On hereditarily decomposable hereditarily equivalent non-metric continua

by

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Abstract. We give a method for constructing hereditarily equivalent Hausdorff arcs. The method yields four topologically distinct examples, two of which are known and two new. An example is given of a hereditarily decomposable, hereditarily equivalent continuum which is not an arc. This example shows that theorems of Henderson, Mahavier and Thomas and Oversteegen and Tymchatyn will not generalize to the non-metric setting.

§ 1. Introduction. A continuum is a compact connected Hausdorff space. A continuum is said to be decomposable if it is the union of two of its proper subcontinua and hereditarily decomposable if each of its non-degenerate subcontinua is decomposable. A continuum is said to be hereditarily equivalent if it is homeomorphic to each of its non-degenerate subcontinua. Clearly decomposable hereditarily equivalent continua are hereditarily decomposable. In 1921 Mazurkiewicz [Maz] raised a question which has still not been fully answered, namely whether there are any hereditarily equivalent metric continua other than the arc [0, 1]. Henderson [H] has shown that the arc is the only decomposable example. Moise [Moi] has shown that the pseudo-arc is hereditarily equivalent, thus answering Mazurkiewicz’ original question. However, the question whether there are any examples beyond these two remains open and appears to be very difficult.

In this paper we investigate non-metric decomposable hereditarily equivalent continua. All of the examples we have been able to find of such continua in the literature are Hausdorff arcs. These examples are due to Arens [Are] and Babcock [B], all described in [B], and a new example due to Hart and van Mill [H-V]. A recent paper of Ward [W] contains a lengthy and interesting discussion of general Hausdorff arcs, including some results on homogeneity closely related to Babcock’s. In § 2 below we give a method for constructing hereditarily equivalent Hausdorff arcs which yields the arc [0, 1], Arens’ original example and two new examples. An extension of the method, to be described in a future paper, will produce all of the Arens-Babcock examples and many others. Other new examples may well be possible from our method described in this paper (see Theorem 2.7 and Question 2.10). In § 3 we give an example of a decomposable

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Theorem 2.5. Let $X$ be a hereditarily equivalent $T_2$-arc and suppose $A \subset X$ is homogeneously embedded. Then $(X, A)_\omega$ is a hereditarily equivalent $T_2$-arc.

Proof. It follows immediately from the construction that $(X, A)_\omega$ is a $T_2$-arc. Hence it remains to be shown that $(X, A)_\omega$ is hereditarily equivalent. Suppose $X = X_0 = [a_0, b_0]$ and $X_n = [a_n, b_n]$. Note that for each $p < q$ and $r < s$ in $X_n$, where $(p, q, r, s) \subseteq A$, $\text{Cl}(f^{-n}_{r,s}(p, q, r, s)) \approx \text{Cl}(f^{-n}_{r,s}(p, q, r, s))$ (under an order preserving homeomorphism). Moreover, we may assume that this homeomorphism maps $f^{-n}_{r,s}(p, q, r, s)$ homeomorphically onto $f^{-n}_{r,s}(p, q, r, s)$.

Let $Y \subset (X, A)_\omega$ be a nondegenerate subcontinuum and let $n_k$ be minimal such that $Y_k = f^{-n_k}_{r,s}(Y)$ is non-degenerate, where $f^{-n_k}_{r,s}(Y) = [a_k, b_k]$ and $Y = [a, b]$. Choose $m_0 \in (a_k, b_k) \cap A_0$ and $k_0 \in (a_k, b_k) \cap A_k$. It suffices to show that $f^{-n_k}_{r,s}(a_k, b_k) \approx f^{-n_k}_{r,s}(a_k, b_k) \cap Y$ and $f^{-n_k}_{r,s}(m_0, b_k) \approx f^{-n_k}_{r,s}(m_0, b_k) \cap Y$ under homeomorphisms which agree on $f^{-n_k}_{r,s}(m_0)$. We will show that $f^{-n_k}_{r,s}(a_k, b_k) \approx f^{-n_k}_{r,s}(a_k, b_k) \cap Y$.

Suppose first that $a_0 \notin A$ and $(f^{-n_k}_{r,s})^{-1}(a_k, b_k) \cap Y_{n_k+j}

Then $f^{-n_k}_{r,s}(a_k, b_k) = \text{Cl}(f^{-n_k}_{r,s}(a_k, b_k)) = \text{Cl}(f^{-n_k}_{r,s}(a_k, b_k)) = f^{-n_k}_{r,s}(a_k, b_k)$ and we are done.

Suppose next that $a_0 \in A$ and $(f^{-n_k}_{r,s})^{-1}(a_k, b_k) \cap Y_{n_k+j} = f^{-n_k}_{r,s}(a_k, b_k)$ for each $j \geq 0$.

