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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Relative congruence distributivity within quasivarieties of nearly associative Φ-algebras

by

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Abstract. The recent papers [3], [5], [6] and [16] contain some results that concern finite axiomatizability of relatively congruence distributive quasivarieties of abstract algebras. However the scope of their applicability is not yet satisfactorily recognized. The reason is that a satisfactory characterization of relative congruence distributivity is unknown, though several general methods of establishing the property do exist. In this paper one more such method is proposed and then it is applied to characterize relative congruence distributivity within quasivarieties of nearly associative Φ-algebras.

0. Introduction. By a quasivariety we mean any class of abstract algebras of the same type, say, τ, which are models of some set of universally quantified first-order sentences whose matrices are of the form $p = q$ or of the form

$$p_0 = q_0 \,\&\ldots\&\, p_k = q_k \to p = q\,,$$

where $p_0, \ldots, p_k, q_0, \ldots, q_k, p, q$ are arbitrary terms expressed in the language of τ. Or equivalently, a class of abstract algebras of type τ is a quasivariety if it is closed under isomorphisms (I), subalgebras (S), direct products (P) and ultraproducts (P_U). An example of a quasivariety important in our discussion is the class of all Φ-algebras which satisfy the sentence $\forall x\,[x^2 = 0 \to x = 0]$, where by a Φ-algebra we understand an arbitrary (not necessarily associative) ring endowed with a structure of unitary left module over a fixed associative and commutative ring Φ with unity and whose multiplication is linked with scalar multiplication by elements of Φ by some natural requirements.

Given a quasivariety $\underline{K}$ of abstract algebras. For a member A of $\underline{K}$ by $\mathrm{Con}_{\underline{K}} A$ we denote the set of all congruence relations Θ on A such that the quotient algebra A/Θ belongs to $\underline{K}$. As $\underline{K}$ is closed under I, S and P and contains a 1-element algebra, the set $\mathrm{Con}_{\underline{K}} A$ forms a complete lattice with respect to lattice meets and joins induced by inclusion. We say that $\underline{K}$ is *relatively congruence distributive* (RCD for short) if, for each A of $\underline{K}$, the lattice $\mathrm{Con}_{\underline{K}} A$ is distributive. When $\underline{K}$ is a variety, the lattice $\mathrm{Con}_{\underline{K}} A$ coincides with the lattice of all congruence relations on A. So for varieties, relative congruence distributivity coincides with the notion of congruence distributivity widely investigated in the literature. If a variety is RCD then we shall simply say that it is CD.

One of the main results concerning CD varieties of abstract algebras is due to Baker [1] and [2]. He has proved that every finitely generated CD variety of finite type is finitely axiomatizable. This result has been strengthened by B. Jónsson in [10] to the following: If the class of finitely subdirectly irreducible members of a CD variety of finite type is finitely axiomatizable then so is the whole variety. In view of these two important results a natural idea was to extend them or at least some portion of them to quasivarieties. This has been realized in [3], [5], [6] and [16]. A final effort made in the quoted papers can be summarized in the following two theorems, where $Q(M)$ denotes the least quasivariety containing M.

THEOREM 0.1 ([16]). *For a finite set M of finite abstract algebras of finite type, if $Q(M)$ is RCD then $Q(M)$ is finitely axiomatizable.*

THEOREM 0.2 ([5]). *For a quasivariety $\underline{K}$ of abstract algebras of finite type, if $\underline{K}$ is RCD and finitely subdirectly irreducible algebras in $\underline{K}$ form a universal class which is finitely axiomatizable then $\underline{K}$ is finitely axiomatizable as well.*

Both these theorems have many applications. However the scope of their applicability is not yet satisfactorily recognized at least in comparison with the corresponding results for varieties. The main reason is that a satisfactory characterization of relative congruence distributivity is unknown, though several general methods of establishing the property do exist and they appear to be quite effective in many cases. In this paper, one more such general method is proposed and then it is applied to characterize relative congruence distributivity within quasivarieties of nearly associative Φ-algebras. We show that a quasivariety $\underline{K}$ of conditionally associative Φ-algebras, in particular, associative, alternative, or right alternative Φ-algebras satisfying $\forall x\,[2x = 0 \to x = 0]$, is RCD if and only if $\underline{K} = \mathrm{ISPP}_U(\underline{M})$ for some class $\underline{M}$ of Φ-algebras without nonzero divisors of zero. Moreover, we show that a finite set M of finite conditionally associative Φ-algebras generates a RCD quasivariety if and only if each algebra of M satisfies the quasiidentity $\forall x\,[x^2 = 0 \to x = 0]$. If M consists of Jordan rings, or right alternative Φ-algebras then $Q(M)$ is RCD if and only if so is the subquasivariety of $Q(M)$ determined by the identity $\forall x\,[2x = 0]$. A portion of these results is proved partly on the base of a result of Rjabuhin [17]. But we show that Rjabuhin's result is a consequence of Lemmas 3.2 and 3.3 of [5] which were crucial in the proof of Theorem 0.2. To accomplish characterizations of RCD quasivarieties we also mention CD varieties of Φ-algebras. As any ring can be viewed as an algebra over the ring of integers, all our results remain also valid for rings.

1. General characterization. Given a quasivariety $\underline{K}$ of abstract algebras. An algebra A of $\underline{K}$ is said to be *finitely subdirectly irreducible relative to $\underline{K}$* if id_A, the identity relation on A, is finitely meet irreducible in $\mathrm{Con}_{\underline{K}}$, that is, whenever

$\Theta_0, \ldots, \Theta_{k-1} \in \mathrm{Con}_{\underline{K}} A$ and $\bigcap (\Theta_i : i<k) = \mathrm{id}_A$ then $\Theta_i = \mathrm{id}_A$ for some $i<k$, and A is said to be *finitely subdirectly irreducible in the absolute sense* if id_A is finitely meet irreducible in the lattice of all congruence relations on A. By $\underline{K}_{\mathrm{RFSI}}$ we denote the class of all finitely subdirectly irreducible relative to $\underline{K}$ members of $\underline{K}$, and by $\underline{K}_{\mathrm{FSI}}$ the class consisting of those members of $\underline{K}$ which are finitely subdirectly irreducible in the absolute sense. Evidently, $\underline{K}_{\mathrm{FSI}} \subseteq \underline{K}_{\mathrm{RFSI}}$. However if $\underline{K}$ is RCD then by a result of [6] we have the equality $\underline{K}_{\mathrm{FSI}} = \underline{K}_{\mathrm{RFSI}}$. A least element of $\mathrm{Con}_{\underline{K}} A$ containing H, where $H \subseteq A \times A$, will be denoted by $\Theta_{\underline{K}}(H)$, and instead of $\Theta_{\underline{K}}(\{(a,b)\})$ we shall write $\Theta_{\underline{K}}(a, b)$. By $\Theta(a, b)$, where $a, b \in A$, we denote a least congruence relation on A that contains (a, b).

An abstract algebra A is said to have *permutable congruences* if, for any congruences Θ and ψ on A, the relational product $\Theta \circ \psi$ of Θ and ψ coincides with $\psi \circ \Theta$. Among algebras with permutable congruences are groups, rings, modules and, in particular, algebras over an arbitrary associative and commutative ring with unity.

A method on which our description of RCD quasivarieties of Φ-algebras will be based refers to the following proposition.

PROPOSITION 1.1. *For a quasivariety $\underline{K}$ of abstract algebras with permutable congruence relations the following conditions are equivalent:*

(i) *$\underline{K}$ is RCD;*

(ii) *$\underline{K}_{\mathrm{RFSI}} = \underline{K}_{\mathrm{FSI}}$ and there exists a finite sequence*

$$\langle p_i(x, y, z), q_i(x, y, z), r_i(x, y, z) \rangle,$$

$i<k$, of triples of 3-ary terms such that the following quasiequation and equations are valid in $\underline{K}$:

(1) $\&\,(p_i(x, y, z) = r_i(x, y, z) \colon i<k) \to x = z$;

(2) $p_i(x, x, z) = q_i(x, x, z),\ q_i(x, z, z) = r_i(x, z, z)$;

(3) $p_i(x, y, x) = q_i(x, y, x) = r_i(x, y, x)$

where $i = 0, \ldots, k-1$.

