_1}^{-1}([c_{n_1}, k_{n_1}]) = \mathrm{Cl}\Big(\bigcup_{j=0}^{\infty} f_{n_1}^{-1}((r_{j+1}, r_j])\Big).$$

There are three remaining cases to consider. The first is the case where $f_0^{-1}(a_0) = \{\text{point}\}$ and there are cofinally many $n_1 < n_2 < \ldots$ such that $(f_{n_j-1}^{n_j})^{-1}(c_{n_j-1}) \cap Y_{n_j} = [c_{n_j}, e_{n_j}]$ is non-degenerate. Choose a sequence $m_j \in X_0 \cap A_0$ such that $m_0 > m_1 > m_2 \ldots$ and $\lim m_j = a_0$. Put $e_{n_1} = k_{n_1}$. Then $f_0^{-1}((m_j, m_{j-1}]) \underset{h_j}{\approx} (f_{n_j})^{-1}((c_{n_j}, e_{n_j}])$ for each j. (See the second case above.) It is not difficult to see that these homeomorphisms define a homeomorphism of

$$f_0^{-1}((a_0, m_0]) = \bigcup_j f_0^{-1}((m_j, m_{j-1}]) \quad \text{onto} \quad \bigcup_j f_{n_j}^{-1}((c_{n_j}, e_{n_j}])$$

which extends to a homeomorphism of the closures of these sets. Hence $f_0^{-1}([a_0, m_0]) \approx f_{n_1}^{-1}([c_{n_1}, k_{n_1}]) \cap Y$, as required. The remaining cases where $f_0^{-1}(a_0)$ is non-degenerate and $(f_{n_j-1}^{n_j})^{-1}(c_{n_j-1}) \cap Y$ is non-degenerate for some $n_j \geqslant n_1$ and the case where $f_0^{-1}(a_0)$ is a point and $(f_{n_j-1}^{n_j})^{-1}(c_{n_j-1}) \cap Y$ is non-degenerate for finitely many n_j are similar to the above cases and are left to the reader.

In a future paper, the method of construction in the previous theorem will be extended to ordinals α other than ω. All of the examples given by Babcock [B] will be seen to be of the form $(I, I)_\alpha$. Using Theorem 2.5 we now construct four examples of hereditarily equivalent T_2-arcs. The arc $(I, Q)_\omega$ is the only metric hereditarily equivalent arc, where $Q \subset I$ denotes the rational numbers. The arc $(I, I)_\omega$ was constructed by Arens [Are1] who also proved its hereditary equivalence.

THEOREM 2.6. *The arcs $(X, Q)_\omega$, $(I, I)_\omega$, $(I, S)_\omega$ and $(I, M)_\omega$ are hereditarily equivalent. Here I denotes the metric unit interval* $[0, 1]$, *Q denotes the rational numbers in I, S denotes the irrational numbers in I and M a set constructed by van Mill and van Engelen* (see [V–V]).

Proof. It is easy to see that Q, I and S are homogeneously embedded in I. It was proved by van Engelen and van Mill [V–V] that the set M has the following properties:

(1) M is uncountable, (2) M does not contain an uncountable analytic set and (3) M is homogeneously embedded.

By Theorem 2.5, each of the above examples is a hereditarily equivalent T_2-arc.

The following theorem shows that if $(I, A)_\omega$ is any T_2-arc which is homeomorphic to either $(I, I)_\omega$ or $(I, S)_\omega$, then A contains an uncountable analytic set. Since M does not contain an uncountable analytic set, neither of these examples is homeomorphic to $(I, M)_\omega$.

THEOREM 2.7. *Let $(I, A)_\omega$ and $(I, B)_\omega$ be two homeomorphic T_2-arcs where B is an uncountable, dense subset of I. Then A contains an uncountable image of a G_δ subset of B. In fact there exists a countable set $Q \subset B$ and a monotone map φ: $B \setminus Q \to A$ such that $\varphi(B \setminus Q)$ is uncountable.*

Proof. Let $(I, B)_\omega = \varprojlim(X_n, f_m^n)$, and let f_n: $(I, B)_\omega \to X_n$ and g_0: $(I, A)_\omega \to I$ be the natural projections and let h: $(I, B)_\omega \to (I, A)_\omega$ be a homeomorphism. We may assume, without loss of generality, that A is also an uncountable set. Note that $X_0 = I$. Let φ_0: $X_0 \to I$ be defined by $\varphi_0(x) = g_0 \circ h \circ f_0^{-1}(x)$, then φ_0 is upper semi-continuous and continuum valued.