Choose a sequence $k_1 > r_1 > r_2 > \ldots > c_k$ such that $r_j \in A_k$ and $\lim r_j = c_k$. Then $f^{-n_k}_{r_k}(r_j, c_k) \approx f^{-n_k}_{r_k}(r_j, c_k)$ for each $j \geq 1$. $f^{-n_k}_{r_k}(r_j, c_k) \approx f^{-n_k}_{r_k}(a_k, b_k)$, where $(f^{-n_k}_{r_k})^{-1}(a_k, b_k) = (a_k, b_k)$. It is easy to see that the homeomorphisms $h_k$ define a homeomorphism of
There are three remaining cases to consider. The first is where \( f_0^{-1}(a_0) = \{\text{point} \} \) and there are cofinally many \( n_1 < n_2 < \cdots \) such that \( f_0^{-1}(a_0) \cap I_{n_1} = \{a_{n_2}, e_{n_2}\} \) is non-degenerate. Choose a sequence \( m_n \in X_\omega \cap A_0 \) such that \( m_0 > m_1 > m_2 \cdots \) and \( \lim m_\omega = a_0 \). Put \( e_{n_2} = k_{n_2} \). Then \( f_0^{-1}(m_j, m_{j+1}) \) \( \not\subseteq \) \( f_0^{-1}(e_{n_2}, e_{n_2}) \) for each \( j \). (See the second case above.) It is not difficult to see that these homeomorphisms define a homeomorphism of

\[
\bigcup_j f_0^{-1}(m_j, m_{j+1}) \quad \text{onto} \quad \bigcup_j f_0^{-1}(e_{n_2}, e_{n_2})
\]

which extends to a homeomorphism of the closures of these sets. Hence \( f_0^{-1}(a_0, m_0) \\approx f_0^{-1}(e_{n_2}, e_{n_2}) \cap Y \), as required. The remaining cases where \( f_0^{-1}(a_0) \) is non-degenerate and \( f_0^{-1}(e_{n_2}, e_{n_2}) \cap Y \) is non-degenerate for some \( n_1 \geq n_2 \) and the case where \( f_0^{-1}(a_0) \) is a point and \( f_0^{-1}(e_{n_2}, e_{n_2}) \cap Y \) is non-degenerate for finitely many \( n_1 \) are similar to the above cases and are left to the reader.

If the method of construction in the previous theorem will be extended to ordinals other than \( \omega \). All of the examples given by Babcock [B] will be seen to be of the form \( (I, \nu) \). Using Theorem 2.5 we now construct four examples of hereditarily equivalent \( T_\text{arc} \)-arcs. The arc \((Q, \nu)\) is the only metric hereditarily equivalent arc, where \( Q = I \) denotes the rational numbers. The arc \((I, \nu)\) was constructed by Arens [Ar1] who also proved its hereditary equivalence.

**Theorem 2.6.** The arcs \((X, \nu)\) on \((I, \nu)\), \((I, \nu)\), \((I, \nu)\), and \((M, \nu)\) are hereditarily equivalent. Here \( I \) denotes the metric unit intervals \([0, 1]\), \( Q \) denotes the rational numbers in \( I \), \( S \) denotes the irrational numbers in \( I \) and \( M \) is a set constructed by van Mill and van Engelen (see [Y-V]).

**Proof.** It is easy to see that \( Q, I \) and \( S \) are homeomorphically embedded in \( I \). It is proved by van Mill and van Engelen [Y-V] that the set \( M \) has the following properties:

1. \( M \) is uncountable,
2. \( M \) does not contain an uncountable analytic set, and
3. \( M \) is homogeneously embedded.

By Theorem 2.5, each of the above examples is a hereditarily equivalent \( T_\text{arc} \)-arc. The theorem shows that if \((I, \nu)\) is any \( T_\text{arc} \)-arc which is homeomorphic to either \((I, \nu)\) or \((I, \nu)\) then \( A \) contains an uncountable analytic set. Since \( M \) does not contain an uncountable analytic set, neither of these examples is homeomorphic to \((I, \nu)\).

**Theorem 2.7.** Let \((I, \nu)\) and \((I, \nu)\) be two homeomorphic \( T_\text{arc} \)-arcs where \( B \) is an uncountable, dense subset of \( I \). Then \( A \) contains an uncountable image of a \( G_\delta \) subset of \( B \).

In fact there exists a countable set \( \mathbb{Q} \subset B \) and a monotone map \( \varphi: B \to A \) such that \( \varphi(B \cap \mathbb{Q}) \) is uncountable.