Proof. (i)⇒(ii): Assume that $\underline{K}$ is RCD. Then, by Theorem 2 of [6], $\underline{K}_{\mathrm{RFSI}} = \underline{K}_{\mathrm{FSI}}$. Denote by F a free algebra in $\underline{K}$ with x^F, y^F, z^F as only free generators. As $\mathrm{Con} F_{\underline{K}}$ is distributive, $\Theta_{\underline{K}}(x^F, z^F) = \Theta(x^F, z^F)$, $\Theta_{\underline{K}}(x^F, y^F) = \Theta(x^F, y^F)$ and $\Theta_{\underline{K}}(y^F, z^F) = \Theta(y^F, z^F)$, we have

$$\langle x^F, z^F \rangle \in \Theta(x^F, z^F) \cap \Theta(x^F, y^F) +_{\underline{K}} \Theta(x^F, z^F) \cap \Theta(y^F, z^F),$$

where $+_{\underline{K}}$ denotes the lattice join in $\mathrm{Con}_{\underline{K}}\ F$. This yields

$$\langle x^F, z^F \rangle \in \Theta_{\underline{K}}([\Theta(x^F, z^F) \cap \Theta(x^F, y^F)] \circ [\Theta(x^F, z^F) \cap \Theta(y^F, z^F)])$$

because the latter congruence coincides with

$$\Theta(x^F, z^F) \cap \Theta(x^F, y^F) +_{\underline{K}} \Theta(x^F, z^F) \cap \Theta(y^F, z^F).$$

Hence, as the lattice $\mathrm{Con}_{\underline{K}}F$ is algebraic and its compact elements are of the form $\Theta_{\underline{K}}(H)$ where H is a finite subset of ${}^2F$, there exists a finite sequence

$$\langle p_i(x,y,z), r_i(x,y,z)\rangle ,$$

$i<k$, of pairs of 3-ary terms having the following two properties:

$$\langle x^F, z^F\rangle \in \Theta_{\underline{K}}\big(\{(p_i^F(x,y,z), r_i^F(x,y,z))\colon i<k\}\big)$$

and

$$\langle p_i^F(x,y,z), r_i^F(x,y,z)\rangle \in [\Theta(x^F,z^F)\cap\Theta(x^F,y^F)]\circ[\Theta(x^F,z^F)\cap\Theta(y^F,z^F)]$$

for all $i<k$. The first property implies

$$\underline{K}\models \forall xyz\,[\&\,(p_i(x,y,z)=r_i(x,y,z)\colon i<k)\to x=z]$$

while from the second follows the existence of 3-ary terms $q_i(x,y,z)$, $i<k$, such that the equations of (2) and (3) composed with terms p_i, q_i, r_i, $(i<k)$ are valid in $\underline{K}$. This proves the part (i) implies (ii).

(ii)$\Rightarrow$(i): First assuming (ii) we claim that, for all A of $\underline{K}$ and congruence relations Θ, Θ_0, Θ_1 on A, it holds:

$$(*)\qquad \Theta_{\underline{K}}(\Theta\cap\Theta_0+\Theta\cap\Theta_1)=\Theta_{\underline{K}}(\Theta\cap(\Theta_0+\Theta_1)),$$

where $+$ denotes the join in the lattice of congruences on A. The inclusion $\subseteq$ of $(*)$ is clear. To prove the inverse inclusion it suffices to show that $\Theta\cap(\Theta_0+\Theta_1)$ $\leqslant\Theta_{\underline{K}}\quad\Theta_0+\Theta\cap\Theta_1)$. Let $(a,c)\in\Theta\cap(\Theta_0+\Theta_1)$. Then, as congruences on A permute, we have $a\Theta c$, $a\Theta_0 b$ and $b\Theta_1 c$ for some $b\in A$. By the equations of (2) and (3) it follows that $p_i(a,b,c)\,\Theta(a,c)\cap\Theta(a,b)\,q_i(a,b,c)\,\Theta(a,c)\cap\Theta(b,c)\,r_i(a,b,c)$ for all $i<k$. Hence $(p_i(a,b,c), r_i(a,b,c))\in\Theta\cap\Theta_0+\Theta\cap\Theta_1$ for all i. Therefore, by (1), we get $(a,c)\in\Theta_{\underline{K}}(\Theta\cap\Theta_0+\Theta\cap\Theta_1)$ which completes the proof of the claim.

Now, let $A\in\underline{K}$, $\Theta_0,\Theta_1,\psi\in\mathrm{Con}_{\underline{K}}A$, $\Theta_0\cap\Theta_1\leqslant\psi$ and let ψ be finitely meet irreducible in $\mathrm{Con}_{\underline{K}}A$. As the lattice $\mathrm{Con}_{\underline{K}}A$ is algebraic, in order to prove that it is distributive it suffices then to show that $\Theta_0\leqslant\psi$ or $\Theta_1\leqslant\psi$. Applying the above claim we get $\Theta_{\underline{K}}([\psi+\Theta_0]\cap[\psi+\Theta_1])=\psi$. Hence $[\psi+\Theta_0]\cap[\psi+\Theta_1]\leqslant\psi$ and, therefore, $[\psi+\Theta_0]\cap[\psi+\Theta_1]=\psi$. So $\Theta_0\leqslant\psi$ or $\Theta_1\leqslant\psi$ because, by $A/\psi\in\underline{K}_{\mathrm{RFSI}}$ and $\underline{K}_{\mathrm{RFSI}}=\underline{K}_{\mathrm{FSI}}$, ψ is finitely meet irreducible in the lattice of congruences on A. Thus (ii) implies (i).

The assumption of Proposition 1.1 saying that algebras of $\underline{K}$ have permutable congruences was used in the above proof only in the proof of (ii) $\Rightarrow$ (i). We do not know however how to prove this part without this assumption. Perhaps it is impossible at all. But at least in the case $\underline{K}$ is a variety this can be done. Indeed, by the equations of (2) and (3), we have $(p_i(a,b,c), r_i(a,b,c))\in\Theta(a,c)\cap\Theta(a,b)+\Theta(a,c)\cap\Theta(b,c)$ for all $i<k$, where $a,b,c\in A$ and $A\in\underline{K}$, and so, by (1), $(a,c)\in\Theta(a,c)\cap\Theta(a,b)+$ $+\Theta(a,c)\cap\Theta(b,c)$. Thus, by a result of [9], $\underline{K}$ is congruence distributive.

We begin now to characterize relative congruence distributivity within quasivarieties of Φ-algebras.

Let Φ be an associative and commutative ring with unity. By a Φ-algebra we mean (see [20]) any abstract algebra of the form $(R,+,-,0,\circ,k_r(r\in\Phi))$, where $(R,+,-,0,\circ)$ is a ring (not necessary associative), $(R,+,-,0,k_r(r\in\Phi))$ is a unitary left module over Φ, and, for each $r\in\Phi$ and $a,b\in R$, $k_r(a\circ b)=k_r(a)\circ b$ $=a\circ k_r(b)$. Instead of $k_r(a)$ we usually write ra. By a Φ-ideal on a Φ-algebra R we mean any subset of R which is a two-sided ideal on the ring of R and as well as a submodule of the left Φ-module of R. In further parts of the paper Φ will always stand to denote an associative and commutative ring with unity. All results concerning Φ-algebras that are established in this paper are also valid for rings since any ring R becomes an algebra over the ring Z of integers whenever, for $n\in Z$ and $a\in R$, we put $k_n(a)=0$ if $n=0$, $k_n(a)=a+\ldots+a$ (n times) if $n>0$, and $k_n(a)=(-a)+\ldots$ $+(-a)$ (n times) if $n<0$.

