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An approximate analog of a theorem of Khintchine

by

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Abstract The following theorem is established: If f is a real valued measurable function on the reals, then f has a finite approximate derivative almost everywhere on the set where the upper approximate symmetric derivate is less than infinity. This theorem is the approximate analog of a theorem of A. Khintchine.

In 1927, Khintchine [2] proved the following:

THEOREM. *If f is a real valued measurable function on the reals, then f has a finite ordinary derivative almost everywhere on the set where the upper symmetric derivate is less than infinity.*

In this paper we prove:

THEOREM 1. *If f is a real valued measurable function on the reals, then f has a finite approximate derivative almost everywhere on the set where the upper approximate symmetric derivate is less than infinity.*

An earlier proof of Theorem 1 (Russo and Valenti [6]) makes the oversight of assuming that not density zero implies positive density. This theorem will almost immediately give a Denjoy–Young–Saks Theorem for the approximate symmetric derivative.

In [1], Belna, Evans, and Humke constructed an additive subgroup, G, of the reals so that both the set G and its complement contain an element from every perfect set, and thus both have inner measure zero. The characteristic function of the set G, therefore, has symmetric derivative zero at every point in G, while the ordinary derivative does not exist at any point in G, showing that the assumption of measurability cannot be dropped in Khintchine's Theorem. Their example, however, leaves open the possibility that a non-measurable version of Khintchine's Theorem might be true if "almost everywhere" were replaced by "except on a set of inner measure zero." This improvement has been established by the following result of Uher [9]:

THEOREM. *If a function f has a finite upper symmetric derivate on a measurable set E, then f is almost everywhere differentiable on E.*

The example of Belna, Evans, and Humke similarly shows that the measurability condition of f in Theorem 1 of this work cannot be dropped, since the characteristic function of their set G also does not have an approximate derivative at any point in G. However, an improvement such as Uher's cannot be proven in ZFC for the approximate case since, with the aid of the continuum hypothesis, Sierpiński [8] constructed a set whose characteristic function has an approximate symmetric derivative zero everywhere and which is nowhere approximately differentiable.

We will use the following notation. We use λ and λ_2 for Lebesgue measure on the line and in the plane respectively. The approximate derivative (upper right, lower right) of f is denoted by $f'_{\mathrm{ap}}(f^+_{\mathrm{ap}}, f_{+\mathrm{ap}})$. The approximate symmetric derivative of f is $f^{(1)}_{\mathrm{ap}}(x) = \lim \mathrm{app}_{h\to 0} \left(f(x+h)-f(x-h)\right)/2h$. The upper and lower approximate symmetric derivates $(\bar{f}^{(1)}_{\mathrm{ap}}, \underline{f}^{(1)}_{\mathrm{ap}})$ are defined similarly.

Proof of Theorem 1. Let E be the set on which $\bar{f}^{(1)}_{\mathrm{ap}} < \infty$ and f'_{ap} does not exist. Since $\bar{f}^{(1)}_{\mathrm{ap}}$ is measurable (Larson [3]), as are the extreme approximate derivates (Saks [7]), E is a measurable set. Suppose that E has positive measure. We may assume that, for some n, $\bar{f}^{(1)}_{\mathrm{ap}} < n$ on E since E is the countable union of such sets, and also assume E is contained in some finite interval. By subtracting the function nx from f, we may further assume that $\bar{f}^{(1)}_{\mathrm{ap}} < 0$ on E.

We first construct a function g and a set $G \subset E$ so that

(1) $\lambda(G) \geqslant \lambda(E)/2$ and

(2) for each x in G and $\varepsilon > 0$,

$$\lambda(\{h \in (0, \delta) |\ f(x+h) < f(x-h)\}) \geqslant (1-\varepsilon)\delta \text{ for all } \delta \leqslant g(\varepsilon).$$

For the construction of G and g we first need to demonstrate, for fixed positive integers n and k, the measurability of

$$A^{nk} = \{x |\ \lambda(\{h \in (0, \delta) |\ f(x+h) < f(x-h)\}) \geqslant (1-1/n)\delta \text{ for all } \delta \text{ in } (0, 1/k]\}.$$