**Proof.** Let \((I, \nu)\) be the natural projections and let \( \varphi: (I, \nu) \to (I, \nu) \) be a homeomorphism. We may assume, without loss of generality, that \( A \) is also an uncountable set. Note that \( X_0 = I \). Let \( \varphi_0: X_0 \to X_0 \) be defined by \( \varphi_0(x) = \varphi_0(\varphi_0(x)) \) (for each \( x \)), then \( \varphi_0 \) is upper semi-continuous and continuous valued.

Let \( Q_0 = \{ x \in X_0 : \varphi_0(x) \text{ is non-degenerate} \} \). Note that \( Q_0 \subset B_0 \). If \( q \in \varphi_0(Q_0) \) then \( \text{Int}(\varphi_0(Q_0)) \cap \text{Int}(\varphi_0(q)) = \emptyset \). Since \( I \) is second countable, \( Q_0 \) is countable. Hence \( B \cap Q_0 \) is a \( G_\delta \) subset of \( B_0 \). If \( \varphi_0(B \cap Q_0) \) is uncountable we are done (note that \( \varphi_0(q) \in \mathcal{E} \) for each \( q \in \varphi_0(Q_0) \). Hence we may assume \( \varphi_0(B \cap Q_0) \) is countable.

Let \( J_1 = (f_1^{-1}(Q_0) \cap B \cap \mathbb{Q}) \) and define \( \varphi_1: J_1 \to I \) by \( \varphi_1 = \varphi_0 \circ \varphi(f_1^{-1}(Q_0)) \). Then \( J_1 \) is a countable union of metric arcs each of which contains a copy of \( B \). Let \( B_1 \) denote the union of all the copies of \( B \) in \( J_1 \). Then \( B_1 \) is \( \varphi_1 \) in \( J_1 \) defined by \( \varphi_1 = \varphi_0 \circ \varphi(f_1^{-1}(Q_0)) \). If for some minimal \( n, \varphi(n, \varphi_1(Q_0)) \) is uncountable, we are done.

Hence, assume that for all \( n, \varphi(n, \varphi_1(Q_0)) \) is countable.

For each \( n \) and each \( q \in Q_0 \), let \( E_n(q) \) denote the set of end points of \( \varphi_0(q) \) and \( E_n = \bigcup E_n(q) \). Then \( \text{Int}(\bigcup(E_n(q) \cup E_n)) \) is a countable set. Since \( A \) is uncountable, there exists a point \( a \in A \). We claim that there exists a point \( q \in \varphi_0(Q_0) \) such that \( \varphi_0(a) \in \text{Int}(\varphi_0(q)) \). Note that this implies that the point \( q \) is unique. We will prove this by induction on \( n \). Suppose \( \varphi_0(a) \in \varphi_0(Q_0) \).

Since \( \varphi_0(a) \in \varphi_0(Q_0) \).

Hence \( q \in \varphi_0(Q_0) \). Inductively assume that \( \text{Int}(\varphi_0(q)) \). Hence \( q \) is unique and \( \varphi_0(a) \in \varphi_0(Q_0) \). If \( \varphi_0(a) \in \varphi_0(Q_0) \) then it follows as above that \( \text{Int}(\varphi_0(a)) \) for some \( x \in Q_0 \). Since \( \varphi_0(a) \in \varphi_0(Q_0) \). Inductively assume that \( \text{Int}(\varphi_0(q)) \). Hence \( q \) is unique and \( \varphi_0(a) \in \varphi_0(Q_0) \). Since \( \varphi_0(a) \in \varphi_0(Q_0) \).

This proves that there exists a sequence \( (q_n, q_1, q_2, \ldots) \) such that \( \text{Int}(\varphi_0(q_n)) \). Hence \( g_0^{-1}(a) \in \text{Int}(\varphi_0(q_n)) \). Since \( g_0^{-1}(a) \in \text{Int}(\varphi_0(q_n)) \).

Since \( g_0^{-1}(a) \) is non-degenerate, this contradicts the fact that \( h \) is a function and the proof of the theorem is complete.

In [8], Babcock constructed for each countable ordinal \( \alpha \geq \omega \) a \( (\omega, \omega) \)-arc \( T_\alpha \) (\( T_\alpha \) is the same continuum as the continuum \((I, \nu)\) in this paper) and showed that \( T_\alpha \) is homeomorphic to \( \alpha \). We will use this fact to show that \( T_\alpha \) is homeomorphic to \( \alpha \).
THEOREM 2.8 (Babcock). Let \( \alpha < \beta \) be countable ordinal numbers. Then there does not exist a monotone surjection \( \varphi: P^\alpha \to P^\beta \).

COROLLARY 2.9. None of the examples \((I, N)_\alpha\) of Theorem 2.6 is homeomorphic to \(P^\alpha\) for any \(\alpha > \omega\).