With the help of Proposition 1.1 we obtain

THEOREM 1.2. *For a quasivariety $\underline{K}$ of Φ-algebras the following two conditions are equivalent:*

(i) *$\underline{K}$ is* RCD;

(ii) $\underline{K}_{\mathrm{RFSI}}=\underline{K}_{\mathrm{FSI}}$ *and* $\underline{K}\models\forall x\ [x^2=0\to x=0]$.

Proof. (i)$\Rightarrow$(ii): Assume that $\underline{K}$ is RCD. By Proposition 1.1, $\underline{K}_{\mathrm{RFSI}}=\underline{K}_{\mathrm{FSI}}$. We show $\underline{K}\models\forall x\ [x^2=0\to x=0]$. Suppose that $R\not\models\forall x\ [x^2=0\to x=0]$ for some $R\in\underline{K}$. Then $a\circ a=0$ and $a\neq0$ for some $a\in R$. Let S be a subalgebra of R generated by a. Notice that the elements of S can all be written in the form $n_1(r_1a)+\ldots+n_k(r_ka)$, where $n_1,\ldots,n_k$ are integers and $r_1,\ldots,r_k\in\Phi$. This yields $S\models\forall xy\ [x\circ y=0]$ and hence the set $J=\{(b,b)\colon b\in S\}$ forms a Φ-ideal on $S\times S$. Notice that Φ-ideals J, J_1 and J_2, where $J_1=\{(b,0)\colon b\in S\}$ and $J_2=\{(0,b)\colon b\in S\}$, generate a diamond that is contained in the lattice of Φ-ideals of $S\times S$. Obviously, $S\times S/J_1$ and $S\times S/J_2$ belong to $\underline{K}$, and an isomorphism $\varphi\colon S\to S\times S/J$ defined by $\varphi(b)=(0,b)+J$ ensures that the algebra $S\times S/J$ also belongs to $\underline{K}$. Thus the diamond generated by J, J_1 and J_2 is isomorphic to a sublattice of $\mathrm{Con}_{\underline{K}}S\times S$ which, by $S\times S\in\underline{K}$, gives that $\underline{K}$ is not RCD, a contradiction. Thus (i) implies (ii).

(ii)$\Rightarrow$(i): Assume (ii) and define the following terms $p(x,y,z):=(x-z)(x-z)$, $q(x,y,z):=(x-z)(y-z)$ and $r(x,y,z):=0$. By (ii),

$$\underline{K}\models\forall xyz\ [p(x,y,z)=r(x,y,z)\to x=z].$$

On the other hand, notice that in any Φ-algebra the following equations are fulfilled:

$$p(x,x,z)=q(x,x,z),\quad q(x,z,z)=r(x,z,z)\ \text{and}$$
$$p(x,y,x)=q(x,y,x)=r(x,y,x).$$

Thus, by (ii) and Proposition 1.1, $\underline{K}$ is RCD.

We want to mention that during the Algebraic Conference in Karlove Vary, 15–20 August, 1988, Professor Keith Kearnes has kindly informed the author that he proved the following result: A quasivariety $\underline{K}$ of abstract algebras contained in a congruence modular variety is RCD if and only if $\underline{K}_{\mathrm{RFSI}} = \underline{K}_{\mathrm{FSI}}$ and there are no nonzero abelian congruences on members of $\underline{K}$. Applying this result instead of our Proposition 1.1 one can also prove the above Theorem 1.2.

We show later on that in Theorem 1.2 the equality $\underline{K}_{\mathrm{RFSI}} = \underline{K}_{\mathrm{FSI}}$ cannot be removed and even replaced by $\underline{K}_{\mathrm{RSI}} = \underline{K}_{\mathrm{SI}}$, where $\underline{K}_{\mathrm{RSI}}$ and $\underline{K}_{\mathrm{SI}}$ are the respective classes of members of $\underline{K}$ which are subdirectly irreducible in the relative and absolute sense.

For a class M of abstract algebras by $Q(M)$ we shall denote the least quasivariety containing M. By a result of [8], we know that $Q(M) = \mathrm{ISPP_U}(M)$ and, in particular, $Q(M) = \mathrm{ISP}(M)$ whenever M is finite and consists of finite algebras only. From Theorem 1.2 the following corollary immediately follows.

COROLLARY 1.3. *For a finite set M of finite Φ-algebras the following two conditions are equivalent*:

(i) $Q(M)$ *is* RCD;

(ii) $M \models \forall x\ [x^2 = 0 \to x = 0]$ *and each member of $S(M)$ is either subdirectly irreducible in the absolute sense or can be decomposed into a subdirect product of members of $S(M)$ which are subdirectly irreducible in the absolute sense.*

We want now to look at the congruence distributive varieties of Φ-algebras. As for each variety $\underline{K}$ the equality $\underline{K}_{\mathrm{RFSI}} = \underline{K}_{\mathrm{FSI}}$ is fulfilled, by Theorem 1.2 it follows that a variety $\underline{M}$ of Φ-algebras is CD if and only if $\underline{M} \models \forall x\ [x^2 = 0 \to x = 0]$. However we want to characterize CD varieties of Φ-algebras in terms of their equations. In order to do that we need some preparations.

By $T(x, y)$ we denote the set consisting of all groupoid terms which contain the variables x and y and possible others. For an element t of $T(x, y)$ we write $t(x, y, \bar{z})$ to indicate that the term t is composed with variables x, y, $(\bar{z} =)\ z_1, \ldots, z_k$.

For a Φ-algebra R and non-empty subsets A, B of R we put

$$A \circ B = \{a \circ b\colon\ a \in A \text{ and } b \in B\},$$

$$A * B = \{t^R(a, b, \bar{c})\colon\ a \in A,\ b \in B,\ (\bar{c} =)\ c_1, \ldots, c_k \in R \text{ and } t(x, y, \bar{z}) \in T(x, y)\},$$

and by ΣA we denote the $+$-closure of A. It is clear that $\Sigma A * B$ is a Φ-ideal on R whenever A and B are Φ-ideals. Notice that the lattice join $A+B$ of Φ-ideals A and B coincides with the set $\{a+b\colon\ a \in A \text{ and } b \in B\}$. A binary operation $[\ldots, \ldots]$ which is the largest among those defined on the lattices of Φ-ideals that satisfy: for any A, B and a surjective homomorphism π, $[A, B] \subseteq A \cap B$ and $\pi([A, B]) = [\pi(A), \pi(B)]$ is called the modular commutator (see [7]). As in [7] we have

LEMMA 1.4. *For any Φ, $(A, B) \to \Sigma A * B + \Sigma B * A$ defines the modular commutator on the lattices of Φ-ideals.*

For a groupoid term t we put $d(t) = 1$ when t is a variable, and $d(t) = d(u) + d(w)$ when t is of the form $u \circ w$. The value $d(t)$ is called the *degree* of t.

The proof of the next theorem refers to a result of Modular Commutator Theory stating that a congruence modular variety $\underline{K}$ is CD if and only if the modular commutator on congruence lattices of members of $\underline{K}$ satisfies the equality $[x, y] = x \wedge y$ (see [7]). The following theorem for associative Φ-algebras was noted in [12].

THEOREM 1.5. *For a variety $\underline{K}$ of Φ-algebras, $\underline{K}$ is* CD *iff there exist elements $r_1, \ldots, r_k$ of Φ and groupoid terms $t_1, \ldots, t_k$ each of degree at least 2 and involving only the variable x such that $x = r_1 t_1 + \ldots + r_k t_k$ is an equation valid in each member of $\underline{K}$.*

Proof. $\Rightarrow$: Assume that $\underline{K}$ is CD, and let F be a free algebra in $\underline{K}$ with x^F as only free generator. By Lemma 1.4, $x^F \in \Sigma F * F$ and hence there exist the required elements $r_1, \ldots, r_k$ of Φ and groupoid terms $t_1, \ldots, t_k$ with $x = r_1 t_1 + \ldots + r_k t_k \in \mathrm{Id}(\underline{K})$.

$\Leftarrow$: This implication is obvious.

