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On partitioner-representability of Boolean algebras

by

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Abstract. It is proved that — consistently with the Martin Axiom — the power-set algebra $P(\omega_1)$ may not be partitioner-representable.

0. Baumgartner and Weese in [B–W] introduced the notion of partitioner-representability of Boolean algebras: if E is m.a.d. (a maximal, almost disjoint family of subsets of ω) then a set $A \subseteq \omega$ is called a *partitioner of* E if for each $e \in E$ either $e \subseteq_* A$ or $e \cap A =_* \emptyset$ (i.e. $e \setminus A$ or $e \cap A$, respectively is finite); the union, intersection and difference of partitioners is again a partitioner, and hence the family $\mathrm{Pt}(E)$ of all the partitioners of E is a Boolean subfield of $P(\omega)$. A Boolean algebra is said to be *partitioner-representable* if for some m.a.d. E it is isomorphic to the factor algebra $\mathrm{Pt}(E) \bmod J$ where J is the ideal generated by fin (the finite sets) and E.

The finite sets and finite unions $e_1 \cup \ldots \cup e_n$ (and their complements) are called *trivial partitioners*. Thus, $\mathrm{Pt}(E) \bmod J$ may be called the *algebra of non-trivial partitioners* of E.

The fundamental theorem in [B–W] (see also [F–Z_1]) says that — under CH (the continuum hypothesis) — each algebra of cardinality $\leqslant c = 2^\omega$ is partitioner-representable. A question arises if the same is true if CH is replaced by MA (Martin Axiom). In this note we prove that this is not the case:

THEOREM. *There is a generic extension of the constructible universe in which* MA *holds and* $c = \varkappa$, *for a given regular* $\varkappa > \omega_1$, *and the algebra* $P(\omega_1)$ *is not partitioner-representable.*

Originally, we had $c = \omega_2$ in our model.

We are grateful to the referee who pointed out how to get the more general case.

In [F–Z_1] it is proved that partitioner-representability of $P(\omega_1)$ implies the existence of Q-sets. From the theorem it follows that the converse is not true.

The idea of proof is, roughly, as follows. Extend $V = L$ (the constructible sets) via a finite support, c.c.c.-iteration of length $\varkappa$ and assume, for contradiction, that $P(\omega_1)$ is representable in $V[G]$. Each (ω_1, ω_1)-chain $C = \langle \{x_\alpha\}; \{y_\beta\} \rangle$, gives then rise to a species of a Hausdorff gap $H = \langle \{a_\alpha\}; \{b_\beta\} \rangle$, which — at a given stage of iteration — cannot be filled. Now, there are two forcing notions Q

and $\mathbf{E}$, connected with H. We force with Q, which adjoins an uncountable antichain to $\mathbf{E}$. Thus, in $V[G]$, $\mathbf{E}$ does not have c.c.c. On the other hand, the chain C can be filled in $P(\omega_1)$, which in turn implies that $\mathbf{E}$ must have c.c.c., a contradiction. A similar idea was used by Kunen and others, see also [F].

1. Suppose that $P(\omega_1)$ is partitioner-representable. Thus, we have an isomorphism

$$f\colon P(\omega_1) \to \mathrm{Pt}(E) \bmod J.$$

Each value $f(x)$ is an equivalence class mod$J$. If $A \in f(x)$, we shall write shortly

$$A \mathrm{pt} x \quad (A \text{ is a partitioner of } E \text{ corresponding to } x).$$

If $A \mathrm{pt} x$, $B \mathrm{pt} y$ and $x \subseteq y$, then $A \subseteq B \bmod J$, i.e. $A \setminus B \in J$. Thus

$$A \setminus B =_* \bigcup F, \quad \text{for some finite } F \subseteq E.$$

Similarly, if $x \cap y = \emptyset$, then $A \cap B \in J$, and hence

$$A \cap B =_* \bigcup F, \quad \text{for some finite } F \subseteq E.$$

Let us denote

$$E(A) = \{e \in E\colon e \subseteq_* A\}.$$

Lemma 1. *Assume that $P(\omega_1)$ is partitioner-representable on a m.a.d. E. If $A \mathrm{pt} x$ and x is infinite, then $E(A)$ has cardinality c.*