Let $Q_0 = \{x \in X_0 |\ \varphi_0(x)$ is non-degenerate$\}$. Note that $Q_0 \subset B_0 = B$. If $q_1 \neq q_2 \in Q_0$, then $\mathrm{Int}(\varphi_0(q_1)) \cap \mathrm{Int}(\varphi_0(q_2)) = \emptyset$. Since I is second countable, Q_0 is countable. Hence $B \setminus Q_0$ is a G_δ subset of B. If $\varphi_0(B \setminus Q)$ is uncountable we are done (note that $\varphi_0(b) \in A$ for each $b \in B \setminus Q_0$). Hence we may assume $\varphi_0(B \setminus Q_0)$ is countable. Let $J_1 = (f_0^1)^{-1}(Q_0)$ and define φ_1: $J_1 \to I$ by $\varphi_1 = g_0 \circ h \circ (f_1)^{-1}$. Then J_1 is a countable union of metric arcs each of which contains a copy of B. Let B_1 denote the union of all these copies of B. Let $Q_1 = \{q \in J_1 |\ \varphi_1(q)$ is non-degenerate$\}$, as above Q_1 is countable. If $\varphi_1(B_1 \setminus Q_1)$ is uncountable, then there exists $q \in Q_0$ such that $\varphi_1([(f_0^1)^{-1}(q) \cap B_1] \setminus Q_1)$ is uncountable and we are done. Hence we may assume $\varphi_1(B_1 \setminus Q_1)$ is countable. Construct inductively $J_n = (f_{n-1}^n)^{-1}(Q_n)$, where $Q_{n-1} \subset J_{n-1}$ is the countable set $\{q \in J_{n-1} |\ \varphi_{n-1}(q) = g_0 \circ h \circ (f_{n-1})^{-1}(q)$ is non-degenerate$\}$, B_n which is the union of all the copies of B in J_{n-1} and φ_n: $J_n \to I$ defined by $\varphi_n = g_0 \circ h \circ (f_n)^{-1}$. If for some minimal n, $\varphi_n(B_n \setminus Q_n)$ is uncountable, we are done.

Hence, assume that for all n, $\varphi_n(B_n \setminus Q_n)$ is countable.

For each n and each $q \in Q_n$, let $E_n(q)$ denote the set of end points of $\varphi_n(q)$ and let $E_n = \bigcup_{q \in Q_n} E_n(q)$. Then $Y = \bigcup_n (\varphi_n(B_n \setminus Q_n) \cup E_n)$ is a countable set. Since A is uncountable, there exists a point $a \in A \setminus Y$. We claim that there exists a point $(q_0, q_1, q_2, q_3, \ldots) \in (I, B)_\omega$ such that for each n, $q_n \in Q_n$ and $a \in \mathrm{Int}(\varphi_n(q_n))$. Note that this implies that the point q_n is unique. We will prove this by induction on n. Suppose $a \notin \varphi_0(Q_0)$. Since $a \notin \varphi_0(B_0 \setminus Q_0)$, $a \notin \varphi_0(B_0)$. Hence there exists $s \in I \setminus B_0 = I \setminus B$ such that $a \in \varphi_0(s)$. Since $f_0^{-1}(s)$ is a point, $g_0^{-1}(a)$ is non-degenerate and B is dense, there exists $b \in B_0$ such that $f_0^{-1}(b) \cap h^{-1} \circ g_0^{-1}(a) \neq \emptyset$. Then $a \in \varphi_0(b)$. This contradiction shows that $a \in \varphi_0(q_0)$ for some $q_0 \in Q_0$. Since $a \notin E_0$, $a \in \mathrm{Int}(\varphi_0(q_0))$. Inductively assume that $a \in \mathrm{Int}\, \varphi_n((q_n))$. Hence q_n is unique and $a \in \varphi_{n+1} | (f_n^{n+1})^{-1}(q_n)$. If $a \notin \varphi_{n+1}(Q_{n+1} \cap (f_n^{n+1})^{-1} q_n)$ then it follows as above that $a \in \varphi_{n+1}(s)$ for some $s \in (f_n^{n+1})^{-1}(q_n) \setminus B_{n+1}$ and there exists $b \in B_{n+1} \cap (f_n^{n+1})^{-1}(q_n)$ such that $\varphi_{n+1}(b) = a$. This contradiction shows that $a \in \varphi_{n+1}(q_{n+1})$ for some $q_{n+1} \in Q_{n+1} \cap (f_n^{n+1})^{-1}(q_n)$. Since $a \notin E_{n+1}$, $a \in \mathrm{Int}(\varphi_{n+1}(q_{n+1}))$. This proves that there exists a sequence $(q_0, q_1, \ldots)$ such that $a \in \mathrm{Int}(\varphi_n(q_n))$ for all n. Hence