Another result of the Modular Commutator Theory we want to apply says that a finite set M of finite algebras taken from a congruence modular variety generates a CD variety if and only if the modular commutator fulfils the equality $[x, y] = x \wedge y$ on congruence lattices of members of $S(M)$. Applying this result to Φ-algebras we obtain

COROLLARY 1.6. *For a finite set M of finite Φ-algebras the following conditions are equivalent*:

(i) $\mathrm{HSP}(M)$ *is* CD;

(ii) $\mathrm{HS}(M) \models \forall x\ [x^2 = 0 \to x = 0]$.

Proof. (i)$\Rightarrow$(ii): Use Theorem 1.2. (ii)$\Rightarrow$(i): By Lemma 1.4, it suffices to show that $A \cap B \subseteq \Sigma A * B + \Sigma B * A$, where A and B range over Φ-ideals of a member R of $S(M)$. But this easily follows from the observation due to (ii) that $R \circ R = R$.

In view of Corollaries 1.3 and 1.6 it is natural to pose the following two questions: Given a finite set M of finite Φ-algebras without nonzero nilpotent elements, that is, Φ-algebras satisfying the quasiidentity $\forall x\ [x^2 = 0 \to x = 0]$,

(1) Is the quasivariety generated by M relatively congruence distributive?

(2) Is the variety generated by M congruence distributive?

It turns out that both these questions have negative answers, though within a large collection of Φ-algebras they can be answered in the affirmative (see Proposition 2.3, Corollary 2.7 and Corollary 3.3). Here we provide only an answer to question (2). The question (1) is answered in section 2.

Let R be a 3-dimensional vector space over Z_3, the 3-element field of integers modulo 3, and let the vectors v_0, v_1, v_2 form a basis for R. Define the multiplication $\circ$ on $\{v_0, v_1, v_2\}$ by $a \circ b = v_1$ when $a = b$ and $a \in \{v_0, v_1\}$, and $a \circ b = v_0$ otherwise, and next extend it in the obvious way on the whole R. As a result we get a structure of Z_3-algebra on R. We show that R has no nonzero nilpotent elements and that the variety generated by R is not CD. Indeed, $(r_0 v_0 + r_1 v_1 + r_2 v_2)^2 = 0$ implies $r_0^2 + r_1^2 = 0$ and $r_0(r_1 + r_2) + r_1(r_0 + r_2) + r_2(r_0 + r_1) + r_2^2 = 0$ in Z_3. By the former equation we have $r_0 = r_1 = 0$ and therefore, by the latter one, $r_2 = 0$. Thus $r_0 v_0 + r_1 v_1 + r_2 v_2 = 0$, showing that R is without nonzero nilpotent elements.

As $R \circ R = \{r_0 v_0 + r_1 v_1 \colon r_0, r_1 \in Z_3\}$, we get $R/R \circ R \nvDash \forall x\,[x^2 = 0 \to x = 0]$. Thus, by Corollary 1.6, the variety $\mathrm{HSP}(R)$ is not CD.

We conclude this section by showing that in Theorem 1.2 the equality $\underline{K}_{\mathrm{RFSI}} = \underline{K}_{\mathrm{FSI}}$ cannot be replaced by $\underline{K}_{\mathrm{RSI}} = \underline{K}_{\mathrm{SI}}$.

Let R be the subring of those rational numbers which can be expressed in the form i/m, where i is an arbitrary integer and m is an odd integer (see [11, page 113]). By Proposition 2.2 of section 2, the quasivariety $\underline{K} := Q(R)$ is RCD. We show that $\underline{K}_{\mathrm{RSI}} \neq \underline{K}_{\mathrm{SI}}$. Let A_i, $i \in I$, be members of $\underline{K}_{\mathrm{RSI}}$ which decompose R into a subdirect product. As R is commutative and the Jacobson radical of R differs from (0), it follows that R cannot be decomposed into a subdirect product of fields. Hence among A_i's there must exist a ring, say, A_{i_0}, which is not a field. The ring A_{i_0} does not belong to $\underline{K}_{\mathrm{SI}}$ since every commutative and subdirectly irreducible ring in the absolute sense and without nonzero nilpotent elements is a field (see [11, Theorem 3.14]). Thus $\underline{K}_{\mathrm{RSI}} \neq \underline{K}_{\mathrm{SI}}$.

2. Associative and power-associative algebras. The following lemma localizes finitely subdirectly irreducible in the relative sense members of a quasivariety. It will be helpful in our considerations.

LEMMA 2.1 ([5]). *If $\underline{K} = Q(\underline{M})$ then $\underline{K}_{\mathrm{RFSI}} \subseteq \mathrm{ISP}_U(\underline{M})$.*

An algebra R is said to be *without nonzero divisors of zero if*, for all a, b of R, $a \circ b = 0$ implies $a = 0$ or $b = 0$.

As a consequence of Theorem 1.2 and Lemma 2.1 we have

PROPOSITION 2.2. *For a quasivariety $\underline{K}$ of associative Φ-algebras, $\underline{K}$ in* RCD *iff $\underline{K} = Q(\underline{M})$ for some class $\underline{M}$ of Φ-algebras without nonzero divisors of zero.*

Proof. $\Rightarrow$: It suffices to show that each member of $\underline{K}_{\mathrm{RFSI}}$ is without nonzero divisors of zero. Let $R \in \underline{K}_{\mathrm{RFSI}}$ and $a, b \in R$ be such that $a \circ b = 0$. As R is associative, $(ba) \circ (ba) = 0$, and, by Theorem 1.2, $R \vDash \forall x\,[x^2 = 0 \to x = 0]$. This yields $ba = 0$. By associativity of R and $ba = 0$, we get $(acb) \circ (acb) = 0$ where $c \in R$. So, by $R \vDash \forall x\,[x^2 = 0 \to x = 0]$, it follows that $acb = 0$ for all $c \in R$. This implies $(a) \cap (b) = (0)$ where (d) denotes a Φ-ideal on R generated by d. Indeed, for $e \in (a) \cap (b)$ we have $e = nf + sa + at + \Sigma s_i a t_i = n'g + s'b + bt' + \Sigma s_i' b t_i'$ where n, n' are integers, $f = ra$, $g = r'b$ for some $r, r' \in \Phi$, and $s, s', t, t', s_i, s_i', t_i, t_i'$ are certain elements of R. Hence, by $acb = 0$ for all $c \in R$, it follows that $e^2 = 0$ which in turn implies $e = 0$. Thus $(a) \cap (b) = (0)$, and consequently $a = 0$ or $b = 0$, since due to Theorem 1.2 R is finitely subdirectly irreducible in the absolute sense, proving that R is without nonzero divisors of zero.

$\Leftarrow$: Assume that each member of $\underline{M}$ is without nonzero divisors of zero and $\underline{K} = Q(\underline{M})$. Evidently, $\underline{K} \vDash \forall x [x^2 = 0 \to x = 0]$. We show $\underline{K}_{\mathrm{RFSI}} \subseteq \underline{K}_{\mathrm{FSI}}$. Let $R \in \underline{K}_{\mathrm{RFSI}}$. Then, by Lemma 2.1, R is without nonzero divisors of zero. As $ab \in (a) \cap (b)$ for any a, b of R, we obtain that $(a) \cap (b) = (0)$ implies $a = 0$ or $b = 0$, showing that R belongs to $\underline{K}_{\mathrm{FSI}}$. Now, by Theorem 1.2, $\underline{K}$ is RCD.

In Section 3 there is proved a slightly stronger result than stated in Proposition 2.2. However, we have decided to present the above proposition in a separate form because it is a direct consequence of Theorem 1.2. To accomplish the above proposition we want to mention that congruence distributive varieties of associative Φ-algebras in terms of their generators are characterized in [12]. By Theorem 8.2 (1) of [12] we know that a variety of associative Φ-algebras in CD if and only if it is generated by associative division Φ-algebras of bounded finite cardinality. In the case of associative rings congruence distributive varieties are exactly those generated by finite sets of finite fields (see [14]).