Proof. Let us divide $x = x_\emptyset$ into two infinite parts $x_\emptyset = x_{\langle 0\rangle} \cup x_{\langle 1\rangle}$ and repeat division up to obtaining a binary tree

$$\{x_s\colon s \in \bigcup_n \{0, 1\}^n\}$$

such that

$$\text{if } s \subseteq t, \quad \text{then } x_s \supseteq x_t$$

and

$$\text{if } s(i) \neq t(i), \quad \text{then } x_{s|i+1} \cap x_{t|i+1} = \emptyset.$$

Let us choose partitioners A_s: $A_s \mathrm{pt} x_s$ and $A_\emptyset = A$. We have

$$s \subseteq t \text{ implies } A_t \subseteq A_s \bmod J$$

and

$$s(i) \neq t(i) \text{ implies } A_{s|i+1} \cap A_{t|i+1} = \emptyset \bmod J.$$

Modifying the sets A_s by trivial partitioners (inductively, along the levels $\{0, 1\}^n$), we may assume that

$$s \subseteq t \text{ implies } A_t \subseteq A_s$$

and

$$s(i) \neq t(i) \text{ implies } A_{s|i+1} \cap A_{t|i+1} = \emptyset.$$

For each branch $g \in \{0, 1\}^\omega$ we choose an infinite set B_g such that

$$B_g \subseteq_* A_{g|n}, \quad \text{for each } n < \omega.$$

Since E is maximal, there is an $e_g \in E$ such that $e_g \cap B_g$ is infinite. It follows that $e_g \subseteq_* A_{g|n}$, for each $n < \omega$, since $A_{g|n}$ are partitioners. In particular, for each g we have $e_g \subseteq_* A$, $e_{g_1} \subseteq_* A_{g_1|n}$, $e_{g_2} \subseteq_* A_{g_2|n}$ and hence $e_{g_1} \cap e_{g_2} \subseteq_* A_{g_1|n} \cap A_{g_2|n} = \emptyset$, which proves that all the e_g's are different, which finishes the proof.

Now, we generalize the notion of a Hausdorff gap as follows. Let D be an almost disjoint family (not necessarily maximal) and let

$$H = \langle \{a_\alpha\}_{\alpha<\omega_1};\ \{b_\beta\}_{\beta<\omega_1}\rangle$$

be a system of uncountable partitioners of D (i.e. the sets $D(a_\alpha)$, $D(b_\beta)$ are uncountable) satisfying the following conditions:

(1) $a_\alpha \cap b_\beta =_* \emptyset$, for all $\alpha, \beta < \omega_1$ and $a_\alpha \cap b_\alpha = \emptyset$, for all $\alpha \leqslant \omega_1$;

(2) $a_\alpha \subseteq_D a_\beta$ and $b_\alpha \subseteq_D b_\beta$, for all $\alpha \leqslant \beta$, where $a \subseteq_D b$ means that $D(a \setminus b)$ is a finite or countably infinite family.

We say that a set $S \subseteq \omega$ *separates* (or *fills*) the gap H if $a_\alpha \subseteq_* S$ and $b_\beta \cap S =_* \emptyset$, for all $\alpha, \beta < \omega_1$.

More generally, we say that S *D-separates* H if $a_\alpha \subseteq_D S$ and $b_\beta \subseteq_D \omega \setminus S$, for all $\alpha, \beta < \omega_1$.