Let $B = \{(x, h, \delta) |\ 0 < h < \delta \text{ and } f(x+h) < f(x-h)\}$, a measurable subset of R^3. Then $b(x, \delta) = \lambda(\{h |\ (x, h, \delta) \in B\})/\delta$ is a measurable function by Fubini's Theorem. Therefore, $C = \{(x, \delta) |\ b(x, \delta) \geqslant 1-1/n\}$ is a measurable subset of R^2. For each rational q in $(0, 1]$, let $D_q = \{(x, y) |\ (x, qy) \in C\}$. Then $D = \bigcap D_q$ is measurable. Since $b(x, \delta)$ is continuous in δ, $D = \{(x, y) |\ y \geqslant 0 \text{ and } (x, \delta) \in C \text{ for all } 0 < \delta \leqslant y\}$. Observe that $\{y |\ (x, y) \in D\}$ is the same as the interval $[0, d(x)]$. Therefore, by Fubini's Theorem, $d(x) = \lambda(\{y |\ (x, y) \in D\})$ is a measurable function, and since $A^{nk} = d^{-1}([1/k, \infty))$, A^{nk} is measurable. Observe, for each n, $E = \bigcup_k (E \cap A^{nk})$.

To construct G, let $A_1 = E$ and $k_1 = 1$. We define A_n and k_n inductively by picking $k_n > k_{n-1}$ so that $A_n = A_{n-1} \cap A^{nk_n}$ has measure greater than $\lambda(E)/2$. Then $\{A_n\}$ is a nested decreasing sequence of sets and $G = \bigcap_{n=1}^{\infty} A_n$ has measure greater than or equal to $\lambda(E)/2$. Define the function g to be $1/k_n$ on each interval $[1/n, 1/(n-1))$. Then G and g clearly satisfy conditions (1) and (2) above.

By the Denjoy–Young–Saks Theorem for approximate derivatives (Jeffery [5]), $f^+_{\mathrm{ap}} = \infty$ and $f_{+\mathrm{ap}} = -\infty$ almost everywhere on G, since $G \subset E$. By ignoring a set of measure 0, we may assume that these derivates are infinite at every point in G and that every point in G is a density point of G.

For x in G and $r > x$, let $T(x, r)$ be the right triangle, together with its interior, with vertices $(x, 0)$, $(r, 0)$, and $((x+r)/2, (r-x)/2)$. If P is any point in $T(x, r)$ off the x-axis, then P is a vertex of some $T(z, y) \subset T(x, r)$. Let $D(x, r)$ consist of all such points P for which $f(z) > f(y)$ and let $U(x, r)$ be $T(x, r) \setminus D(x, r)$. Observe that, if $f^{(1)}_{\mathrm{ap}}(c) < 0$ for c between x and r then $D(x, r)$ has linear density 1 at the point $(c, 0)$ along the vertical line through $(c, 0)$.

The function g provides the uniformity needed to establish the following limit we will need later in the proof. For $x \in G$

$$(*) \qquad \lim_{r\to x^+} \lambda_2(U(x, r))/\lambda_2(T(x, r)) = 0.$$

To see this, let $\varepsilon > 0$. Since x is a density point of G, we can pick $\delta \leqslant g(\varepsilon)$ so that $\lambda(G \cap (x, r))/(r-x) > 1-\varepsilon$ for $0 < r-x < \delta$. By property (2), if $y \in G \cap (x, r)$ and $r-x < \delta$, the linear measure of $D(x, r)$ on the vertical line segment through the point $(y, 0)$ inside of $T(x, r)$ is at least $1-\varepsilon$ times the height of $T(x, r)$ above $(y, 0)$, since this height is less than $g(\varepsilon)$. Thus, the total measure of $D(x, r)$ is greater than or equal to $(1-\varepsilon)^3\lambda_2(T(x, r))$, so $\lim_{r\to x^+} \lambda_2(D(x, r))/\lambda_2(T(x, r)) = 1$, establishing $(*)$.

Fix $x \in G$, let l be the line through the point $(x, 0)$ with slope 1. The line l contains a side of $T(x, r)$ for all $r > x$. For any point P, let $l_{-1}(P)$ be the line through P with slope -1. Observe that if the upper vertices of $T(x, y)$ and $T(x, z)$ are in $D(x, z)$ and $U(x, z)$ respectively, then the upper vertex of $T(y, z)$ is in $U(x, z) \cap l_{-1}(Z)$, where Z is the point $(z, 0)$. This says that, if $x < b < c$, then for each point P in $(U(x, c) \setminus T(x, b)) \cap l$, $\lambda(U(x, c) \cap l_{-1}(P)) \geqslant \lambda(D(x, b) \cap l)$. Thus, if $x < b < c$

$$(**) \qquad \lambda_2(U(x, c)) \geqslant \lambda(D(x, b) \cap l)\lambda((U(x, c) \setminus T(x, b)) \cap l).$$

