Non-multidimensional theories without groups

by

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Abstract. We prove that a non-multidimensional theory which does not interpret an infinite group is superstable. This result is closely related to Hrushovski's result which states that a unidimensional theory is superstable.

0. Introduction. In [2], Hrushovski has proven that every unidimensional theory is superstable. In his treatment of unidimensional theories, he has two cases: In the first, there is an infinite definable group, which helps him to obtain a superstable formula; in the second, where there are no such groups, the structure of the models of the theory is very simple, since everything lies in the algebraic closure of the minimal type over some parameters. Our result is a generalization of this second case. We prove that a non-multidimensional theory without an infinite definable group is superstable. (In fact we can find a basis of regular superstable types whose algebraic closure is the whole model.)

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1. Preliminaries. We fix a stable theory T and work in a big model $\mathfrak{C}$ of T. Our notations are fairly standard. Types are complete types with parameters, and they are denoted by $p, q, \ldots$ The non-forking extension of a stationary type p to the domain A is denoted by $p|A$. The type of a over A is denoted by $\mathrm{tp}(a/A)$. If p and q are stationary types over A, then $p \otimes q$ denotes the type $\mathrm{tp}(a\hat{\ }b/A)$, where a realizes p and b realizes $q|A \cup \{a\}$. $U(p)$ is the U-rank (Lascar rank) of a type p. We simply write $U(a/A)$ instead of $U(\mathrm{tp}(a/A))$. A theory T is superstable if and only if every type has an ordinal U-rank. The following inequality of U-ranks is quite important: $U(a\hat{\ }b/A) \leqslant U(b/A) \oplus U(a/A \cup \{b\})$, where $\alpha \oplus \beta$ is the natural sum of ordinals α and β. The set of realizations of a type p in $\mathfrak{C}$ is denoted by $p^{\mathfrak{C}}$.

DEFINITION 1.1. We say a theory T is *non-multidimensional*, if there is a bound to the size of families of pairwise orthogonal types.

Our definition of non-multidimensionality slightly differs from Shelah's original one. (See [4, Ch. V, § 2].)

2. Main Theorem.

THEOREM 2.1. *A non-multidimensional theory T which does not interpret an infinite group is superstable.*

Proof. First by induction we construct a sequence $\{p_i | i<\alpha\}$ of non-algebraic types with the following conditions:

(a) $U(p_i)<\infty$;

(b) Any two types p_i and p_j are orthogonal;

(c) If $\mathrm{tp}(a/B)$ is a forking extension of p_i, and I_j is a basis of $p_j|B^{\mathfrak{C}}$ for $j<i$, then a is in the algebraic closure of $B \cup \bigcup \{I_j | j<i\}$.

Let p_0 be an arbitrary type with $U(p_0)=1$. (By stability we can easily find such a type.) Suppose we have already constructed the types $\{p_i | i<\beta\}$. If no types are orthogonal to the types $\{p_i | i<\beta\}$, we stop the construction. So let r be a type which is orthogonal to the types $\{p_i | i<\beta\}$. We may assume

$$\mathrm{dom}(r) \supset \bigcup \{\mathrm{dom}(p_i) | i<\beta\}.$$

We can choose a maximally long sequence $\{r_i | i<\gamma\}$ of non-algebraic types such that

(i) $r_0=r$;

(ii) r_j is a forking extension of r_i $(i<j<\gamma)$; and

(iii) r_i is orthogonal to the types $\{p_j | j<\beta\}$ $(i<\gamma)$.

We show that γ is a successor ordinal. In other words, we show:

CLAIM. *$s=\bigcup\{r_i | i<\gamma\}$ is orthogonal to the types $\{p_i | i<\beta\}$.*

If this is not the case, then there is a type p_i $(i<\beta)$ which is not orthogonal to the type s. So there are elements a realizing s and $b \in \mathrm{acl}(a, B)-\mathrm{acl}(B)$ such that $\mathrm{tp}(b/B)$ is $p_i|B$-internal, where B is the domain of s. (See Proposition 2.23 of [3].) Consider the linking group G of $\mathrm{tp}(b/B)$ over $p_i|B$. (See [3, Ch2, 2e] for the definition of the linking group.) By our assumption, G must be finite. So b is algebraic over $B \cup P_i$, where $P_i=p_i|B^{\mathfrak{C}}$. Now there is a $k<\gamma$ such that

(d) a and P_i are dependent over $\mathrm{dom}(r_k)$.