There are two forcing notions, introduced by Kunen, associated with a gap H. The first one, denoted by $\mathbf{E}$, consists of pairs $p = \langle s_p, t_p\rangle$, where s_p, t_p are finite functions from ω_1 into ω satisfying the property

$$\bigcup_{\alpha \in \mathrm{dm}(s_p)} a_\alpha \setminus s_p(\alpha) \cap \bigcup_{\beta \in \mathrm{dm}(t_p)} b_\beta \setminus t_p(\beta) = \emptyset.$$

The ordering on $\mathbf{E}$ is defined thus

$$p \leqslant q \quad \text{iff} \quad s_p \supseteq s_q \text{ and } t_p \supseteq t_q.$$

Lemma 2. *If H can be filled, then $\mathbf{E}$ has c.c.c.*

Proof. Suppose, on the contrary, that there is an uncountable antichain $\{p_\xi\colon \xi < \omega_1\}$ in $\mathbf{E}$. Using the Δ-system lemma, we may assume that $\max \mathrm{dm}(s_{p_\xi}) < \min \mathrm{dm}(s_{p_\eta})$, for $\xi < \eta$ and similarly — for $\mathrm{dm}(t_p)$. If

$$k_\xi = \bigcup_{\alpha \in \mathrm{dm}(s_{p_\xi})} a_\alpha \setminus s_{p_\xi}(\alpha) \quad \text{and} \quad l_\xi = \bigcup_{\beta \in \mathrm{dm}(t_{p_\xi})} b_\beta \setminus t_{p_\xi}(\beta)$$

then incompatibility means that the set

$$p_\xi * p_\eta = (k_\xi \cap l_\eta) \cup (l_\xi \cap k_\eta)$$

is nonempty, cf. [F–Z$_2$]. By assumption, there is a set S which separates H, and hence we can find an $m < \omega$ such that

$$k_\xi \setminus m \subseteq S \quad \text{and} \quad (S \setminus m) \cap l_\xi = \emptyset,$$

for $\xi \in X_0$, where $X_0 \subseteq \omega_1$ is uncountable. Let $\xi_0 = \min X_0$ and find an uncountable $X_1 \subseteq X_0$ such that $p_{\xi_0} * p_\eta$ is constant for $\eta \in X_1$. Again, if $\xi_1 = \min X_1$, then we can find an uncountable $X_2 \subseteq X_1$, such that $p_{\xi_1} * p_\eta$ is constant on X_2 etc. Thus, for each $n < \omega$ we have an uncountable X_n, such that if $\xi_n = \min X_n$, then the nonempty sets

$$p_{\xi_0} * p_{\xi_n}, \ldots, p_{\xi_{n-1}} * p_{\xi_n}$$

are pairwise disjoint and lie below m, which is impossible and the proof is finished.

The other forcing, denoted by Q, consists of finite sets $q \subseteq \omega_1$ such that:

$$\text{for all } \alpha \neq \beta \in q, \quad a_\alpha \cap b_\beta \neq \emptyset \text{ or } a_\beta \cap b_\alpha \neq \emptyset .$$

Q is ordered by the reverse inclusion.

Each function $g\colon \omega_1 \to \omega_1$, with $g(\alpha) \geq \alpha$, determines a subgap $H(g)$ of H, where

$$H(g) = \langle \{a_\alpha \cap a_{g(\alpha)}\};\ \{b_\beta \cap b_{g(\beta)}\}\rangle .$$

Let $Q(g)$ and $E(g)$ denote the forcings Q and E, respectively associated with $H(g)$. Finally, let Q^* be the finite support product:

$$Q^* = \prod_g Q(g) .$$

LEMMA 3. *If no set D-separates H, then Q^* has the c.c.c. and for each $g \in V$:*

$$Q^* \Vdash E(g) \text{ has an uncountable antichain.}$$

Proof. For the countable chain condition, it is sufficient to prove that each finite subproduct

$$Q(g_1) \times \ldots \times Q(g_n)$$

has this property. Assume, for contradiction, that

$$q^\xi = \langle q_1^\xi, \ldots, q_n^\xi\rangle, \quad \xi < \omega_1 ,$$

is an uncountable antichain. Applying the Δ-system Lemma n times we may assume that

$$\max q_i^\xi < \min q_i^\eta, \quad \text{for } \xi < \eta \text{ and } i = 1, \ldots, n .$$