We will show that $f^+_{\mathrm{ap}} = \infty$ and $f_{+\mathrm{ap}} = -\infty$ on G lead to a contradiction of $(*)$. Let α and β be $1/\sqrt{2}$ times the lower and upper densities of the set $U(x, r) \cap l$ at the point $(x, 0)$ (The factor of $1/\sqrt{2}$ is used due to the fact that $r-x$ is along the x-axis while $U(x, r) \cap l$ is along l). Thus $\limsup_{r\to x^+} \left(\lambda(U(x, r) \cap l)\right)/(r-x) = \beta$ and $\liminf_{r\to x^+} \left(\lambda(U(x, r) \cap l)\right)/(r-x) = \alpha$. If the right approximate derivates of f are ∞ and $-\infty$, then $U(x, r) \cap l$ and $D(x, r) \cap l$ have positive upper linear density at $(x, 0)$, and hence $0 < \beta$ and $\alpha < 1/\sqrt{2}$. We distinguish two cases.

Case 1. If $\alpha < \beta$, let $m = (\alpha+\beta)/2$. We pick two sequences $\{a_n\}$ and $\{b_n\}$ with $b_{n+1} < a_{n+1} < b_n < a_n$, where n ranges over the set of positive integers for which $\beta - 1/n > m$. We pick a_n so that

$$\lambda(U(x, a_n) \cap l) = (\beta - 1/n)(a_n - x) \quad \text{and}$$

$$b_n = \sup\{b \in (x, a_n) |\ \lambda(U(x, b) \cap l) = m(b-x)\}.$$

Then, for $\beta-1/n>m$, $\lambda(U(x,a_n)\cap l)=(\beta-1/n)(a_n-x)=\lambda(U(x,b_n)\cap l)+$ $+k_n(a_n-b_n)$ where $m<k_n\leqslant 1/\sqrt{2}$. This gives $\beta-1/n=m(b_n-x)/(a_n-x)+$ $+k_n(a_n-b_n)/(a_n-x)$. We can assume that $\lim(a_n-b_n)/(a_n-x)$ exists by passing to a subsequence if necessary. This limit cannot be zero, since then $(b_n-x)/(a_n-x)$ would approach 1, and the equation above would yield $\beta=m$. Thus for some $\delta>0$ and all n, $a_n-b_n\geqslant\delta(a_n-x)\geqslant\delta(b_n-x)$. Since a_n-b_n may be much larger than $\delta(b_n-x)$, we let $c_n=b_n+\delta(b_n-x)$ so that $(b_n-x)/(c_n-x)=1/(1+\delta)$. We also have $\lambda(D(x,b_n)\cap l)=(1/\sqrt{2}-m)(b_n-x)$ and $\lambda((U(x,c_n)\setminus T(x,b_n))\cap l)>m(c_n-b_n)$ $=m\delta(b_n-x)$. Thus, by (**), the measure of $U(x,c_n)$ is at least

$$(1/\sqrt{2}-m)(b_n-x)m\delta(b_n-x).$$

Since the measure of $T(x,c_n)$ is $(c_n-x)^2/4$, the relative measure of $U(x,c_n)$ in $T(x,c_n)$ is at least $(1/\sqrt{2}-m)(b_n-x)^2 4m\delta/(c_n-x)^2=4(1/\sqrt{2}-m)m\delta/(1+\delta)^2$, contradicting limit (*).

Case 2. If $\alpha=\beta$, define $b_n=x+1/2^n$, and $c_n=b_{n-1}$, so that $(c_n-b_n)=(b_n-x)$. Since the density of $U(x,r)\cap l$ at $(x,0)$ exists, the relative measures

$$\lambda(D(x,b_n)\cap l)/(b_n-x) \quad\text{and}\quad \lambda((U(x,c_n)\setminus T(x,b_n))\cap l)/(c_n-b_n)$$

approach $1/\sqrt{2}-\alpha$ and α respectively. By (**), the liminf of the sequence $\lambda_2(U(x,c_n))/\lambda_2(T(x,c_n))$ is at least $\alpha(1/\sqrt{2}-\alpha)>0$, contradicting (*). This finishes the proof of our theorem.

Theorem 1 immediately gives the following approximate symmetric analog of the Denjoy–Young–Saks Theorem.

THEOREM 2. *For a measurable function f, almost everywhere*

(1) $f_{\mathrm{ap}}^{(1)}$ *exists (finite) or*

(2) $\overline{f}_{\mathrm{ap}}^{(1)}=\infty$ *and* $\underline{f}_{\mathrm{ap}}^{(1)}=-\infty$.

Proof. By Theorem 1, f'_{ap} exists (finite) almost everywhere on the complement of the set for which (2) holds, and $f_{\mathrm{ap}}^{(1)}=f'_{\mathrm{ap}}$ when the approximate derivative exists.

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