$$g_0^{-1}(a) \subset \bigcap_n h \circ f_n^{-1}(q_n) = h((q_0, q_1, q_2, \ldots)).$$

Since $g_0^{-1}(a)$ is non-degenerate, this contradicts the fact that h is a function and the proof of the theorem is complete.

In [B], Babcock constructed for each countable ordinal $\alpha \geqslant \omega$ a (non-metric) T_2-arc I^α (I^ω is the same continuum as the continuum $(I, I)_\omega$ in this paper) and showed that I^α is hereditarily equivalent for uncountably many α. He used the following theorem to show that if I^α is homeomorphic to I^β, then $\alpha = \beta$. We will use it to show that I^α $(\alpha > \omega)$ is not homeomorphic to $(X, Q)_\omega$, $(I, I)_\omega$, $(I, S)_\omega$ or $(I, M)_\omega$.

THEOREM 2.8 (Babcock). *Let $\alpha < \beta$ be countable ordinal numbers. Then there does not exist a monotone surjection $\varphi\colon I^\beta \twoheadrightarrow I^\alpha$.*

COROLLARY 2.9. *None of the examples $(I, N)_\omega$ of Theorem 2.6 is homeomorphic to I^α for any $\alpha > \omega$.*

Proof. Let $X = (I, N)_\omega$ where N is either the rational numbers, Q, the irrational numbers, S, the interval, I, or the set M and let $\alpha > \omega$. Suppose X is homeomorphic to I^α. Since $\alpha > \omega$, I^α is not metric. Hence $N \neq Q$. It is easy to see that there exists a monotone surjection $\varphi'\colon (I, I)_\omega \to (I, N)_\omega$. Since $I^\omega = (I, I)_\omega$ and we assume that X is homeomorphic to I^α, φ' induces a monotone surjection $\varphi\colon I^\omega \twoheadrightarrow I^\alpha$. This contradicts Theorem 2.8. ■

In light of Theorem 2.6 (for the purpose of constructing distinct hereditarily equivalent T_2-arcs) the following question is of interest.

QUESTION 2.6. *Do there exist infinitely many homogeneously embedded subsets $R^\alpha \subset I$ such that no R_α contains an uncountable subset which is the monotone image of a set $C_\beta \subset R_\beta$ $(\alpha \neq \beta)$ such that $R_\beta \backslash C_\beta$ is countable?*

Hence we have established that the hereditarily equivalent T_2-arcs $(I, I)_\omega$, $(I, S)_\omega$ and $(I, M)_\omega$ are distinct from any of the examples I^α $(\alpha > \omega)$ constructed by Babcock. The following theorem shows that they are distinct from each other.

THEOREM 2.11. *The hereditarily equivalent arcs $(I, Q)_\omega$, $(I, I)_\omega$, $(I, S)_\omega$ and $(I, M)_\omega$ constructed in Theorem 2.6 are topologically distinct.*

Proof. Since $(I, Q)_\omega$ is separable metric, this example is clearly distinct from the remaining three examples. It follows immediately from (2) and Theorem 2.7 that $(I, M)_\omega$ is not homeomorphic to $(I, I)_\omega$ or to $(I, S)_\omega$. We will show next that $(I, S)_\omega$ is not homeomorphic to $(I, I)_\omega$. Note that $(I, S)_\omega$ contains a countable set D $(= f_0^{-1}(Q)$, where $Q = I \backslash S)$ such that each point of D (except the two end points corresponding to $f_0^{-1}(0)$ and $f_0^{-1}(1)$) is a limit from two sides of points of D and every other point of $(I, S)_\omega$ is a limit from at most one side of points of D. It suffices to show that $(I, I)_\omega$ does not contain such a countable set.