Since q^ξ, q^η are incompatible,

$$q^\xi \perp q^\eta, \quad \text{for } \xi \neq \eta ,$$

we have that $q_i^\xi \perp q_i^\eta$, for some i and hence there are $\alpha \in q_i^\xi$ and $\beta \in q_i^\eta$ such that

$$(a_\alpha \cap a_{g_i(\alpha)}) \cap (b_\beta \cap b_{g_i(\beta)}) = \emptyset \quad \text{and} \quad (a_\beta \cap a_{g_i(\beta)}) \cap (b_\alpha \cap b_{g_i(\alpha)}) = \emptyset .$$

Thus, if

$$S_\xi = \bigcap \{a_\alpha \cap a_{g_i(\alpha)}\colon \alpha \in q_i^\xi \text{ and } i = 1, \ldots, n\}$$

and

$$T_\eta = \bigcap \{b_\beta \cap b_{g_i(\beta)}\colon \beta \in q_i^\eta \text{ and } i = 1, \ldots, n\} ,$$

then $S_\xi \cap T_\eta = \emptyset$, for $\xi \neq \eta$, and $S_\xi \cap T_\xi = \emptyset$ as well, since $a_\alpha \cap b_\alpha = \emptyset$, for each $\alpha < \omega_1$.

For $\alpha \leq \min q_1^\xi, \ldots, \min q_n^\xi$ we have $a_\alpha \subseteq_D S_\xi$, and hence, if $S = \bigcup \{S_\xi\colon \xi < \omega_1\}$, then $a_\alpha \subseteq_D S$, for each $\alpha < \omega_1$. Symmetrically, if $T = \bigcup \{T_\eta\colon \eta < \omega_1\}$, then $b_\beta \subseteq_D T$, for each $\beta < \omega_1$ and since $S \cap T \neq 0$, we see that the set S D-separates H, a contradiction.

In particular, each $Q(g)$ has c.c.c. and consequently we may assume that for each $q \in Q(g)$ the set

$$\{\alpha < \omega_1\colon g \cup \{\alpha\} \in Q(g)\}$$

is uncountable. Let $G \subseteq Q^*$ be a generic filter. The projection $G(g)$ onto gth coordinate is then generic in $Q(g)$ and the sets

$$\{q\colon \exists \beta \ulcorner \beta \geq \alpha \text{ and } \beta \in q \urcorner\}$$

are dense. It follows that $G(g)$ contains uncountably many singletons $\{\alpha\}$. For each such α define $p_\alpha = \langle s_\alpha, t_\alpha\rangle \in E(g)$ as follows:

$$\operatorname{dm}(s_\alpha) = \operatorname{dm}(t_\alpha) = \{\alpha\} \quad \text{and} \quad s_\alpha(\alpha) = t_\alpha(\alpha) = 0 .$$

Now, p_α and p_β are incompatible in $E(g)$, since $\{\alpha\}$ and $\{\beta\}$ are compatible in $Q(g)$ as elements of $G(g)$. Thus $\{p_\alpha\colon \alpha < \omega_1\}$ is an uncountable antichain and the proof is complete.

2. In this section we show how to construct a particular gap H from a "partial" representation of $P(\omega_1)$. It is more convenient to deal with $P(\omega_1 \times \omega_1)$, rather than with $P(\omega_1)$.

For an uncountable set $X \subseteq \omega_1$, let $(X)_\alpha$ denote its initial segment of order type α. Suppose that also $\omega_1 \setminus X$ is uncountable and define

$$x_\alpha = \bigcup \{\{\xi\} \times \omega_1\colon \xi \in (X)_\alpha\}, \quad y_\beta = \bigcup \{\{\eta\} \times \omega_1\colon \eta \in (\omega_1 \setminus X)_\beta\} .$$

Let $B(X)$ be the subalgebra containing the sets x_α, y_β, for $\alpha, \beta < \omega_1$ and $\{\xi\} \times \omega_1$, all $\xi < \omega_1$.