Proof. Let \(X = (I, N)\), where \(N\) is either the rational numbers, \(Q\), the irrational numbers, \(S\), the interval, \(I\), or the set \(M\) and let \(\alpha > \omega\). Suppose \(X\) is homeomorphic to \(P^\alpha\). Since \(\alpha > \omega\), \(P^\alpha\) is not metric. Hence \(N \neq Q\). It is easy to see that there exists a monotone surjection \(\varphi: (I, N) \to (I, N)\). Since \(P^\alpha = (I, N)\), we assert that \(X\) is homeomorphic to \(P^\alpha\). \(\varphi\) induces a monotone surjection \(P^\alpha \to P^\alpha\), which contradicts Theorem 2.8. \[
\]

In light of Theorem 2.6 (for the purpose of constructing distinct hereditarily equivalent \(T_2\)-arcs) the following question is of interest.

QUESTION 2.6. Do there exist infinitely many homogeneously embedded subsets \(R^\alpha < I\) such that no \(R^\alpha\) contains an uncountable subset which is the monotone image of a set \(C^\alpha < R^\alpha (\alpha \neq \beta)\) such that \(R^\alpha \setminus C^\alpha\) is countable?

Hence we have established that the hereditarily equivalent \(T_2\)-arcs \((I, N), (I, S), (I, M)\) are distinct from any of the examples \(P^\alpha (\alpha > \omega)\) constructed by Babcock. The following theorem shows that they are distinct from each other.

THEOREM 2.11. The hereditarily equivalent arcs \((I, N), (I, S), (I, M)\) constructed in Theorem 2.6 are topologically distinct.

Proof. Since \((I, Q)\) is separable metric, this example is clearly distinct from the remaining three examples. It follows immediately from (2) and Theorem 2.7 that \((I, M)\) is not homeomorphic to \((I, N)\) or to \((I, S)\). We will show next that \((I, S)\) is not homeomorphic to \((I, N)\). Note that \((I, S)\) contains a countable set \(D = f_0^{-1}(Q)\), where \(Q = I \setminus S\) such that each point of \(D\) (except the two end points corresponding to \(f_0^{-1}(0)\) and \(f_0^{-1}(1)\)) is a limit from two sides of points of \(D\) and every other point of \((I, S)\) is a limit from at most one side of points of \(D\). It suffices to show that \((I, N)\) does not contain such a countable set.

CLAIM 1. Suppose \(E \subseteq (I, N)\) is a countable set such that each point of \(E\) (except the end points of \((I, N)\)) is a limit from two sides of points of \(E\). Then there exists a point in \((I, N)\) which is a limit from two sides of points of \(E\).

Proof of Claim 1. Let \(E \subseteq (I, N)\) be a countable set such that each point of \(E\) (except the end points of \((I, N)\)) is a limit from both sides. Then there exists a increasing sequence \(n_i < n_2 < \ldots\) and finite subsets \(D_i \subseteq f_0(E)\) such that \(D_i\) does not contain an end point of \(X_{n_i} = f_0([l, n_i])\) and \((f_0^{-1})^{-1}(d) \cap D_{n_i+1} = 2\) for each \(d \in D_i\). Let \(D = \bigcup_{i=1}^{\infty} D_i\), then \(D\) is clearly homeomorphic to the Cantor set and \(D < E\).

Since \(E\) is countable, \(D\) is an uncountable separable metric space with the order topology. Hence (see [Moo], Theorem 6, p. 3), there exists a point \(d \in D\) which is a limit point from both sides. Since \(D < E\), \(d\) is also a limit from both sides of points of \(E\). This completes the proof of the theorem.

Finally, we note that Vázquez and Zubieta [V-Z] have given an example related to our examples in §2, but not satisfying the condition of compactness. It is a hereditarily equivalent totally ordered spaces in which every increasing (or decreasing) sequence has a limit.

§ 3. A hereditarily decomposable, hereditarily equivalent Hausdorff continuum which is not an arc.

In this section we construct the example indicated in the title. This situation cannot arise for metric continua because of Henderson's theorem [H] that every decomposable hereditarily equivalent metric continuum is an arc, indeed is homeomorphic to the unit interval [0,1] (since there are no metrizable arc topologically distinct from [0,1]). Our example shows that a related theorem of Mahavier [Ma] and Thomas [T] (generalized by Oversteegen and Tymchatyn [O-T]) that any chainable hereditarily decomposable metric continuum contains a subcontinuum with a degenerate tranche, will not generalize to the non-metric setting. (See the remarks at the end of this section.) We will not need the concept of a tranche in this paper, but we will occasionally use the word to help the reader who is familiar with the concept visualize the construction. For a discussion of tranches see [Ku], §48, iv or [T].

