The dimension of products of complete separable metric spaces

by

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Abstract. For each $n \in \omega$, a complete, separable, totally disconnected metric space X_n is described satisfying $\dim(X_n) = n$ and $\dim(X_n^\omega) = n$. The space X_n can be chosen to be a subspace of R^{n+1} homeomorphic to its own square.

0. Introduction. It is well known that if A and B are separable, metrizable spaces with positive dimension, then

$$\dim(A \times B) > \max\{\dim(A), \dim(B)\},$$

provided either A or B is compact. In 1967, in [AK], Anderson and Keisler proved:

THEOREM. *For each $n \in \omega$ there is a separable, metrizable subspace Y_n of R^{n+1} satisfying* $\dim(Y_n) = \dim(Y_n^\omega) = n$.

It follows from above that an example as in this theorem cannot be compact for $n > 0$. In this paper, we improve on the Anderson and Keisler result by showing:

THEOREM 1. *For each $n \in \omega$ there is a completely metrizable, totally disconnected subspace X_n of R^{n+1} satisfying* $\dim(X_n) = \dim(X_n^\omega) = n$.

In fact X_n^ω can be shown to embed in R^{n+1}; hence the example can additionally be made homeomorphic to its square.

Examples for $n = 0$ and $n = 1$ are well known; for $n = 0$ the Cantor set suffices while for $n = 1$ the set of irrational points in l_2 will do.

The examples provided will actually be graphs of functions from a Cantor set into an n-cube. Dimension theory techniques of Rubin, Schori, and Walsh [RSW], as noticed by R. Pol in [P] will easily give n dimensional complete graphs. We choose the graphs carefully so that the products will not have greater dimension than the original graphs.

In Section 2 a direct proof of Theorem 1 is given, while Sections 3 and 4 are intended to isolate and generalize the techniques used in the proof of Theorem 1.

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In Section 5 an example is presented which shows that some of the care used in getting the spaces for Theorem 1 is necessary.

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1. Preliminaries. Let R denote the real numbers, let I denote the closed interval [0, 1], and let C denote the usual Cantor set in I. If X is a set then $X^n = \{f: f \text{ is a function, domain}(f) = n, \text{ and image}(f) \subseteq X\}$. We identify $f \in X^n$ with the ordered n-tuple $(f(0), f(1), \ldots, f(n-1))$, and let $f|_m$ denote the restriction of f to m. For $x, y \in R^n$, $x \geqslant_l y$ if x is greater than or equal to y in the lexicographic ordering on R^n. If $f: X \to Y$ is a function then $\mathrm{Gr}(f)$ denotes the graph of f; the topology on $\mathrm{Gr}(f)$ is the subspace topology where $\mathrm{Gr}(f) \subseteq X \times Y$. A topological space is totally disconnected if each point is the intersection of all of the simultaneously closed and open subsets containing it. If X is a topological space, and G is a collection of pairwise disjoint compact subsets of X whose union is X, then G is an upper semicontinuous (usc) decomposition of X if and only if for each closed subset C in X, $\bigcup \{g \in G: g \cap C \neq \emptyset\}$ is closed in X.

2. Proof of Theorem 1. In [RSW], for each $n \in \omega$ a compact subset W_n of $C \times I^n \subseteq I^{n+1}$ is described where W_n projects onto C, and any subset of W_n which projects onto C has dimension n. For each $c \in C$, let p_c denote the least element of $(\{c\} \times I^n) \cap W_n$ in the lexicographic ordering, and let $X_n = \{p_c: c \in C\}$. Then X_n is the graph of a function $f: C \to I^n$, where $p_c = (c, f(c))$, and $\dim(X_n) = n$. The referee has pointed out that X_n is essentially the set in the example on pages 80 and 81 of [L]. In fact, in [L] it is shown that such a metric spaces is complete; completeness will also follow from our Theorem 3.

We want to show that $\dim(X_n^\omega) = n$; it suffices to show $\dim(X_n^m) = n$ for all m. For maximal simplicity we do the case $m = 2$; for larger m the proof is analogous. We actually show X_n^2 embeds in $C \times R^n$, and the result is immediate. Consider the function $h: X_n^2 \to C \times C \times R^n$ given by

$$h((c_1, f(c_1)), (c_2, f(c_2))) = (c_1, c_2, f(c_1) + f(c_2)).$$

If h is an injective homeomorphism we are done because $C \times C$ is homeomorphic to C. To show that h is a homeomorphism we need only show its inverse is continuous because h is obviously injective and continuous. Supposing the inverse is not continuous, we have $(c_i, d_i, f(c_i) + f(d_i)) \to (c, d, f(c) + f(d))$ but $(c_i, f(c_i)) \to (c, r)$ and $(d_i, f(d_i)) \to (d, s)$ where not both $r = f(c)$ and $s = f(d)$, for some sequence $\{(c_i, d_i, f(c_i) + f(d_i))\}$. Since W_n is compact, by the choice of f, it follows that $r \geqslant_l f(c)$ and $s \geqslant_l f(d)$, with at least one of the inequalities being strict. But then $r + s >_l f(c) + f(d)$, and this implies that $(c_i, d_i, f(c_i) + f(d_i)) \to (c, d, r+s)$, violating the original choice. ■