DEFINITION. If D is an a.d. family, then a *D-representation* of an (ω_1, ω_1)-chain

$$k(X) = \langle \{x_\alpha\};\ \{y_\beta\}\rangle$$

is a function $r\colon B(X) \to P(\omega)$ such that:

(1) For each $x \in B(X)$, $r(x)$ is a partitioner of D and for $x \neq \emptyset$ the set $D(r(x)) = \{e \in D\colon e \subseteq_* r(x)\}$ is uncountable.

(2) r is congruent with Boolean operations, in particular conditions $x \subseteq y$ and $x \cap y = \emptyset$ imply that the sets $D(r(x) \setminus r(y))$ and $D(r(x) \cap r(y))$ are finite respectively.

Let r be a given D-representation of $k(X)$. If $A_\alpha = r(x_\alpha)$, $B_\beta = r(y_\beta)$ then we have a corresponding system

$$K(X) = \langle \{A_\alpha\};\ \{B_\beta\}\rangle$$

which need not be a D-gap, since the intersections $A_\alpha \cap B_\beta$ are, in general, infinite.

We say that a D-gap $H = \langle \{a_\alpha\};\ \{b_\beta\}\rangle$ is *contained in* $K(X)$ if

$$A_\alpha \subseteq_D a_\alpha \subseteq_* A_\alpha, \quad a_\alpha \cap B_\beta =_* \emptyset$$

and

$$B_\beta \subseteq_D b_\beta \subseteq_* B_\beta, \quad b_\beta \cap A_\alpha =_* \emptyset$$

for all $\alpha, \beta < \omega_1$; H is said to be *regular* if

$$a_\alpha \cap a_\gamma \subseteq_* a_\alpha \cap a_\beta \quad \text{and} \quad b_\alpha \cap b_\gamma \subseteq_* b_\alpha \cap b_\beta$$

for all $\alpha < \beta < \gamma < \omega_1$.

If there are enough Cohen reals and dominating functions, then we can always find a regular, unfilled D-gap H, contained in $K(X)$, for some X. Recall the dominating forcing $\boldsymbol{D}$: the conditions are pairs $P = \langle s_p, F_p\rangle$, where s_p is a finite sequence of natural numbers and F_p is a finite set of functions $f\colon \omega \to \omega$. The ordering on $\boldsymbol{D}$ is defined as follows: $p \leqslant q$ iff $s_p \supseteq s_q$ and $F_p \supseteq F_q$ and $s_p(i) > f(i)$, for each $i \in \mathrm{dm}(s_p) \setminus \mathrm{dm}(s_q)$ and $f \in F_q$.

If $G \subseteq \boldsymbol{D}$ is a generic filter, then the function $g = \bigcup \{s_p\colon p \in G\}$ dominates each function f from the ground model: $g(i) > f(i)$, for all but finitely many i's. Moreover, $\boldsymbol{D}$ always has the c.c.c.

LEMMA 4. *Let $D \in V$ and $\boldsymbol{P}_\gamma = \sum_{\alpha<\gamma} \boldsymbol{P}_\alpha$, where $\mathrm{cf}(\gamma) = \omega_1$, be a finite support iteration such that each $\boldsymbol{P}_\alpha$ has the c.c.c. and for some cofinal in γ sequences $\langle \alpha_\xi\colon \xi < \omega_1\rangle$, $\langle \beta_\xi\colon \xi < \omega_1\rangle$ we have $\boldsymbol{P}_{\alpha_\xi+1} = \boldsymbol{P}_{\alpha_\xi} * C$ (the Cohen forcing), and $\boldsymbol{P}_{\beta_\xi+1} = \boldsymbol{P}_{\beta_\xi} * \boldsymbol{D}$. Then, in $V[G]$, there is an X such that for an arbitrary D-representation r of $k(X)$ there is a reqular, D-unfilled gap H contained in $K(X)$.*