The desired space \(X\) will be an inverse limit of metric chainable continua \(X_n\) where \(F\) is an element of the directed set \(\mathcal{F}\) to be described below. The construction is reminiscent of Janiszewski's famous example [J] of a hereditarily decomposable chainable continuum which does not contain an arc. In that example countably many points of an arc are replaced by limit segments of \(\sin(1/x)\) curves, which are also built into the space. The procedure is then iterated on the inserted limit arcs, countably many times. Our procedure is the same except that every point of every arc must be replaced by a \(\sin(1/x)\)-limit arc. The rather complicated directed set \(\mathcal{F}\) described below is needed to keep track of these insertions. If \(F_1, F_2, F_3, \ldots, F_n\) is an immediate successor of \(F_1\), then \(X_{F_n}\) will be \(X_{F_1}\) with a limit arc inserted at a location indicated by the function \(F_2\).

The directed set \(\mathcal{F}\) will be the collection of "all finite rooted tree-sequences in the closed interval [-1,1]." A finite rooted tree is a finite partially ordered set \(D\) with a unique minimal element \(d_0\) called the root of \(D\), and such that if \(d \in D\), then there is a unique sequence \(d_0, d_1, \ldots, d_n = d\) of elements of \(D\) such that for every \(i = 1, 2, \ldots, n\), \(d_i\) is an immediate successor of \(d_{i-1}\). (This is equivalent to the condition that each element \(d \neq d_0\) of \(D\) has a unique immediate predecessor.) Let \(D_1\) and \(D_2\) be finite rooted trees. Then \(D_2\) is said to be an extension of \(D_1\) if there is a one-to-one function \(f: D_1 \to D_2\) carrying roots to roots and immediate successors to immediate successors. A finite rooted tree-sequence in \([-1,1]\) is a function \(F\) from a finite rooted tree, denoted \(D(F_0)\), into \([-1,1]\) carrying the root of \(D(F)\) to 0 and such that if \(d_0, d_1, \ldots, d_n\) are distinct immediate successors of some \(d \in D(F)\), then \(F(d_0) \neq F(d_1)\). Let \(F_1\) and \(F_2\) be finite rooted tree-sequences in \([-1,1]\). Then \(F_2\) is said to be an extension of \(F_1\) if there is a function \(f: D(F_1) \to D(F_2)\) as above (i.e., \(D(F_2)\) is an extension of \(D(F_1)\)) and for every \(d \in D(F_1)\), \(F_2(f(d)) = f_1(d)\). Note that the function \(f\) is necessarily unique.

Let \(\mathcal{F}\) be a set containing exactly one isomorphic copy of each finite rooted
tree-sequence in $[-1, 1]$. Partially order $\mathcal{F}$ by extension. Note that $\mathcal{F}$ has a unique minimal element. It is straightforward to verify that $\mathcal{F}$ is a directed set.

We are now ready to define our factor spaces $X_\pi$. We begin by defining a function $s: (0, \infty) \rightarrow [-1, 1]$ similar to the sin(1/x) function. $s$ is defined on the interval $[1/2, 1]$ as follows:

$$ s(x) = \begin{cases} 
\frac{8x}{2} & \text{if } x \in [1/2, 5/8], \\
\frac{8x + 6}{2} & \text{if } x \in [5/8, 7/8], \\
\frac{8x}{2} & \text{if } x \in [7/8, 1]. 
\end{cases} $$

Extend $s$ to $(0, \infty)$ by imposing the condition

$$(*) \quad s(2^n x) = s(x) \quad \text{for all integers } n.$$ 

Let $F \in \mathcal{F}$. $X_F$ will be a subcontinuum of the finite product $\prod_{d \in D(F)} I_d$, where for each $d \in D(F)$, $I_d$ denotes the interval $[-1, 1]$ with the usual topology. For $x$ an element of the product, let $x_d$ denote the $d$th coordinate of $x$. $X_F$ may be thought of as a chainable continuum produced as follows: Let $d_0$ be the root of $D(F)$. Start with the unit interval $I = [-1, 1]$. For each immediate successor $d_1$ of $d_0$, replace the point $F(d_1) \in I$ by an interval perpendicular to $I$ and “parallel” to $I_d$. Make the formation of $I_{d_1}$ converge onto these inserted intervals by adding “s(c) oscillations”, in the direction of $I_{d_1}$ to the arc $I$. Continue this process recursively for the arcs just inserted.