Remark. It will also follow from Theorem 2 that X_n^m embeds in R^{n+1}. In fact it can be shown that X_n^ω embeds in R^{n+1}. For $i \in \omega$, let $f_i: C \to I^n$ be defined by

$f_i(x) = f(x)/2^i$, where f is as in Theorem 1. Then $\mathrm{Gr}(f_i)$ is homeomorphic to $\mathrm{Gr}(f)$, which is homeomorphic to X_n and so $\prod \mathrm{Gr}(f_i)$ is homeomorphic to X_n^ω. The function $p: \prod \mathrm{Gr}(f_i) \to C^\omega \times R^n$ defined by

$$p((c_0, f(c_0)), (c_1, f(c_1)), \ldots) = ((c_0, c_1, \ldots), f_0(c_0) + f_1(c_1) + \ldots)$$

is a homeomorphism. But C^ω is homeomorphic to C, hence X_n^ω embeds in $C \times R^n$.

3. Bounding the dimension of products of graphs. We start by characterizing graphs for which a particular function will be a homeomorphism.

DEFINITION 1. Let $f: X \to R^n$, and $g: Y \to R^n$ be functions. The pair $\langle f, g \rangle$ is said to be *properly spread* if the conditions that $(x, u) \in \mathrm{cl}(\mathrm{Gr}(f))$, $(y, v) \in \mathrm{cl}(\mathrm{Gr}(g))$ and $u + v = f(x) + g(y)$, imply that $u = f(x)$ and $v = g(y)$.

LEMMA 1. *If X and Y are metrizable, $f: X \to R^n$ and $g: Y \to R^n$ are bounded functions, then the function $e: \mathrm{Gr}(f) \times \mathrm{Gr}(g) \to X \times Y \times R^n$ defined by*

$$e((x, f(x)), (y, g(y))) = (x, y, f(x) + g(y))$$

is an injective homeomorphism if and only if $\langle f, g \rangle$ is properly spread.

Proof. Obviously e is continuous and injective. The function $e^{\leftarrow}$ is continuous if and only if, whenever $(x_i, y_i, f(x_i) + g(y_i)) \to (x, y, f(x) + g(y))$ it follows that $(x_i, f(x_i)) \to (x, f(x))$ and $(y_i, g(y_i)) \to (y, g(y))$, which is true if and only if $\langle f, g \rangle$ is properly spread. ■

In general, if f and g are continuous or $n = 1$ and f and g are both lower (or upper) semicontinuous, then e will be a homeomorphism. We proceed to generalize semicontinuity in a way which allows for the application of Lemma 1, and obviously includes functions like the f used in Theorem 1.

DEFINITION 2. If X is a topological space, and $f: X \to R^n$ is a function, then f is n *lower semicontinuous* (n-lsc) provided, if $p = (x, r) \in \{x\} \times R^n$ is a limit point of $\mathrm{Gr}(f)$ (in $X \times R^n$), then $r \geqslant_l f(x)$.

One can define n upper semicontinuity analogously and get all of our results. We stay with n lower semicontinuity.

LEMMA 2. *If X and Y are metrizable spaces and $f: X \to R^n$, $g: Y \to R^n$ are n-lsc, then $\langle f, g \rangle$ is properly spread.*

Proof. Suppose $(x, u) \in \mathrm{cl}(\mathrm{Gr}(f))$, $(y, v) \in \mathrm{cl}(\mathrm{Gr}(g))$ and $u + v = f(x) + g(y)$. By the n-lsc condition, $u \geqslant_l f(x)$ and $v \geqslant_l g(y)$, so $u + v \geqslant_l f(x) + g(y)$, with equality holding exactly when $u = f(x)$ and $v = g(y)$. Thus $\langle f, g \rangle$ is properly spread. ■

LEMMA 3. *If X and Y are metrizable spaces, $f: X \to R^n$ and $g: Y \to R^n$ are bounded n-lsc functions, then $h: X \times Y \to R^n$ defined by $h(x, y) = f(x) + g(y)$ is a bounded n-lsc function.*