Proof. Let c_ξ be the Cohen real added at the stage $\alpha_\xi+1$. Define $X \subseteq \omega_1$ as the Cohen subset determined by the sequence $\langle c_\xi\colon \xi < \omega_1\rangle$:

$$X = \{\omega\cdot\xi + i\colon c_\xi(i) = 1\}.$$

Thus, $X \in V[G]$ but, for no $\alpha < \gamma$, X is in the submodel $V[G_\alpha]$. Consequently, any D-gap

$$H = \langle \{a_\alpha\};\ \{b_\beta\}\rangle$$

contained in $K(X)$ must be D-unfilled.

Indeed, suppose that a set T D-separates H and let $C_\xi = r(\{\xi\}+\omega_1)$. Then we have that $\xi \in X$ implies $C_\xi \subseteq_D a_\alpha$ for some α and

$$\xi \notin X \text{ implies } C_\xi \subseteq_D b_\beta, \text{ for some } \beta.$$

It follows that

$$X = \{\xi < \omega_1\colon C_\xi \subseteq_D T\},$$

and hence $X \in V[G_\alpha]$, for some $\alpha < \gamma$, which is impossible.

The gap H will be defined inductively. Assume that we have already sets a_ξ, b_ξ, for $\xi < \alpha$ and define a_α, b_α as follows. The family

$$D_\xi = \{e \in D\colon \exists_\eta[\xi < \eta < \alpha \text{ and } e \subseteq_* a_\xi \setminus a_\eta]\}$$

is at most countable, since $a_\xi \subseteq_D a_\eta$. Using a dominating function, we can find a partitioner S_ξ of D such that

$$a_\xi \setminus a_\eta \subseteq_* S_\xi, \quad \text{for each } \xi < \eta < \alpha$$

and

$$e \cap S_\xi =_* \emptyset, \quad \text{for each } e \in D \setminus D_\xi$$

in a usual way: the family

$$R_\xi = \{a_\xi \cap (a_\eta \setminus a_{\eta+1})\colon \xi \leqslant \eta < \alpha\}$$

is disjoint and countable. Changing its elements, if necessary, on a finite set, we may assume that $\bigcup R_\xi = \omega$. Choose a 1-1 onto function $j\colon \omega \to \omega\times\omega$ such that the images of the elements of R_ξ are the vertical axes $\{n\}\times\omega$. Then, the sets $j[e]$ for $e \in D \setminus D_\xi$ are finite on each axis, and hence the functions

$$f_e(n) = \max\{j[e] \cap (\{n\}\times\omega)\}$$

are well defined. All this takes place in some submodel $V[G_\beta]$ and therefore we have a function g dominating each f_e. Now, if

$$F = \{\langle n, i\rangle\colon i \leqslant g(n)\}$$

then $S_\xi = \omega \setminus j^{-1}[F]$ is as required.

In a similar way we find an S such that

$$S_\xi \subseteq_* S, \quad \text{for each } \xi < \alpha$$

and

$$e \cap S =_* \emptyset, \quad \text{for each } e \in D \setminus \bigcup_{\xi<\alpha} D_\xi.$$

Finally, let

$$D_\alpha = \{e \in D\colon \exists_{\eta<\alpha}[e \subseteq_* A_\alpha \cap B_\eta]\}.$$

There is a partitioner T such that

$$A_\alpha \cap B_\eta \subseteq_* T, \quad \text{for each } \eta < \alpha$$

and

$$e \cap T =_* \emptyset, \quad \text{for each } e \in D \setminus D_\alpha.$$

It is easy to see that $a_\alpha = A_\alpha \setminus (S \cup T)$ is as required. The set b_α is defined symmetrically and the proof is complete.