For each $j \leqslant i$, let I_j be a basis of $p_j|\mathrm{dom}(r_k)^{\mathfrak{C}}$. Using the induction hypothesis (c) for $i<\beta$, we have

$$P_i \subset \mathrm{acl}(\mathrm{dom}(r_k)^{\mathfrak{C}} \cup \{I_j | j \leqslant i\}).$$

From this and (d), we see that a and $\bigcup\{I_j | j\leqslant i\}$ are dependent over $\mathrm{dom}(r_k)$. Trivially a realizes the type r_k. So some p_l $(l \leqslant i)$ must be non-orthogonal to r_k. This contradicts our choice of r_k. (End of Proof of Claim.)

Thus we see that s is orthogonal to the types $\{p_i | i<\beta\}$, and every non-algebraic forking extension of s is non-orthogonal to some p_i $(i<\beta)$. Let $p_\beta = s$. We have to show that the conditions (a) and (c) are satisfied by p_β. First we show (c): If (c) is false for p_β, we have a non-algebraic forking extension $\mathrm{tp}(a/C)$ of p_β with $a \notin \mathrm{acl}(C \cup \bigcup\{I_j | j<\beta\})$, where I_j $(j<\beta)$ are bases of the sets $(p_j|C)^{\mathfrak{C}}$. Choose

small subsets J_j $(j<\beta)$ of I_j such that $\mathrm{tp}(a/C \cup \bigcup\{J_j | j\leqslant\beta\})$ does not fork over $C \cup \bigcup\{J_j | j \leqslant \beta\}$. By our choice of $s=p_\beta$, $\mathrm{tp}(a/C\cup\bigcup\{J_j | j\leqslant\beta\})$ is non-orthogonal to some p_i $(i<\beta)$. Since T does not interpret an infinite group, by the same argument as in the above claim, we have:

$$a \text{ and } R_i \text{ are dependent over } C \cup \bigcup\{J_j | j<\beta\},$$

where R_i is the set of realizations of $p_i|(C\cup\bigcup\{J_j | j\leqslant\beta\})$. Note that I_k-J_k is a basis of $p_k|(C\cup\bigcup\{J_j | j\leqslant\beta\})$. So by the induction hypothesis (c) for i,

$$a \text{ and } \bigcup\{I_j-J_j | j\leqslant i\} \text{ are dependent over } C\cup\bigcup\{J_j | j<\beta\}.$$

This contradicts our choice of J_j's. Thus we get (c). For (a), let $\mathrm{tp}(a/C)$ be an arbitrary forking extension of s. By (c), $U(a/C)<\infty$. In fact, we have

$$U(a/C) \leqslant \sup\{U(p_{i_1}) \oplus \ldots \oplus U(p_{i_m}) | m<\omega;\ i_j<\alpha\} < \infty.$$

By the definition of U-ranks, we have $U(p_\beta)<\infty$.

Since T is non-multidimensional, the construction of p_i's must stop at some stage α. Then every type t is non-orthogonal to some p_i $(i<\alpha)$. Thus by the same argument for (a) and (c) above, we have $U(t)<\infty$.

Remarks 2.2. (1) In the above theorem, the U-rank of the βth type p_β $(\beta<\alpha)$ is clearly bounded by ω^β. So the rank of the theory is bounded by ω^α.

(2) By (c) the types p_β are regular types. In fact we have proven that a non-multidimensional theory without an infinite group has a basis of regular superstable types whose algebraic closure is the whole model.