The answers to questions (1) and (2) within associative Φ-algebras are affirmative. This easily follows from Proposition 2.2. From that proposition it also follows that each finitely generated quasivariety of associative Φ-algebras is a variety. This observation however is not true within arbitrary Φ-algebras. Here is a counterexample.

Let S be a 4-dimensional vector space over Z_2, the 2-element field of integers modulo 2, and let the vectors v_0, v_1, v_2, v_3 form a basis for S. Define on the set $\{v_0, v_1, v_2, v_3\}$ a groupoid by the table

	v_0	v_1	v_2	v_3
v_0	v_0	v_2	v_2	v_1
v_1	v_2	v_1	v_0	v_3
v_2	v_2	v_0	v_2	v_1
v_3	v_1	v_3	v_1	v_3

and extend this groupoid in the obvious way on the whole S. The resulting structure on S is a Z_2-algebra, further denoted also by S. Notice that the set

$$I = \{r(v_0 + v_2) \colon r \in Z_2\}$$

forms a Z_2-ideal on S and that the quotient algebra S/I is not embeddable into S. Hence $S/I \notin Q(S)$, since S/I is subdirectly irreducible in the absolute sense, and thus $Q(S)$ does not coincide with $V(S)$. Using Corollary 1.3 one may easily show that the quasivariety $Q(S)$ is RCD.

The answer to question (2) is also positive within the so-called power-associative algebras. An algebra is said to be *power-associative* if every one generated subalgebra of it is associative.

PROPOSITION 2.3. *If M is a finite set of finite power-associative Φ-algebras without nonzero nilpotent elements then there exists $n \geqslant 2$ with $x = x^n \in \mathrm{Id}(M)$ and $V(M)$ is* CD.

Proof. As M is finite and each member of M is finite too, there must exist natural numbers $k > m \geqslant 1$ such that $x^k = x^m \in \mathrm{Id}(M)$. If $m = 1$ then, by Theorem 1.5, $V(M)$ is CD else applying power-associativity and the equation $x^k = x^m$ we obtain $(x^{k-1}-x^{m-1})(x^{k-1}-x^{m-1}) = 0 \in \mathrm{Id}(M)$ and consequently, by $M \models \forall x\,[x^2 = 0 \to x = 0]$, $x^{k-1} = x^{m-1} \in \mathrm{Id}(M)$. Proceeding in this way, we get $x^{k-m+1} = x \in \mathrm{Id}(M)$. Thus, by Theorem 1.5, $V(M)$ is CD.

We want to add that within power-associative algebras question (1) has a negative answer. A counterexample is given at the end of this section.

To extend the above proposition we also notice the following

PROPOSITION 2.4. *For a variety $\underline{K}$ of power-associative Φ-algebras the following two conditions are equivalent*:

(i) $\underline{K}$ *is* CD;

(ii) $x = x^n \in \mathrm{Id}(\underline{K})$ *for some* $n \geqslant 2$.

Proof. (i)$\Rightarrow$(ii): Let $\underline{K}$ be CD and let F be a free algebra in $\underline{K}$ with exactly one free generator. Evidently, $V(F)$ is CD. By power-associativity of $\underline{K}$, $V(F)$ consists of associative Φ-algebras only. Hence, by Theorem 8.2 (1) of [12], $x = x^n \in \mathrm{Id}(F)$ for some $n \geqslant 2$ and consequently $x = x^n \in \mathrm{Id}(\underline{K})$.

(ii)$\Rightarrow$(i): Use Theorem 1.5.

We want now to show that for a finite set M of finite power-associative Φ-algebras question (1) can be reduced to the same question but concerning some finite collections of finite Φ-algebras uniquely determined by M. The details are contained in the next proposition. At first however we notice the following

LEMMA 2.5. *Let R be a finite nontrivial power-associative Φ-algebra without nonzero nilpotent elements. Then there exist prime numbers $p_1, \ldots, p_k$ and Φ-subalgebras $S_1, \ldots, S_k$ of R such that $S_i \models p_i x = 0$, for all $i = 1, \ldots, k$, and $R \cong S_1 \times \ldots \times S_k$.*

Proof. Define $S_p = \{a \in R\colon p^n a = 0$ for some natural number $n\}$, where p is a prime number, and as $S_1, \ldots, S_k$ take those from among S_p's that satisfy $S_p \neq 0$.

For a finite set M of finite Φ-algebras and a prime number p let $p|M$ consist of those members of $S(M)$ which satisfy $px = 0$. Notice that $p|M$ is never empty, since a 1-element Φ-algebra always belongs to $p|M$, and, as M is finite and each algebra of M is finite too, only finite number collections from among $p|M$'s may contain nontrivial algebras. The reduction we have announced looks now as follows.

PROPOSITION 2.6. *Let M be a finite set of finite power-associative Φ-algebras without nonzero nilpotent elements. Then $Q(M)$ is* RCD *iff, for each prime p, so is $Q(p|M)$.*

Proof. $\Rightarrow$: By Theorem 1.2 and Lemma 2.1.

$\Leftarrow$: By Lemma 2.5, $Q(M) = Q(\bigcup(p|M\colon p$ is prime$))$ which, by Lemma 2.1, imp[illegible] $(M)_{\mathrm{RFSI}} \subseteq \bigcup(Q(p|M)_{\mathrm{RFSI}}\colon p$ is prime$)$, and hence the assumption that each $Q(p|M)$ is RCD gives us that each member of $Q(M)_{\mathrm{RFSI}}$ is finitely subdirectly irreducible in the absolute sense. Thus by Theorem 1.2, $Q(M)$ is RCD.

A *Jordan algebra* is any algebra satisfying the equations $xy = yx$ and $(x^2y)x = x^2(yx)$. Jordan algebras and, in particular, Jordan rings are power-associative. Moreover, by Theorem 15.11 of Osborn [15] we know that any finite Jordan ring satisfying $x = x^n$ for some $n \geqslant 2$ and of characteristic different from 2 is isomorphic to a direct product of simple Jordan rings. So applying Proposition 2.6 we obtain the following corollary.

COROLLARY 2.7. *For a finite set M of finite Jordan rings without nonzero nilpotent elements the following two conditions are fulfilled*:

(i) $Q(M)$ *is* RCD *iff so is* $Q(2|M)$;

(ii) *If $2|M$ consists of a zero ring only then $Q(M)$ coincides with $V(M)$.*

Proof. By Proposition 2.3, $V(M)$ is CD. So in view of Proposition 2.6 it suffices only to show condition (ii) because for $p \neq 2$ in the set $p|M$ only zero ring satisfies $2x = 0$.

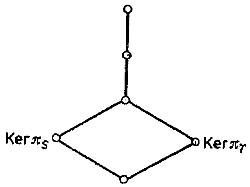
By a result of Jónsson [9], $V(M)_{\mathrm{FSI}} \subseteq \mathrm{HS}(M)$. Hence assuming that $2|M$ consists of a zero ring only, by Proposition 2.3 and Theorem 15.11 of Osborn [15], we get $V(M)_{\mathrm{FSI}} \subseteq \mathrm{IS}(M)$ which yields that $Q(M)$ coincides with $V(M)$.

The above corollary suggests that an example of finite set of finite algebras answering question (1) in the negative may be found within Jordan algebras over the field of integers modulo 2. Having this hint in mind we start now to prepare such an example.