The precise definition of $X_{\pi}$ is as follows: A point $x \in \prod_{d \in D(F)} I_d$ is in $X_{\pi}$ if and only if its coordinates $x_d$ satisfy the following conditions: Let $d_0, d_1, \ldots, d_n, d$ be the unique sequence of elements of $D(F)$ associated with $d$, as described in the definition of finite rooted trees. Then $x_d$ can be defined once $x_{d_0}, \ldots, x_{d_{n-1}}$ have been defined (i.e. the following definition is recursive).

(1) $x_{d_0}$ is arbitrary in $[-1, 1]$.

(2) If $x_{d_{n-1}} = F(d)$ for all $i = 1, \ldots, n-1$ and $x_{d_n} = F(d)$, then $x_d$ is arbitrary in $[-1, 1]$. (These are the “tranches” and “tranches-within-tranches”.)

(3) If $x_{d_{n-1}} = F(d)$, but $x_{d_n} \neq F(d)$ for some $i = 1, \ldots, n-1$, then $x_d = 0$.

(4) If $d$ is an immediate successor of $d_n$, then $x_d = s(x_{d_n} - F(d))$ (unless $x_{d_n} = F(d)$, in which case (2) applies).

(1)-(4) are special degenerate situations. The generic case is (5) below, which applies in all other situations. Note that in cases (3), (4) and (5) the value of $x_d$ is forced by the values of $x_{d_0}, \ldots, x_{d_{n-1}}$.

$$ x_d = |x_{d_n} - F(d)| \cdot \frac{s(x_{d_n} - F(d))}{s(x_{d_n} - F(d))}. $$

Let $F_1, F_2 \in \mathcal{F}$, $F_1 \leq F_2$. Then the bonding map from $X_{F_1}$ to $X_{F_2}$ will be the natural projection of $\prod_{d \in D(F)} I_d$ to $\prod_{d \in D(F)} I_d$ restricted to $X_{F_1}$. This map will always be atomic in the sense of Cook (C). (This is easiest to see in the case where $D(F)$ contains just one more element than $D(F_2)$. Other cases follow from the fact that compositions of atomic maps are atomic.) Therefore, if $\pi$ is the projection map from $X$ to some factor space $X_{\pi}$ and $M$ is a subcontinuum of $X$ such that $\pi(M)$ is non-degenerate, then $M = \pi^{-1}[\pi(M)]$. (i.e. $\pi$ is atomic.)

We are now ready to show that $X$ is hereditarily decomposable. Let $M$ be a non-degenerate subcontinuum of $X$. Consider the special case in which $\pi(M)$ is non-degenerate, where $\pi$ is the projection map from $X$ to the first factor space $X_0 = [-1, 1]$, in the inverse system. Then $\pi(M) = [a, b]$, a subinterval of $[-1, 1]$, and $M = \pi^{-1}([a, b])$.

We begin by showing that $M$ is homeomorphic to $\pi^{-1}([-c, e])$, where $2e = b-a$. If $F$ is a finite rooted tree-sequence in $[-1, 1]$, then we denote by $M_F$ the projection of $M$ in $X_F$ and by $E_F$ the projection of $\pi^{-1}([-c, e])$ in $X_F$. Then $M$ is the inverse limit of the spaces $M_F$ and $\pi^{-1}([-c, e])$ is the inverse limit of the spaces $E_F$ under the appropriately restricted bonding maps from the system of $X_F$'s. We will induce a mapping of $M$ to $\pi^{-1}([-c, e])$ by mapping the spaces $M_F$ homeomorphically onto appropriately chosen factor spaces $E_F$. Let $h$ be an isometry carrying $[a, b]$ onto $[-c, e]$. Let $F$ be a finite rooted tree-sequence in $[-1, 1]$. Let $d$ be the root of $D(F)$ and let $d_1, d_2, \ldots, d_{n-1}$ be the immediate successors of $d_n$ in $D(F)$. Suppose that for each $i = 1, 2, \ldots, n$, $F(d_i) \in [a, b]$. Then define the tree-sequence $F$ by setting $F(d_i) = F(d)$ for each $i = 1, 2, \ldots, n$, and for $d$ a strict successor of $d_{n-1}, F(d) = F(d_n)$ can then be mapped homeomorphically (indeed isometrically) onto $E_F$ by the mapping $h^* \xi d_{n-1}$ defined as follows: Let $x \in M_F$. Then $x_{d_n} \in [a, b]$. Let $h(x_{d_n}) = h_{d_{n-1}}(x_{d_n})$. For $d \in D(F) = D(F), d \neq d_0$, let $h^* \xi d_{n-1} = x_d$. It is straightforward to check that $h^*$ is a homeomorphism of $M_F$ onto $E_F$.