CLAIM 1. *Suppose $E \subset (I, I)_\omega$ is a countable set such that each point of E (except the end points of $(I, I)_\omega$) is a limit from two sides of points of E. Then there exists a point in $(I, I)_\omega \backslash E$ which is a limit from two sides of points of E.*

Proof of Claim 1. Let $E \subset (I, I)_\omega$ be a countable set such that each point of E (except the end points of $(I, I)_\omega$) is a limit from both sides. Then there exists a increasing sequence $n_1 < n_2 < \ldots$ and finite subsets $D_j \subset f_{n_j}(E)$ such that D_1 does not contain an end point of

$$X_{n_1} = f_{n_1}((I, I)_\omega) \quad \text{and} \quad |(f_{n_j}^{n_{j+1}})^{-1}(d) \cap D_{j+1}| = 2 \quad \text{for each } d \in D_j.$$

Let $D = \varprojlim(D_j, f_{n_{j-1}}^{n_j}|D_j)$, then D is clearly homeomorphic to the Cantor set and $D \subset \bar{E}$.

Since E is countable, $D \backslash E$ is an uncountable separable metric space with the order topology. Hence (see [Moo], Theorem 6, p. 3), there exists a point $d \in D \backslash E$ which is a limit point from both sides. Since $D \subset \bar{E}$, d is also a limit from both sides of points of E. This completes the proof of the theorem.

Finally, we note that Vázquez and Zubieta [V–Z] have given an example related to our examples in § 2, but not satisfying the condition of compactness. It is a hereditarily equivalent totally ordered spaces in which every increasing (or decreasing) sequence has a limit.

§ 3. A hereditarily decomposable, hereditarily equivalent Hausdorff continuum which is not an arc. In this section we construct the example indicated in the title. This situation cannot arise for metric continua because of Henderson's theorem [H] that every decomposable hereditarily equivalent metric continuum is an arc, indeed is homeomorphic to the unit interval [0, 1] (since there are no metrizable arcs topologically distinct from [0, 1]). Our example also shows that a related theorem of Mahavier [Ma] and Thomas [T] (generalized by Oversteegen and Tymchatyn [O–T]) that any chainable hereditarily decomposable metric continuum contains a subcontinuum with a degenerate tranche, will not generalize to the non-metric setting. (See the remarks at the end of this section.) We will not need the concept of a tranche in this paper, but we will occasionally use the word to help the reader who is familiar with the concept visualize the construction. For a discussion of tranches see [Ku], § 48, iv or [T].

The desired space X will be an inverse limit of metric chainable continua X_F where F is an element of the directed set $\mathscr{F}$ to be described below. The construction is reminiscent of Janiszewski's famous example [J] of a hereditarily decomposable chainable continuum which does not contain an arc. In that example countably many points of an arc are replaced by limit segments of $\sin(1/x)$ curves, which are also built into the space. The procedure is then iterated on the inserted limit arcs, countably many times. Our procedure is the same except that *every* point of every arc must be replaced by a $\sin(1/x)$ limit arc. The rather complicated directed set $\mathscr{F}$ described below is needed to keep track of these insertions. If $F_1, F_2, \in \mathscr{F}$ and F_2 is an immediate successor of F_1, then X_{F_2} will be X_{F_1} with a limit arc inserted at a location indicated by the function F_2.