Let T be a 4-dimensional vector space over the field Z_2. Supply the space T with a structure of Z_2-algebra by extending on the whole T in the obvious way the multiplication defined on a fixed basis v_0, v_1, v_2, v_3 of T by the following table

	v_0	v_1	v_2	v_3
v_0	v_0	v_2	v_2	v_3
v_1	v_2	v_1	v_1	v_0
v_2	v_2	v_1	v_2	v_0
v_3	v_3	v_0	v_0	v_3

A Z_2-algebra obtained in this way, denoted also by T, satisfies the equations: $x^2 = x$ and $xy = yx$, and thus it is Jordan. The set $J = \{r(v_1+v_2)\colon r \in Z_2\}$ forms a Z_2-ideal on T and one may verify that the quotient algebra T/J is isomorphic to S/I, where the algebra S and the Z_2-ideal I are defined after Proposition 2.2. Fix an isomorphism $\varphi\colon S/I \to T/J$ and define $R = \{(a, b) \in S \times T\colon \varphi(a+I) = b+J\}$. Evidently, R is a Jordan Z_2-algebra which is a subalgebra of the direct product $S \times T$. In particular, R is without nonzero nilpotent elements. Moreover, each of the projections π_S, π_T of R onto S and T, respectively, is a surjection. By Proposition 2.3, the variety $V(R)$ is CD. Using this conclusion one may verify that the lattice of Z_2-ideals of R is of the form



This shows that $R \notin (R)_{\mathrm{FSI}}$. We prove that $R \in Q(R)_{\mathrm{RFSI}}$ which, by Theorem 1.2, would mean that the quasivariety $Q(R)$ is not RCD. To this end it suffices to show that $R/\mathrm{Ker}\,\pi_T$ does not belong to $Q(R)$. Supposing otherwise that $R/\mathrm{Ker}\,\pi_T \in Q(R)$, by $R/\mathrm{Ker}\,\pi_T \cong T$ and the fact that the lattice of Z_2-ideals of T is of the form

we conclude that T is embeddable, say, *via* an embedding ψ, into R. As T is not isomorphic to S, $J \subseteq \mathrm{Ker}\,\pi_S \circ \psi$. In fact, $J \subsetneq \mathrm{Ker}\,\pi_S \circ \psi$ because $T/J \cong S/I$ and S/I is not embeddable into S. This together with $|T| = 16$ and $|J| = 2$ implies that the image of T under $\pi_S \circ \psi$ has at most 4 elements. But every subalgebra of R whose image under π_S contains at most 4 elements has no more than 8 elements. Therefore, as $|T| = 16$, it follows that T cannot be embedded into R. Thus R must belong to $Q(R)_{\mathrm{RFSI}}$.

The above example also shows that the quasiidentity $\forall x\ [x^2 = 0 \to x = 0]$ cannot solely characterize RCD quasivarieties of Φ-algebras. Thus the equality $\underline{K}_{\mathrm{RFSI}} = \underline{K}_{\mathrm{FSI}}$ in condition (ii) of Theorem 1.2 cannot be removed.

3. Conditionally associative algebras. An algebra is said to be *conditionally associative* if it satisfies the following two quasiidentities:

$$\forall xyz\ [(xy)z = 0 \to x(yz) = 0] \text{ and } \forall xyz\ [x(yz) = 0 \to (xy)z = 0].$$

This notion is due to Rjabuhin [17] as well as the following result:

A Φ-algebra is isomorphic to a subdirect product of Φ-algebras without nonzero divisors of zero if and only if it is conditionally associative and has no nonzero nilpotent elements.

Denote by $\underline{D}_\Phi$ the class of all Φ-algebras without nonzero divisors of zero, and by $\mathrm{Mod}_\Phi \Sigma$ the class of all Φ-algebras that satisfy each first-order sentence from a set Σ. As the class $\underline{D}_\Phi$ is closed under substructures and ultraproducts, the above quoted Rjabuhin's result can be phrased as follows.

THEOREM 3.1 (Rjabuhin [17], rephrased). $Q(\underline{D}_\Phi) = \mathrm{Mod}_\Phi\ \{\forall x\ [x^2 = 0 \to x = 0],\ \forall xyz\ [(xy)z = 0 \to x(yz) = 0],\ \forall xyz\ [x(yz) = 0 \to (xy)z = 0]\}$.

We show in section 4 that Rjabuhin's result is a particular instance of a more general result implicitly stated in [5].

In this section we want to collect with the help of Rjabuhin's result some observations concerning relative congruence distributivity and congruence distributivity as well.

We begin with the following proposition which strengthens Proposition 2.2 from section 2.

PROPOSITION 3.2. *Let $\underline{K}$ be a quasivariety of conditionally associative Φ-algebras. Then $\underline{K}$ is* RCD *iff $\underline{K} = Q(\underline{M})$ for some class $\underline{M}$ of Φ-algebras without nonzero divisors of zero.*

Proof. $\Rightarrow$: By the assumption, Theorem 1.2 and 3.1, it follows that $\underline{K} \subseteq Q(\underline{D}_\Phi)$. Hence, by RCD of $\underline{K}$, Theorem 1.2 and Lemma 2.1, $\underline{K}_{\mathrm{RFSI}} \subseteq \underline{D}_\Phi$. So, as M we can take $\underline{K}_{\mathrm{RFSI}}$.

$\Leftarrow$: Apply the arguments used in the proof of the "if" part of Proposition 2.2.

An algebra is said to be *right alternative* if it satisfies the equation $x(yy) = (xy)y$, and if in addition it satisfies $(xx)y = x(xy)$ then it is called *alternative*, or equivalently, an algebra is alternative if every its subalgebra generated by two elements is associative. By Proposition 3.1 of Rjabuhin [17] we know that any alternative algebra without nonzero nilpotent elements is conditionally associative. The same is also true for right alternative algebras satisfying $\forall x\ [2x = 0 \to x = 0]$ since due to Miheev [13] any such algebra fulfils the equation $[(xx)y - x(xy)]^4 = 0$ and, therefore, it is alternative and thus conditionally associative as well. This together with Theorem 1.2 and Proposition 3.2 allows us to state that a quasivariety $\underline{K}$ of alternative or right alternative algebras satisfying $\forall x\ [2x = 0 \to x = 0]$ is RCD if and only if $\underline{K} = Q(M)$ for some class M of algebras without nonzero divisors of zero.

We know that within associative algebras questions (1) and (2) have affirmative answers. It turns out that the answers are also affirmative within conditionally associative algebras and thus within alternative and right alternative algebras satisfying $\forall x\ [2x = 0 \to x = 0]$. This directly follows from the following corollary.

COROLLARY 3.3. *Let M be a finite set of finite conditionally associative Φ-algebras, and let each member of M be without nonzero nilpotent elements. Then $V(M)$ is* CD *and, moreover, $V(M)$ coincides with $Q(M)$.*

Proof. Define inductively the term $\langle x, n\rangle$ as follows, where x is a fixed individual variable and $n \geqslant 2$: $\langle x, 2\rangle := x \circ x$ and $\langle x, n+1\rangle := (\langle x, n\rangle) \circ x$ for $n \geqslant 2$. As M is finite and each member of M is finite too, there must exist natural numbers

$k > m \geqslant 2$ such that $\langle x, k\rangle = \langle x, m\rangle \in \mathrm{Id}(M)$. Since due to Theorem 3.1 each member of M can be decomposed into a subdirect product of algebras without nonzero divisors of zero, we can see that $\langle x, k-m+1\rangle = x \in \mathrm{Id}(M)$ and hence, by Theorem 1.5, the variety $V(M)$ is CD. In fact, $V(M)$ is arithmetical because Φ-algebras have permutable congruences. As any Φ-algebra contains a trivial subalgebra, to complete the proof that $V(M)$ coincides with $Q(M)$ it suffices to show that the variety $V(M)$ is semi-simple. But this directly follows from Theorem 3.1 because every finite algebra without nonzero divisors of zero is simple and from a result of Jónsson [9] stating that $V(M)_{\mathrm{SI}} \subseteq \mathrm{HS}(M)$.

Similarly as for Jordan rings question (1) in the case of a finite set M of finite right alternative algebras can be reduced to the same question but concerning the set $2|M$. In establishing this the following lemma will be helpful, where the term $\langle x, n\rangle$ is defined in the proof of Corollary 3.3.