Proof. Obviously h is bounded. Suppose $(x, y, r) \in X \times Y \times R^n$ is a limit point of $\mathrm{Gr}(h)$. Then there are r_x and r_y with $(x, r_x) \in \mathrm{cl}(\mathrm{Gr}(f))$ and $(y, r_y) \in \mathrm{cl}(\mathrm{Gr}(g))$

and $r_x+r_y = r$. Since the functions are n-lsc $r_x \geqslant_l f(x)$ and $r_y \geqslant_l g(y)$, so $r \geqslant_l f(x)+g(y) = h(x, y)$. ■

Combining Lemmas 1, 2 and 3:

THEOREM 2. *If X and Y are metrizable f: $X \to \boldsymbol{R}^n$ and g: $Y \to \boldsymbol{R}^n$ are bounded and n-lsc, then $\mathrm{Gr}(f) \times \mathrm{Gr}(g)$ is homeomorphic to $\mathrm{Gr}(h)$ where h: $X \times Y \to \boldsymbol{R}^n$ is defined by $h(x, y) = f(x)+g(y)$, and h is n-lsc.*

Immediately we get:

COROLLARY 1. *For X, Y metrizable, and f, g n-lsc functions as in Theorem* 2, $\dim(\mathrm{Gr}(f) \times \mathrm{Gr}(g)) \leqslant \dim(X)+\dim(Y)+n$.

Proof. By Theorem 2, $\mathrm{Gr}(X) \times \mathrm{Gr}(Y)$ is homeomorphic to $\mathrm{Gr}(h)$ which is a subset of $X \times Y \times \boldsymbol{R}^n$. ■

COROLLARY 2. *If X is a strongly zero dimensional metrizable space and f: $X \to \boldsymbol{R}^n$ is bounded and n-lsc, then $\dim(\mathrm{Gr}(f)^\omega) \leqslant n$.*

Proof. We only need to show $\dim(\mathrm{Gr}(f)^k) \leqslant n$ for all $k \geqslant 1$. But this is obvious using Theorem 2 and induction. ■

In the same way one proves the following.

COROLLARY 3. *If $\{X_i$: $i \in \omega\}$ is a collection of strongly zero dimensional metrizable spaces, and for each $i \in \omega$, f_i: $X_i \to \boldsymbol{R}^n$ is n-lsc, then $\dim(\prod \mathrm{Gr}(f_i)) \leqslant n$.*

4. Getting completeness. In the example of Theorem 1 the use of the lexicographic ordering was twofold. First, it enabled the use of the homeomorphism technique of Section 3, by giving n-lsc functions. The main goal of this section is to show its other use, which is to guarantee completeness; this is the content of Theorem 3 which is quite general and interesting in its own right. Apparently the use of the lexicographic ordering in selection theorems is quite old. (See [BS] for an example.) Also, other people know of Theorem 3; we include a proof because we do not know of a reference and because it can be used to show the spaces X_n from Theorem 1 are complete.

We start with some notation and a preliminary lemma.

If $X \subseteq \boldsymbol{I}^n$ is a compact set, and G is a usc decomposition of X, then for each $g \in G$ let $l(g)$ denote the least element of g in the lexicographic ordering on $\boldsymbol{I}^n$. Then for $0 \leqslant m \leqslant n$, let $g(m) = \{x \in g\colon x|_m = l(g)|_m\}$, and let $G_m = \{g(m)\colon g \in G\}$. Then $G_n = \{\{l(g)\}\colon g \in G\}$.

LEMMA 4. *For X, G as above, G_m is a usc decomposition of $\bigcup G_m$.*

Proof. Fix m. Suppose K is a closed subset of $\bigcup G_m$, and $x \in g(m)$ is a limit point of $H = \bigcup \{g \in G_m\colon g \cap K \neq \emptyset\}$. Then for each $t \in \omega$, there exists $x_t \in g_t(m)$ with $x_t \to x$, and each $g_t(m) \subseteq H$. Now $x_t|_m \to x|_m$. There are also $y_t \in g_t(m) \cap K$, and $y_t|_m = x_t|_m$. By the usc property of G, any limit point of $\{y_t\}$ is in g, and there must be such a point; call it y. Now it follows that $y \in K \cap g(m)$, so $g(m) \subseteq H$. ■

THEOREM 3. *For X and G as above, for each m with $0 \leqslant m \leqslant n$, $\bigcup G_m$ is a G_δ subset of $\boldsymbol{I}^n$. In particular $\{l(g)\colon g \in G\}$ is completely metrizable.*

Proof. Obviously G_0 is G_δ. Suppose $\bigcup G_m$ is G_δ. We show that $\bigcup G_{m+1}$ is. For each $i \in \omega$ and $0 \leqslant j < 2^i$ let $J_{ij} = \{x \in \boldsymbol{I}^n\colon x(m) \in [j/2^i, (j+1)/2^i]\}$, and let $J_i = \{J_{ij}\colon 0 \leqslant j \leqslant 2^i\}$. We consider J_i as ordered by the second subscript. We have:

(1) J_i is a closed subset of $\boldsymbol{I}^n$,

(2) If x, y are both in J_{ij}, then $|x(m)-y(m)| \leqslant 1/2^i$, and

(3) For each $g \in G$, $g(m+1) \subseteq J_{ij}$ where J_{ij} is the least element of J_i intersecting $g(m)$.