The directed set $\mathscr{F}$ will be the collection of "all finite rooted tree-sequences in the closed interval $[-1, 1]$." A *finite rooted tree* is a finite partially ordered set D with a unique minimal element d_0, called the *root* of D, and such that if $d \in D$, then there is a unique sequence $d_0, d_1, \ldots, d_n = d$ of elements of D such that for every $i = 1, 2, \ldots, n$, d_i is an immediate successor of d_{i-1}. (This is equivalent to the condition that each element $d \neq d_0$ of D has a unique immediate predecessor.) Let D_1 and D_2 be finite rooted trees. Then D_2 is said to be an *extension* of D_1 if there is a one-to-one function $f\colon D_1 \to D_2$ carrying roots to roots and immediate successors to immediate successors. A *finite rooted tree-sequence in* $[-1, 1]$ is a function F from a finite rooted tree, denoted $D(F)$, into $[-1, 1]$ carrying the root of $D(F)$ to 0 and such that if d_1 and d_2 are distinct immediate successors of some $d \in D(F)$, then $F(d_1) \neq F(d_2)$. Let F_1 and F_2 be finite rooted tree-sequences in $[-1, 1]$. Then F_2 is said to be an *extension* of F_1 if there is a function $f\colon D(F_1) \to D(F_2)$ as above (i.e., $D(F_2)$ is an extension of $D(F_1)$) and for every $d \in D(F_1)$, $F_2(f(d)) = F_1(d)$. Note that the function f is necessarily unique.

Let $\mathscr{F}$ denote a set containing exactly one isomorphic copy of each finite rooted tree-sequence in $[-1, 1]$. Partially order $\mathscr{F}$ by extension. Note that $\mathscr{F}$ has a unique minimal element. It is straightforward to verify that $\mathscr{F}$ is a directed set.

We are now ready to define our factor spaces X_F. We begin by defining a function $s\colon (0, \infty) \to [-1, 1]$ similar to the $\sin(1/x)$ function. s is defined on the interval $[1/2, 1]$ as follows:

$$s(x) = \begin{cases} 8x-4 & \text{if } x \in [1/2, 5/8], \\ -8x+6 & \text{if } x \in [5/8, 7/8], \\ 8x-8 & \text{if } x \in [7/8, 1]. \end{cases}$$

Extend s to $(0, \infty)$ by imposing the condition

$$(*) \qquad s(2^n x) = s(x) \quad \text{for all integers } n.$$

Let $F \in \mathscr{F}$. X_F will be a subcontinuum of the finite product $\prod_{d \in D(F)} I_d$, where for each $d \in D(F)$, I_d denotes the interval $[-1, 1]$ with the usual topology. For x an element of the product, let x_d denote the dth coordinate of x. X_F may be thought of as a chainable continuum produced as follows: Let d_0 be the root of $D(F)$. Start with the unit interval $I = [-1, 1]$. For each immediate successor d of d_0, replace the point $F(d) \in I$ by an interval perpendicular to I and "parallel" to I_d. Make the rest of I converge onto these inserted intervals by adding "$s(x)$ oscillations", in the direction of I_d, to the arc I. Continue this process recursively for the arcs just inserted. The precise definition of X_F is as follows: A point $x \in \prod_{d \in D(F)} I_d$ is in X_F if and only if its coordinates x_d satisfy the following conditions: Let $d_0, d_1, \ldots, d_{n-1}, d$ be the unique sequence of elements of $D(F)$ associated with d, as described in the definition of finite rooted trees. Then x_d can be defined once $x_{d_0}, \ldots, x_{d_{n-1}}$ have been defined (i.e. the following definition is recursive).

(1) x_{d_0} is arbitrary in $[-1, 1]$.

(2) If $x_{d_{i-1}} = F(d_i)$ for all $i = 1, \ldots, n-1$ and $x_{d_{n-1}} = F(d)$, then x_d is arbitrary in $[-1, 1]$. (These are the "tranches" and "tranches-within-tranches".)

(3) If $x_{d_{n-1}} = F(d)$, but $x_{d_{i-1}} \neq F(d_i)$ for some $i = 1, \ldots, n-1$, then $x_d = 0$.

(4) If d is an immediate successor of d_0, then $x_d = s(|x_{d_0} - F(d)|)$ (unless $x_{d_0} = F(d)$, in which case (2) applies).