EXAMPLES 2.3. (1) Let M be the set $\omega^\omega = \{\mu | \mu: \omega\to\omega\}$ and D be an infinite set. For each $n<\omega$, define the relation E_n on M by:

$$E_n(\mu, \nu) \quad \text{iff} \quad \mu(i)=\nu(i) \text{ for every } i \geqslant n.$$

For each $n<\omega$ and each $X \in M/E_n$, $f_n(X, *)$ is a bijection from the set X/E_{n-1} to D. Now the theory $T=\mathrm{Th}(M\cup D, E_n, f_n)_{n<\omega}$ is bi-dimensional $(\alpha=2)$: The first type p_0 generated by $D(x)$ determines the number of E_{n-1}-classes existing in one E_n-class. p_0 has the rank 1. The second type p_1 generated by $M(x)$ determines the number of elements modulo the equivalence relation $\bigcup\{E_n | n<\omega\}$. This second type has the rank ω.

(2) Let M be the additive group $\mathbf{C}^\omega$. For each $n<\omega$, let E_n be the equivalence relation on M defined by:

$$E_n(x, y) \quad \text{iff} \quad x_n = y_n,$$

where $x=(x_i)_{i<\omega}$ and $y=(y_i)_{i<\omega}$. We shall consider the theory $T=\mathrm{Th}(M, E_n)_{n<\omega}$. For each $\eta \in 2^\omega$, let $p_\eta(x)$ be the non-algebraic type generated by:

$$\{E_n(x, 0) | \eta(n)=0\} \cup \{\neg E_n(x, 0) | \eta(n)=1\}$$

Let $q(x)$ be an arbitrary type. $q(x)$ essentially has the form:

$$\{E_n(x, a) | n \in A\} \cup \{\neg E_n(x, a) | n \in \omega - A\}.$$

If x and y are independent realizations of q, then $x-y$ realizes the type $p_{\eta_A}|a$, where η_A is the defining function of A. Thus q and p_{η_A} are non-orthogonal. Since q is an arbitrary type, T is non-multidimensional. It is clear that T is unsuperstable.

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A subclass of the class MOBI *

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Abstract. Necessary and sufficient conditions are given for a regular space to be an open and compact image of a σ-discrete metacompact Moore space. The class of regular spaces satisfying these conditions is invariant under open mappings with compact metric fibers. This gives a characterization of the minimal class of regular spaces containing all σ-discrete metric spaces and invariant under open and compact mappings.

For a class $\mathscr{K}$ of topological spaces, let $\mathrm{MOBI}_i(\mathscr{K})$ be the minimal class of T_i-spaces containing all metric spaces from $\mathscr{K}$ and invariant under open and compact mappings (see [BCh1]).

It is easy to observe that a T_i-space is in $\mathrm{MOBI}_i(\mathscr{K})$ if and only if it can be obtained as an image of a metric space from $\mathscr{K}$ under a mapping which is a composition of a finite number of open and compact mappings with T_i-domains [B].

If the class $\mathscr{K}$ contains the class of all metric spaces, we write MOBI_i instead of $\mathrm{MOBI}_i(\mathscr{K})$.

The purpose of this note is to prove a characterization of the class MOBI_3 (σ-discrete). This gives a partial solution to the problem of characterizing MOBI_3 (see [A] and [Ch2]) and generalizes the characterization of MOBI_3(scattered) from [BCh2].

There seems to be a pattern that the solutions of problems concerning MOBI_3 are similar to the solutions of the corresponding problems in MOBI_2. The main difference is that the techniques needed in the regular case are more complicated than those required in MOBI_2.

In the present paper we follow this pattern. We prove that a regular space is in MOBI_3(σ-discrete) if and only if it is in MOBI_2(σ-discrete) (see [Ch3]) and has a base of countable order.

It turns out that in characterizing MOBI_i(σ-discrete), the regular case ($i = 3$) is much more complicated than the Hausdorff case ($i = 2$). In fact, the techniques used in this paper have been distilled from [BCh2] rather than from [Ch3].

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