Lemma 3.4. *Let R be a right alternative algebra without nonzero nilpotent elements. Then, for each a of R, if $\langle a, n\rangle = 0$ for some $n \geqslant 2$ then $a = 0$.*

Proof. By induction on n. For $n = 2$ the lemma is obvious. Let us assume that our lemma is valid for each natural number less than n, and let $\langle a, n\rangle = 0$ where $a \in R$. If $n = 2^k$ for some $k \geqslant 2$ then $\langle a, 2^{k-1}\rangle \circ \langle a, 2^{k-1}\rangle = 0$, since the equation $\langle x, 2^k\rangle = \langle x, 2^{k-1}\rangle \circ \langle a, 2^{k-1}\rangle$ is valid in right alternative algebras, and hence $\langle a, 2^{k-1}\rangle = 0$ which, by IH, implies $a = 0$. So assume $n \neq 2^k$ for all $k \geqslant 2$. Then $2^{m-1} < n < 2^m$ for some m, and this yields $\langle a, 2^m\rangle = 0$ because $\langle a, n\rangle = 0$. Thus, by the equation $\langle x, 2^m\rangle = \langle x, 2^{m-1}\rangle \circ \langle x, 2^{m-1}\rangle$ and IH, we get $a = 0$.

Corollary 3.5. *For a finite set M of finite right alternative algebras without nonzero nilpotent elements, $Q(M)$ is* RCD *iff so is $Q(2|M)$.*

Proof. Let R be a nontrivial member of M. Proceeding as in Lemma 2.5 we get that R is isomorphic to a direct product of the algebras S_p, where p is a prime number and $S_p = \{a \in R\colon p^n a = 0$ for some natural number $n\}$. These algebras satisfy $S_p \models px = 0$. Indeed, let $a \in S_p$. Then $p^n a = 0$ for some n, and hence $\langle pa, n\rangle = 0$. Thus, by Lemma 3.4, $pa = 0$ which proves $S_p \models px = 0$. Now as in the proof of Proposition 2.6 we obtain that $Q(M)$ is RCD iff so is $Q(p|M)$ for each p. The corollary then follows from Corollary 3.3 and the observation that $p|M \models \forall x\, [2x = 0 \rightarrow x = 0]$ for $p \neq 2$.

As we mentioned in section 2, congruence distributive varieties of associative Φ-algebras are exactly those generated by collections of associative division Φ-algebras of bounded finite cardinality. The same characterization is also valid for varieties of alternative and right alternative algebras satisfying $\forall x\, [2x = 0 \rightarrow x = 0]$. This results from the following proposition and the previously mentioned Miheev's result.

Proposition 3.6. *Suppose that $\underline{K}$ is a variety of alternative Φ-algebras. Then each member of $\underline{K}$ is associative whenever $\underline{K}$ is* CD.

Proof. By Propositions 3.2 and 2.4, each SI member of $\underline{K}$ has the following two properties: it is without nonzero divisors of zero and satisfies the equation $x^n = x$ for some $n \geqslant 2$. It suffices then to show that each alternative ring A with these two properties is associative. First we claim that, for each nonzero elements a, b of A, $a^{n-1} = b^{n-1}$ which, in particular, would mean that A has a unity. Denote by B the subring of A generated by a and b. Evidently, B is associative, without nonzero divisors of zero and contains a^{n-1} and b^{n-1}. In particular, B is FSI in the absolute sense. Moreover, by the equation $x^n = x$ which is valid in B, the elements a^{n-1} and b^{n-1} are idempotents in B. So, applying Exercise 6 from chapter 3 of [11] which is also valid for associative FSI rings, we get that each of a^{n-1} and b^{n-1} is the unity of B, and thus $a^{n-1} = b^{n-1}$, showing the claim. By the claim and the equation $x^n = x$ it follows that for any a, b of A with $a \neq 0$ each of the equations $ax = b$ and $ya = b$ has a solution which must be unique since A has no nonzero divisors of zero. Thus A is a division ring. Hence referring to the description of alternative division rings given by Bruck, Kleinfeld and Skornjakov (see [4] and [18], also Corollary 2 of [20], chapter 7) we obtain that A is associative, or A is a 8-dimensional algebra over its center. In the latter case, by Theorem 3 from chapter 1 of [20], it follows that the center of A has at most n elements, since otherwise the equation $x^n - x = 0$ would yield the equation $x = 0$ valid in A, which in turn implies that A is finite. As in an alternative division ring any two elements generate an associative division subring, the finiteness of A and the Wedderburn theorem imply that A is commutative. Thus, byŽevlakov's result (see [19], or [20, Theorem 3 of chapter 7]) which says that any simple alternative and commutative algebra is a field, it follows that A is a field. In particular, A is associative which completes the proof.

4. A proof of Rjabuhin's result. A quasivariety $\underline{K}$ of arbitrary abstract algebras is said to have *equationally definable principal meets* (EDPM for short) (see [5]) if there exists a finite system $\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle_{i<k}$ of pairs of 4-ary terms such that $\underline{K}_{\mathrm{RFSI}} \models \forall xyzw\, [\mathop{\&}_{i<k} p_i(x, y, z, w) = q_i(x, y, z, w) \leftrightarrow (x = y \text{ or } z = w)]$. This notion for varieties was proposed in Blok and Pigozzi [3]. With the restriction $|\Delta| = 1$, it was previously considered in Baker [1] where it is noticed that the notion is reminiscent of the property of an integral domain. Indeed, in our setting it is easily seen by the help of Lemma 2.1 that the system $\Pi = \langle (x-y)(z-w), 0\rangle$ realizes EDPM for any quasivariety generated by Φ-algebras without nonzero divisors of zero.

Let $\underline{K}$ be a quasivariety of abstract algebras, and let

$$\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle_{i<k}$$

realizes EDPM for $\underline{K}$. Let Σ be a set of quasiequations such that (*) $\underline{K}_{\mathrm{RFSI}} = \mathrm{Mod}_{\underline{L}}\Sigma \cup \{\forall xyzw\, [\mathop{\&}_{i<k} p_i(x, y, z, w = q_i(x, y, z, w) \rightarrow (x = y \text{ or } z = w)]\}$, where $\underline{L}$ is a fixed variety that contains $\underline{K}$. Moreover, let for a quasiequation $Q := r_0$

$= s_0 \& \ldots \& r_{m-1} = s_{m-1} \to r = s$ the set $\Delta(Q)$ consist of the following quasi-equations:

$$\mathop{\&}_{i<k} \mathop{\&}_{j<m} p_i(r_j, s_j, z, w) = q_i(r_j, s_j, z, w) \to p_n(r, s, z, w) = q_n(r, s, z, w)$$

where $n = 0, \ldots, k-1$ and the variables z, w are assumed to be distinct from the variables occurring in Q. Then for a set $\Gamma(\Delta)$ of quasiequations defined in [5] we have:

PROPOSITION 4.1. $\underline{K} = \mathrm{Mod}_{\underline{L}} \Sigma \cup \Gamma(\Delta) \cup \bigcup (\Delta(Q)\colon\ Q \in \Sigma \cup \Gamma(\Delta))$.

Proof. $\subseteq$: By Lemma 3.2 of [5]. $\supseteq$: By Lemma 3.3 of [5],

$$\mathrm{Mod}_{\underline{L}} \Sigma \cup \Gamma(\Delta) \cup \bigcup (\Delta(Q)\colon\ Q \in \Sigma \cup \Gamma(\Delta))_{\mathrm{RFSI}} \models \forall xyzw\ [\mathop{\&}_{i<k} p_i(x, y, z, w) = q_i(x, y, z, w) \leftrightarrow (x = y \text{ or } z = w)],$$

and hence, by (*), the inclusion follows.