For each $i \in \omega$ and $0 \leqslant j \leqslant 2^i$ let $K_{ij} = (J_{ij} \cap (\bigcup G_m)) \setminus W_{ij}$ where $W_{ij} = \bigcup \{g(m)\colon g(m) \cap J_{ir} \neq \emptyset$ for some $r < j\}$. Then:

(4) K_{ij} is a G_δ set,

(5) If $K_{ij} \cap g \neq \emptyset$, then $g(m+1) \subseteq K_{ij}$, and

(6) For each $i \in \omega$ there is a K_{ij} which intersects g.

The only one at all tricky to verify is (4). By Lemma 4, W_{ij} is a closed subset of $\bigcup G_m$. But J_{ij} is closed, $\bigcup G_m$ is G_δ by hypothesis, so (4) holds.

Let $K_i = \bigcup \{K_{ij}\colon 0 \leqslant j < 2^i\}$; clearly K_i is G_δ. Thus $\bigcap K_i$ is also a G_δ. By (2), (5), and (6), $\bigcup G_{m+1} = \bigcap K_i$. ■

Observe that Theorem 3 remains true with n replaced by ω. Thus Theorem 3 gives a way of finding G_δ sections for mappings on compact metric spaces; simply embed the domain in the Hilbert cube and choose the least element in each point inverse.

5. A false conjecture. One might guess that whenever f: $\boldsymbol{C} \to \boldsymbol{R}^n$ is a fuction $\dim(\mathrm{Gr}(f)^\omega) \leqslant n$. We provide a counterexample as follows. Let X_2 be as in Theorem 1, and let f be the function which X_2 is the graph of. Define, for $i \in \{0, 1\}$, f_i: $\boldsymbol{C} \to \boldsymbol{R}$ by $f_0(c) = x_0$ and $f_1(c) = x_1$ where $f(c) = (x_0, x_1)$. Let M be the free union of $\mathrm{Gr}(f_0)$ and $\mathrm{Gr}(f_1)$. Clearly M is homeomorphic to the graph of a function from $\boldsymbol{C}$ to $\boldsymbol{R}^n$, and so $\dim(M) \leqslant 1$, but M^2 contains a copy of $\mathrm{Gr}(f)$ (along the diagonal), hence has dimension 2.

6. Questions. The following questions present themselves.

(1) Is there a (complete) separable metric space M such that $\dim(M) < \dim(M^\omega) < \infty$?

(2) The examples of this paper give, to some extent, high dimensional analogues of the irrational points in l_2. In order to make the analogue more complete, can these examples be made homogeneous?

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An approximate analog of a theorem of Khintchine

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Abstract The following theorem is established: If f is a real valued measurable function on the reals, then f has a finite approximate derivative almost everywhere on the set where the upper approximate symmetric derivate is less than infinity. This theorem is the approximate analog of a theorem of A. Khintchine.

In 1927, Khintchine [2] proved the following:

THEOREM. *If f is a real valued measurable function on the reals, then f has a finite ordinary derivative almost everywhere on the set where the upper symmetric derivate is less than infinity.*

In this paper we prove:

THEOREM 1. *If f is a real valued measurable function on the reals, then f has a finite approximate derivative almost everywhere on the set where the upper approximate symmetric derivate is less than infinity.*

An earlier proof of Theorem 1 (Russo and Valenti [6]) makes the oversight of assuming that not density zero implies positive density. This theorem will almost immediately give a Denjoy–Young–Saks Theorem for the approximate symmetric derivative.

In [1], Belna, Evans, and Humke constructed an additive subgroup, G, of the reals so that both the set G and its complement contain an element from every perfect set, and thus both have inner measure zero. The characteristic function of the set G, therefore, has symmetric derivative zero at every point in G, while the ordinary derivative does not exist at any point in G, showing that the assumption of measurability cannot be dropped in Khintchine's Theorem. Their example, however, leaves open the possibility that a non-measurable version of Khintchine's Theorem might be true if "almost everywhere" were replaced by "except on a set of inner measure zero." This improvement has been established by the following result of Uher [9]:

THEOREM. *If a function f has a finite upper symmetric derivate on a measurable set E, then f is almost everywhere differentiable on E.*