(1)–(4) are special degenerate situations. The generic case is (5) below, which applies in all other situations. Note that in case (3), (4) and (5) the value of x_d is forced by the values of $x_{d_0}, \ldots, x_{d_{n-1}}$.

$$(5) \qquad x_d = |x_{d_{n-1}} - F(d)|^{\sum_{i=1}^{n-1} |x_{d_{i-1}} - F(d_i)|} s(|x_{d_{n-1}} - F(d)|).$$

Let $F_1, F_2 \in \mathscr{F}$, $F_1 \leqslant F_2$. Then the bonding map from X_{F_2} to X_{F_1} will be the natural projection of $\prod_{d \in D(F_2)} I_d$ to $\prod_{d \in D(F_1)} I_d$ restricted to X_{F_2}. This map will always be atomic in the sense of Cook [C]. (This is easiest to see in the case where $D(F_2)$ contains just one more element than $D(F_1)$. Other cases follow from the fact that compositions of atomic maps are atomic.) Therefore, if π is the projection map from X to some factor space X_F and M is a subcontinuum of X such that $\pi(M)$ is non-degenerate, then $M = \pi^{-1}[\pi(M)]$. (I.e. π is atomic.)

We are now ready to show that X is hereditarily equivalent. Let M be a non-degenerate subcontinuum of X. First consider the special case in which $\pi(M)$ is non-degenerate, where π is the projection map from X to the first factor space, $X_0 = [-1, 1]$, in the inverse system. Then $\pi(M) = [a, b]$, a subinterval of $[-1, 1]$, and $M = \pi^{-1}([a, b])$.

We begin by showing that M is homeomorphic to $\pi^{-1}([-e, e])$, where $2e = b-a$. If F is a finite rooted tree-sequence in $[-1, 1]$, then we denote by M_F the projection of M in X_F and by E_F the projection of $\pi^{-1}([-e, e])$ in X_F. Then M is the inverse limit of the spaces M_F and $\pi^{-1}([-e, e])$ is the inverse limit of the spaces E_F under the appropriately restricted bonding maps from the system of X_F's. We will induce a mapping of M to $\pi^{-1}([-e, e])$ by mapping the spaces M_F homeomorphically onto appropriately chosen factor spaces $E_{F'}$. Let h be an isometry carrying $[a, b]$ onto $[-e, e]$. Let F be a finite rooted tree-sequence in $[-1, 1]$. Let d_0 be the root of $D(F)$ and let $d_{1,1}, d_{1,2}, \ldots, d_{1,n}$ be the immediate successors of d_0 in $D(F)$. Suppose that for each $i = 1, 2, \ldots, n$, $F(d_{1,i}) \in [a, b]$. Then define the tree-sequence F' by setting $D(F') = D(F)$, $F'(d_{1,i}) = h(F(d_{1,i}))$ for each $i = 1, 2, \ldots, n$, and for d a strict successor of any $d_{1,i}$, $F'(d) = F(d)$. M_F can then be mapped homeomorphically (indeed isometrically) onto $E_{F'}$ by the mapping h^* defined as follows: Let $x \in M_F$. Then $x_{d_0} \in [a, b]$. Let $h(x)_{d_0} = h(x_{d_0})$. For $d \in D(F') = D(F)$, $d \neq d_0$, let $h^*(x)_d = x_d$. It is straightforward to check that h^* is a homeomorphism of M_F onto $E_{F'}$.

We claim that the homeomorphisms just described suffice to generate the desired homeomorphism of M onto $\pi^{-1}([-e, e])$. For suppose that F is an arbitrary finite rooted tree-sequence in $[-1, 1]$. Then there is a finite rooted tree-sequence F_m which is a restriction of F such that $F_m(d_1) \in [a, b]$ for each immediate successor d_1 of the root in $D(F_m)$ and such that $D(F_m)$ is maximal with respect to this property among all sub-finite rooted trees of $D(F)$. It is not difficult to show that M_F is homeomorphic to M_{F_m} (the "extra" coordinates of points in M_F are all forced). Thus the mapping $h^*\colon M_{F_m} \to E_{F'_m}$ can be easily lifted to a homeomorphism of M_F onto any $E_{F'}$ where F' is any extension of F'_m such that $D(F') = D(F)$ and every immediate successor d_1 of the root in $D(F')$ such that $F'(d_1) \in [-e, e]$, lies in $D(F'_m)$. Thus the homeomorphisms described in the previous paragraph can be lifted to other factor spaces to produce homeomorphisms of the entire inverse system of spaces M_F onto the entire inverse system of spaces E_F.