We claim that the homeomorphisms just described suffice to generate the desired homeomorphism of $M_F$ onto $\pi^{-1}([-c, e])$. For suppose that $F$ is an arbitrary finite rooted tree-sequence in $[-1, 1]$. There is a finite rooted tree-sequence $F_m$ which is a restriction of $F$ such that $F_m(d) \in [a, b]$ for each immediate successor $d_i$ of $d$ in $D(F_m)$ such that $D(F_m)$ is maximal with respect to this property among all sub-finite rooted trees of $D(F)$. It is not difficult to show that $M_F$ is homeomorphic to $M_{F_m}$ (the “extra” coordinates of points in $M_F$ are all forced). Thus the mapping $h^*: M_{F_m} \rightarrow M_{F_m}$ can be easily lifted to a homeomorphism of $M_F$ onto $E_F$ where $F$ is any extension of $F_m$ such that $D(F) = D(F_m)$ and every immediate successor $d_i$ of $d$ in $D(F)$ such that $F(d_i) \in [-c, e]$ lies in $D(F_m)$. Thus the homeomorphisms described in the previous paragraph can be lifted to other factor spaces to produce homeomorphisms of the entire inverse system of spaces $M_F$ onto the entire inverse system of spaces $E_F$.

So we may assume that $M = \pi^{-1}([-c, e])$ for some $c \in (0, 1)$. Let $c$ be a natural number such that $2^{c} < c$ and let $c = (2^{c-1})^{2^{c-1}}$. Then $c < e$ and $2^{c}(1-c) = c-e$. For each $F \in \mathcal{F}$, $M_F$ denote the projection of $M$ in the factor space $X_F$. Let $F \in \mathcal{F}$. We will map $X_F$ homeomorphically onto $M_F$, where $F$ is defined as follows: $D(F) = D(F)$ and for $d$ an immediate successor of the root, $F(d) = F(d)$ if $d \in [-c, e]$, $F(d) = F(d) + c$ if $d \in [-c, -e]$.

For all other $d \in D(F)$, $F(d) = F(d)$. 

We define $h^*: X_F \rightarrow M_F$ as follows: Let $x \in X_F$ and let $r$ be the root of $D(F) = D(F)$. Then

$$h^*(x) = \begin{cases} x_r & \text{if } x_r \in [-c, c], \\ c + 2^{-m}(x_r - c) & \text{if } x_r > c, \\ c - 2^{-m}(-x_r - c) & \text{if } x_r < -c, \end{cases}$$

Note that $h^*$ is continuous, one-to-one and onto in the root coordinate. Now suppose that $d$ is an immediate successor of $r$ in $D(F) = D(F)$. It may happen that the definition of $h^*(x_d)$ is forced by the definition of $h^*(x_r)$. If not, we define $h^*(x_d) = x_d$. To see that $h^*$ is continuous in the $d$ coordinate, we note that $h^*(x_d)$ will turn out to equal $x_d$ on one of the following closed intervals: $[-1, -c], [-1, -c], [-c, c], [-c, 1]$ and $[c, 1]$. Namely, the smallest of these containing $F(I_d)$ in its interior. Suppose, for example, that $F(I_d) > c$. We claim that $h^*(x_d) = x_d$ on the interval $[c, 1]$. This follows from property $c)$ of the function $s$. Suppose for example, that $x_d \in J$ and $x_r \notin F(I_d)$. Then $h^*(x_d)$ is forced to be $s((h^*(x_d) - F(I_d)))$. But

$$h^*(x_d) - F(I_d) = c + 2^{-m}(x_r - c) - (c + 2^{-m}((F(I_d) - c)) = 2^{-m}(x_r - F(I_d)).$$

So

$$h^*(x_d) = s((2^{-m}(x_r - F(I_d)))) = s(x_r - F(I_d)) = x_d.$$ 

For $d$, a successor (immediate or not) of a $d$ as above, we similarly define $h^*(x_d) = x_d$, in all cases where $h^*(x_d)$ is not forced. It is straightforward but tedious to verify that $h^*$ is a homeomorphic map $X_F$ onto $M_F$, and commutes with the appropriate bonding maps.

Note that the tree sequences $F$ described above all have the property that for immediate successors $d$ of the root, $F(I_d) \notin [-e, e]$. In order to ensure that our mapping into $M_F$ is well-defined, we need to produce homeomorphisms onto factor spaces $M_{F'}$ where $F'$ is arbitrary in $\mathcal{F}$ from appropriately chosen factor spaces $X_{F'}$ of $X_F$. But we claim that, as above, the homeomorphisms just suffices to induce these desired maps. For if $F'$ is an arbitrary element of $\mathcal{F}$, then $D(F')$ contains a maximal subtree $D(F')$, such that $F' = F'_n$ satisfies the restricted condition above. But then $M_{F'}$ is homeomorphic to $M_F$ (the "extra" coordinates for points in $M_{F'}$ are all forced) and consequently the homeomorphism from the appropriate $X_{F'}$ to $M_{F'}$ can be lifted to $M_F$ in such a way that it continues to commute with the bonding maps. This completes the special case in which $M$ projects non-degenerately in $X_F = [-1, 1]$.