3. Let us assume that $P(\omega_1)$ is partitioner-representable on some m.a.d. E.

If $\langle\{x_\alpha\};\ \{y_\beta\}\rangle$ is an (ω_1, ω_1)-chain in $P(\omega_1)$, i.e. the sequences $\{x_\alpha\colon \alpha < \omega_1\}$, $\{y_\beta\colon \beta < \omega_1\}$ are strictly increasing and $x_\alpha \cap y_\beta = \emptyset$, for all $\alpha, \beta < \omega_1$, then we can choose a corresponding system of partitioners

$$K = \langle\{A_\alpha\};\ \{B_\beta\}\rangle$$

and since there are no gaps in $P(\omega_1)$, there is a partitioner S (e.g. S corresponds to the union of the x_α's), such that

$$A_\alpha \setminus S =_* \bigcup U_\alpha, \quad \text{for some finite } U_\alpha \subseteq E$$

and

$$B_\beta \cap S =_* \bigcup W_\beta, \quad \text{for some finite } W_\beta \subseteq E.$$

Lemma 5. *In the notation as above, there are finite sets $U, W \subseteq E$ and, for each $\alpha < \omega_1$, finite sets $u_\alpha, w_\alpha \subseteq \omega_1$, with $\alpha = \min u_\alpha = \min w_\alpha$, and such that*

$$\bigcap_{\xi \in u_\alpha} A_\xi \subseteq_* S \cup \bigcup U, \quad \textit{for each } \alpha < \omega_1$$

and

$$\bigcap_{\eta \in w_\beta} B_\eta \cap S \subseteq_* \bigcup W, \quad \textit{for each } \beta < \omega_1$$

Proof. Each intersection

$$p(\alpha) = \bigcap \{U_\beta\colon \alpha \leqslant \beta < \omega_1\}$$

reduces to a finite one

$$p(\alpha) = U_\alpha \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_n}$$

and we take $u_\alpha = \{\alpha, \alpha_1, \ldots, \alpha_n\}$. On the other hand, since the function $p(\alpha)$ is non-decreasing and its values are finite sets, there can be only finitely many jumps, and hence for some $\alpha_m < \omega_1$ we have

$$p(\alpha) \subseteq p(\alpha_m), \quad \text{for each } \alpha < \omega_1.$$

Define $U = p(\alpha_m)$. Now, since the elements of E are almost disjoint, we have

$$\bigcap_{\xi \in u_\alpha} A_\xi \setminus S =_* \bigcap_{\xi \in u_\alpha} \bigcup U_\xi =_* \bigcup \bigcap_{\xi \in u_\alpha} U_\xi =_* \bigcup p(\alpha) \subseteq_* \bigcup p(\alpha_m) =_* \bigcup U$$

which proves the first part of the lemma.

The sets W and w_β are defined in a similar way.

Consider now a regular $\mathcal{D}$-gap

$$H = \langle\{a_\alpha\};\ \{b_\beta\}\rangle$$

contained in K. Then, since

$$a_\alpha \cap B_\beta =_* \emptyset \quad \text{and} \quad b_\beta \cap A_\alpha =_* \emptyset$$

for all $\alpha, \beta < \omega_1$, we obtain

$$\bigcap_{\xi \in u_\alpha} a_\xi \cap \bigcup U =_* \emptyset \quad \text{and} \quad \bigcap_{\eta \in w_\beta} b_\eta \cap \bigcup W =_* \emptyset$$

for all $\alpha, \beta < \omega_1$.