So we may assume that $M = \pi^{-1}([-e, e])$ for some $e \in (0, 1)$. Let n be a natural number such that $2^{-n} < e$ and let $c = (2^n e - 1)/(2^n - 1)$. Then $c < e$ and $2^{-n}(1-c) = e-c$. For each $F \in \mathscr{F}$, let M_F denote the projection of M in the factor space X_F. Let $F \in \mathscr{F}$. We will map X_F homeomorphically onto $M_{F'}$, where F' is defined as follows: $D(F') = D(F)$ and for d an immediate successor of the root,

$$F'(d) = \begin{cases} F(d) & \text{if } F(d) \in [-c, c], \\ c + 2^{-n}(F(d) - c) & \text{if } F(d) \geqslant c, \\ -c - 2^{-n}(-c - F(d)) & \text{if } F(d) \leqslant -c. \end{cases}$$

For all other $d \in D(F')$, $F'(d) = F(d)$.

We define $h^*\colon X_F \to M_{F'}$ as follows: Let $x \in X_F$ and let r be the root of $D(F') = D(F)$. Then

$$h^*(x)_r = \begin{cases} x_r & \text{if } x_r \in [-c, c], \\ c+2^{-n}(x_r - c) & \text{if } x_r \geqslant c, \\ -c-2^{-n}(-x_r - c) & \text{if } x_r \leqslant -c. \end{cases}$$

Note that h^* is continuous, one-to-one and onto in the root coordinate. Now suppose that d is an immediate successor of r in $D(F') = D(F)$. It may happen that the definition of $h^*(x)_d$ is forced by the definition of $h^*(x)_r$. If not, we define $h^*(x)_d = x_d$. To see that h^* is continuous in the d coordinate, we note that $h^*(x)_d$ will turn out to equal x_d on one of the following closed intervals: $[-1, -c]$, $[-1, c]$, $[-c, c]$, $[-c, 1]$ and $[c, 1]$; namely, the smallest of these containing $F(d)$ in its interior. Suppose, for example, that $F(d) > c$. We claim that $h^*(x)_d = x_d$ on the interval $[c, 1]$. This follows from property (*) of the function s. Suppose for example, that $x_r \in J$ and $x_r \neq F(d)$. Then $h^*(x)_d$ is forced to be $s(|h^*(x)_r - F'(d)|)$. But

$$h^*(x)_r - F'(d) = c+2^{-n}(x_r - c) - \big(c+2^{-n}(F(d)-c)\big) = 2^{-n}\big(x_r - F(d)\big).$$

So

$$h^*(x)_d = s\big(|2^{-n}(x_r - F(d))|\big) = s\big(x_r - F(d)\big) = x_d.$$

For d_1 a successor (immediate or not) of a d as above, we similarly define $h^*(x)_{d_1} = x_{d_1}$, in all cases where $h^*(x)_{d_1}$ is not forced. It is straightforward but tedious to verify that h^* is a homeomorphism of X_F onto $M_{F'}$, and commutes with the appropriate bonding maps.

Note that the tree sequences F' described above all have the property that for immediate successors d of the root, $F'(d) \in [-e, e]$. In order to insure that our mapping into M is well defined, we need to produce homeomorphisms onto factor spaces $M_{F''}$ where F'' is arbitrary in $\mathscr{F}$ from appropriately chosen factor spaces X_F of X. But we claim that, as above, the homeomorphisms just described suffice to induce these desired maps. For if F'' is an arbitrary element of $\mathscr{F}$, then $D(F'')$ contains a maximal subtree D, such that $F' = F''|_D$ satisfies the restricted condition above. But then $M_{F''}$ is homeomorphic to $M_{F'}$ (the "extra" coordinates for points in $M_{F''}$ are all forced) and consequently the homeomorphism from the appropriate X_F to $M_{F'}$ can be lifted to $M_{F''}$ in such a way that it continues to commute with the bonding maps. This completes the special case in which M projects non-degenerately in $X_0 = [-1, 1]$.