Now suppose $M$ is an arbitrary non-degenerate continuum in $X_F$. Let $F$ be a minimal element of $\mathcal{F}$ such that $M$ projects non-degenerately in $M_F$. In fact $F$ is unique. But we do not need this fact. As above, let $M_F$ be the projection of $M$ in $X_F$. $X_F = \bigwedge_{a \in \mathcal{F}} I_a$. Let $d \in D(F)$ be such that the projection (normal projection in the product) of $M_F$ in $I_d$ is non-degenerate. Let $d_0, d_1, \ldots, d_n = d$ be the unique chain associated with $d$ as described in the definition of finite rooted trees. We claim that $(d_0, d_1, \ldots, d_n) = D(F)$. For if not, then $F' = F_{(d_0, d_1, \ldots, d_n)}$ would be a strict predecessor of $F$ in which $M$ would project non-degenerately. By the same line of reasoning, $M_F$ must project non-degenerately in any of the arcs $I_{d_i}, i < n$. Thus $M_F$ is an arc. The rest of the proof is like the proof of the special case, with the arc $M_F \subset X_F$ replacing the arc $[a, b] \subset X_F$.

In a construction somewhat analogous to ours above, starting from a pseudo-arc instead of an arc, Michel Smith [S] has produced an example of an indecomposable non-metric hereditarily equivalent continuum. Smith's example fails to be first countable, and does not have cardinality $c$. However, as noted in §2, all hereditarily equivalent Hausdorff arcs are first countable and have cardinality $c$.

**Question 3.1.** Does every decomposable hereditarily equivalent continuum have cardinality $c$?

**Question 3.2.** Is every decomposable hereditarily equivalent continuum first countable?

Arkhangelskii [Ar] has shown that every compact, first countable Hausdorff space has cardinality at most $c$. Thus a positive answer to 3.2 implies a positive answer to 3.1.

Every hereditarily equivalent continuum is irreducible. Therefore, if it is decomposable, according to a theorem of Gordh [G], it admits a monotone decomposition into "tranches", whose quotient space will necessarily be a hereditarily equivalent Hausdorff arc. Moreover, any such continuum will, by a theorem of the first author [Mol], be chainable. Thus our example above shows that the theorems of Mahavier and Thomas [Ma] and [T] and the second author and Tymchatyn [O-T] mentioned at the beginning of §3, do not generalize to the non-metric setting.

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Intersection properties of partitions of a cardinal

by

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Abstract. We study the properties P and R which are statements about families of functions, and are motivated by the characterization of Bernstein's property B (of families of sets) in terms of characteristic functions. In an earlier paper we applied constraints that were generalizations of those introduced by Erdős and Hajnal for families of sets.

Here we impose conditions that are of an opposite nature and have meaning only for families of functions. Positive results are obtained under weaker conditions, showing that these are more appropriate for families of functions.

Introduction. In this paper we study the properties P and R introduced in [2]. These are statements about families of functions, and are motivated by the characterization of Bernstein's property B (of families of sets A) in terms of characteristic functions \( \chi_A \). In [2] we imposed a condition, denoted by \( B(2, \lambda) \), which is a direct generalization of the condition \( C(2, \lambda) \) for sets, introduced by Erdős and Hajnal [1].

Here we look at families of functions all with the same domain (rather than of arbitrary domain), and constrained by intersection conditions that are in a sense opposite from those dealt with in [2]. The earlier intersection conditions require that like preimages are "well-spaced", while it seems more natural when considering families on a fixed domain to require that different preimages be separated.

We introduce the intersection condition \( C[n, \lambda] \) on such a family, defined to mean that every intersection of the preimages of \( n \) different values is of size less than \( \lambda \). Positive results are ensured even when the conditions are weaker than those of \( C[n, \lambda] \), showing that \( C[n, \lambda] \) is more appropriate for families of functions.

Background. A family of sets \( A \) is said to possess property B if there is a set \( T \) such that \( A \cap T \neq \emptyset \) and yet \( A \not\subseteq T \) for all sets \( A \) in \( A \). Equivalently:

\[ \exists x (\chi_T(x) = \chi_A(x) = 1) \quad \text{and} \quad \exists y (\chi_T(y) \neq \chi_A(y) = 1). \]

Bernstein showed that a family of \( x \) sets each of size \( x \) always possess property B.

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