The set $\Gamma(\Pi)$ for the system $\Pi = \langle (x-y)(z-w), 0\rangle$, which as we mentioned realizes EDPM for any quasivariety generated by Φ-algebras without nonzero divisors of zero, consists of the following quasiequations:

(1) $[(x-y)(z-w)](v-u) = 0 \to (x-y)[(z-w)(v-u)] = 0$,

(2) $(x-y)[(z-w)(v-u)] = 0 \to [(x-y)(z-w)](v-u) = 0$,

(3) $x = y \to (x-y)(z-w) = 0$,

(4) $(x-y)(z-w) = 0 \to (y-x)(z-w) = 0$,

(5) $(x_0-x_1)(z-w) = 0 \& (x_1-x_2)(z-w) = 0 \to (x_0-x_2)(z-w) = 0$,

$(6)_-$ $(x-y)(z-w) = 0 \to [(-x)-(-y)](z-w) = 0$,

$(7)_+$ $(x_0-y_0)(z-w) = 0 \& (x_1-y_1)(z-w) = 0 \to [(x_0+x_1)-(y_0+y_1)](z-w) = 0$,

$(8)_\circ$ $(x_0-y_0)(z-w) = 0 \& (x_1-y_1)(z-w) = 0 \to [(x_0x_1)-(y_0y_1)](z-w) = 0$,

$(9)_r$ $(x-y)(z-w) = 0 \to (rx-ry)(z-w) = 0$,

where r is an element of Φ,

(10) $(x-y)(z-w) = 0 \to (z-w)(x-y) = 0$,

(11) $(x-y)(x-y) = 0 \to x = y$.

As $Q(\boldsymbol{D}_\Phi)_{\mathrm{RFSI}} = \mathrm{Mod}_\Phi \forall xyzw\ [(x-y)(z-w) = 0 \to (x = y \text{ or } z = w)]$, by Proposition 4.1 we have $Q(\underline{D}) = \mathrm{Mod}_\Phi \Gamma(\Pi) \cup \bigcup (\Pi(Q)\colon\ Q \in \Gamma(\Pi))$. So in order to derive Rjabuhin's result from Proposition 4.1 it remains then to prove the following lemma.

LEMMA 4.2. $\mathrm{Mod}_\Phi \Gamma(\Pi) \cup \bigcup (\Pi(Q)\colon\ Q \in \Gamma(\Pi)) = \mathrm{Mod}_\Phi \{\forall x [x^2 = 0 \to x = 0,\ \forall xyz\ [(xy)z = 0 \leftrightarrow x(yz) = 0]\}$.

Proof. Notice that the quasiequation (11) is equivalent to

(I) $$\forall x\ [x^2 = 0 \to x = 0]$$

and that the conjunction of (1) and (2) is equivalent to

(II) $$\forall xyz\ [(xy)z = 0 \leftrightarrow x(yz) = 0].$$

This proves the inclusion $\subseteq$. To prove the inverse inclusion notice that the quasi-equations (3), (4), (5), $(6)_-$, $(7)_+$, $(9)_r$, where $r \in \Phi$, are valid in any Φ-algebra and, in particular, in a Φ-algebra A taken from the class $\mathrm{Mod}_\Phi\{(\mathrm{I}), (\mathrm{II})\}$. Moreover, notice that the remaining quasiequations of $\Gamma(\Pi) \cup \bigcup (\Pi(Q)\colon\ Q \in \Gamma(\Pi))$ are also valid in A whenever in addition A satisfies the following sentences:

(III) $$\forall xy\ [xy = 0 \to yx = 0],$$

(IV) $$\forall xyz\ [(xy)z = 0 \to (yx)z = 0],$$

(V) $$\forall xyzw\ [[(xy)z]w = 0 \leftrightarrow [x(yz)]w = 0].$$

That A indeed satisfies (III), (IV) and (V) can be verified as follows (comp. Rjabuhin [17, Lemma 1.1]).

(III):

1. $xy = 0$ (assumption),
2. $(xy)x = 0$,
3. $x(yx) = 0$ (by (II)),
4. $y[x(yx)] = 0$
5. $yx = 0$ (by (II), (I)).

To verify (IV) we show

(VI) $$\forall xyz\ [xy = 0 \to x(zy) = 0],$$

(VI):

1. $xy = 0$ (assumption),
2. $yx = 0$ (by (III)),
3. $z(yx) = 0$,
4. $x(zy) = 0$ (by (II), (III),

(IV):

1. $(xy)z = 0$ (assumption),
2. $x[z(yz)] = 0$ (by (II), (VI)),
3. $[(xz)y]z = 0$ (by (II)),
4. $(xz)[y(xz)] = 0$ (by (VI), (II)),
5. $y(xz) = 0$ (by (II), (I)),
6. $(yx)z = 0$ (by (II)).

To verify (V) we show

(VII) $$\forall xyzw\ [(xy)(zw) = 0 \to (xz)(yw) = 0],$$

(VII):

1. $(xy)(zw) = 0$ (assumption),
2. $[(xy)z]w = 0$ (by (II)),

3. $[(xy)z][(xz)(yw)] = 0$ (by (VI)),
4. $x[y(z[(xz)(yw)])] = 0$ (by (II)),
5. $[y(z[(xz)(yw)])][y(z[(xz)(yw)])] = 0$ (by (VI), (II)),
6. $y(z[(xz)(yw)]) = 0$ (by (I)),
7. $z[(xz)(yw)] = 0$ (by (VI), (II), (I)),
8. $(xz)(yw) = 0$ (by (VI), (II), (I)),

(V):

1. $[(xy)z]w = 0$ (assumption),
2. $[z(xy)]w = 0$ (by (IV)),
3. $(wz)(xy) = 0$ (by (III), (II)),
4. $(wx)(zy) = 0$ (by (VII)),
5. $(yz)(wx) = 0$ (by (III), (IV)),
6. $w[x(yz)] = 0$ (by (III), (II)),
7. $[x(yz)]w = 0$ (by (III)).

This proves one direction of (V). The proof of the inverse direction of (V) runs from (7) to (1).

We also notice the following

COROLLARY 4.3 (comp. [5, Theorem 3.4]). *For a class M of Φ-algebras without nonzero divisors of zero, the quasivariety $Q(M)$ is finitely axiomatizable relative to the variety of Φ-algebras iff so is the universal class* $\mathrm{ISP_U}(M)$.

Proof. $\Rightarrow$: As each algebra of $Q(M)$ without nonzero divisors of zero is a member of $Q(M)_{\mathrm{RFSI}}$, by Lemma 2.1 it follows that

$$\mathrm{ISP_U}(M) = \mathrm{Mod}_\Phi \Sigma \cup \{\forall xy\,[xy = 0 \to (x = 0 \text{ or } y = 0)]\},$$

where Σ is a finite set of first-order sentences such that $Q(M) = \mathrm{Mod}_\Phi \Sigma$.

$\Leftarrow$: By compactness theorem we may assume that

$$\mathrm{ISP_U}(M) = \mathrm{Mod}_\Phi \Sigma \cup \{\forall xy\,[xy = 0 \to (x = 0 \text{ or } y = 0)]\}$$

for some finite set Σ of universally quantified first-order sentences whose matrices are of the form $\&(r_i = 0\colon i < m) \to OR(s_i = 0\colon i < n)$. For each such sentence U denote by $[U]$ the quasiidentity

$$\forall\,[\&(r_i = 0\colon i < m) \to s_0 \circ (s_1 \circ \ldots (s_{n-2} \circ s_{n-1}) \ldots) = 0],$$

where $\forall$ is the prefix of universal quantifiers. It is clear that

$$\mathrm{ISP_U}(M) = \mathrm{Mod}_\Phi\{[U]\colon U \in \Sigma\} \cup \{\forall xy\,[xy = 0 \to (x = 0 \text{ or } y = 0)]\}.$$

So, as $Q(M)_{\mathrm{RFSI}} = \mathrm{ISP_U}(M)$, by Proposition 4.1 and Lemma 2.1 it follows that

$$Q(M) = \mathrm{Mod}_\Phi\{[U]\colon U \in \Sigma\} \cup \{\Pi([U])\colon U \in \Sigma\} \cup \cup \{\forall x\,[x^2 = 0 \to x = 0],\ \forall xyz\,[(xy)z = 0 \leftrightarrow x(yz) = 0]\}.$$

Thus $Q(M)$ is finitely axiomatizable relative to the variety of Φ-algebras.

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