Now suppose M is an arbitrary non-degenerate subcontinuum of X. Let F be a minimal element of $\mathscr{F}$ such that M projects non-degenerately in M_F. (In fact F is unique. But we do not need this fact.) As above, let M_F be the projection of M in X_F. $X_F \subset \prod_{d \in D(F)} I_d$. Let $d \in D(F)$ be such that the projection (normal projection in the product) of M_F in I_d is non-degenerate. Let $d_0, d_1, \ldots, d_n = d$ be the unique chain associated with d as described in the definition of finite rooted trees. We claim that $\{d_0, d_1, \ldots, d_n\} = D(F)$. For if not, then $F' = F|_{\{d_0, d_1, \ldots, d_n\}}$ would be a strict predecessor of F in which M would project non-degenerately. By the same line of reasoning, M_F must project degenerately in any of the arcs I_{d_i}, $i < n$. Thus M_F is an arc. The rest of the proof is like the proof of the special case, with the arc $M_F \subset X_F$ replacing the arc $[a, b] \subset X_0$.

In a construction somewhat analogous to ours above, starting from a pseudo-arc instead of an arc, Michel Smith [S] has produced an example of an indecomposable non-metric hereditarily equivalent continuum. Smith's example fails to be first countable, and does not have cardinality $\mathfrak{c}$. However, as noted in § 2, all hereditarily equivalent Hausdorff arcs are first countable and have cardinality $\mathfrak{c}$.

QUESTION 3.1. *Does every decomposable hereditarily equivalent continuum have cardinality* $\mathfrak{c}$?

QUESTION 3.2. *Is every decomposable hereditarily equivalent continuum first countable*?

Arkhangel'skii [Ar] has shown that every compact, first countable Hausdorff space has cardinality at most $\mathfrak{c}$. Thus a positive answer to 3.2 implies a positive answer to 3.1.

Every hereditarily equivalent continuum is irreducible. Therefore, if it is decomposable, according to a theorem of Gordh [G], it admits a monotone decomposition into "tranches", whose quotient space will necessarily be a hereditarily equivalent Hausdorff arc. Moreover, any such continuum will, by a theorem of the first author [Moh], be chainable. Thus our example above shows that the theorems of Mahavier and Thomas [Ma] and [T] and the second author and Tymchatyn [O–T] mentioned at the beginning of § 3, do not generalize to the non-metric setting.

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Intersection properties of partitions of a cardinal

by

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Abstract. We study the properties P and R which are statements about families of functions, and are motivated by the characterization of Bernstein's property B (of families of sets) in terms of characteristic functions. In an earlier paper we applied constraints that were generalizations of those introduced by Erdös and Hajnal for families of sets.

Here we impose conditions that are of an opposite nature and have meaning only for families of functions. Positive results are obtained under weaker conditions, showing that these are more appropriate for families of functions.

Introduction. In this paper we study the properties P and R introduced in [2]. These are statements about families of functions, and are motivated by the characterization of Bernstein's property B (of families of sets A) in terms of characteristic functions χ_A. In [2] we imposed a condition, denoted by $C(2, \lambda)$, which is a direct generalization of the condition $C(2, \lambda)$ for sets, introduced by Erdös and Hajnal [1].

Here we look at families of functions all with the same domain (rather than of arbitrary domain), and constrained by intersection conditions that are in a sense opposite from those dealt with in [2]. The earlier intersection conditions require that like preimages are "well-spaced", while it seems more natural when considering families on a fixed domain to require that different preimages be separated.

We introduce the intersection condition $C[\eta, \lambda]$ on such a family, defined to mean that every intersection of the preimages of η different values is of size less than λ. Positive results are ensured even when the conditions are weaker than those of $C(\eta, \lambda)$, showing that $C[\eta, \lambda]$ is more appropriate for families of functions.

Background. A family of sets $\mathbf{A}$ is said to *possess property* B if there is a set T such that $A \cap T \neq \emptyset$ and yet $A \nsubseteq T$ for all sets A in $\mathbf{A}$. Equivalently:

$$\exists x(\chi_T(x) = \chi_A(x) = 1) \quad \text{and} \quad \exists y(\chi_T(y) \neq \chi_A(y) = 1).$$

Bernstein showed that a family of $\varkappa$ sets each of size $\varkappa$ always possess property B.

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