Thus, since U and Q are disjoint, we infer that the partitioner $T = S \cup U \setminus \bigcup W$ separates the subgap

$$\langle\{\bigcap_{\xi \in u_\alpha} a_\xi\};\ \{\bigcap_{\eta \in w_\beta} b_\eta\}\rangle.$$

The regularity property implies

$$\bigcap_{\xi \in u_\alpha} a_\xi = a_\alpha \cap a_{f_1(\alpha)}, \quad \text{where } f_1(\alpha) = \max u_\alpha$$

and

$$\bigcap_{\eta \in b_\beta} b_\eta = b_\beta \cap b_{f_2(\beta)}, \quad \text{where } f_2(\beta) = \max w_\beta.$$

If $f = \max\{f_1, f_2\}$, then $a_\alpha \cap a_{f(\alpha)}$ and $b_\beta \cap b_{f(\beta)}$ are even smaller and hence we have the following.

Corollary. *If $P(\omega_1)$ is partitioner-representable and K, H are as above, then there is an $f\colon \omega_1 \to \omega_1$, with $f(\alpha) \geqslant \alpha$, such that the subgap*

$$H(f) = \langle\{a_\alpha \cap a_{f(\alpha)}\};\ \{b_\beta \cap b_{f(\beta)}\}\rangle$$

can be filled.

4. Now, using the results of the preceding section, we can finish the proof of our theorem.

In the ground model V (the constructible universe), we fix a regular cardinal $\varkappa > \omega_1$ and write $\mathrm{HC}(\varkappa)$ = the sets hereditarily of cardinality $< \varkappa$. Then, we have

$$\mathrm{HC}(\varkappa) = \bigcup \{\mathrm{HC}_\alpha\colon \alpha < \varkappa\}$$

where $\mathrm{HC}_\alpha = \{x \in \mathrm{HC}(\varkappa)\colon \operatorname{rank} x < \alpha\}$.

We shall make use of the following version of Diamond: there is a sequence

$$\{T_\alpha\colon \alpha < \varkappa \text{ and } \operatorname{cf}(\alpha) = \omega_1\}$$

such that for each $F \subseteq \mathrm{HC}(\varkappa)$, the set

$$\{\alpha < \varkappa\colon F \cap \mathrm{HC}_\alpha = T_\alpha\}$$

is stationary.

For each $\alpha < \varkappa$, with $\operatorname{cf}(\alpha) = \omega_1$ we choose a cofinal in α sequence

$$\Gamma_\alpha = \{h_\alpha(\xi)\colon \xi < \omega_1\} \in V$$

such that $\operatorname{cf}(h_\alpha(\xi)) = \omega$, for all $\xi < \omega_1$.

Let us define a finite support iteration $\boldsymbol{P} = \sum_{\alpha < \varkappa} \boldsymbol{P}_\alpha$, of length $\varkappa$ (forcing notions and forcing names for small sets are encoded as elements or subsets of $\mathrm{HC}(\varkappa)$), as follows.

Direct limit are taken at all limit stages and

(1) if $\mathrm{cf}(\alpha) = \omega$, then we take

$$P_{\alpha+1} = P_\alpha * C \quad \text{and} \quad P_{\alpha+2} = P_{\alpha+1} * D;$$

(2) if $\mathrm{cf}(\alpha) = \omega_1$ and T_α is a P_α-name for a p.o. set and

$$P_\alpha \Vdash T_\alpha \text{ has c.c.c.}$$

then we take $P_{\alpha+1} = P_\alpha * T_\alpha$;

(3) if $\mathrm{cf}(\alpha) = \omega_1$ and T_α is a P_α-name such that for some D, $P_\alpha \Vdash T_\alpha$ is a D-representation of $k(X_\alpha)$, where X_α is a Cohen set produced by Cohen reals added at stages $h_\alpha(\xi)+1$, then there is a P_α-name Q_α for

$$Q^* = \prod \{Q(g)\colon g \in V\}$$

where Q is associated with some D-unfilled gap H contained in $K(X_\alpha)$. We take $P_{\alpha+1} = P_\alpha * Q_\alpha$.

In all the remaining cases let $P_{\alpha+1} = P_\alpha$.

Thus, by Lemma 3, each P_α and P have the c.c.c.

Let $G \subseteq P$ be a generic filter. Thus, in $V[G]$ the cardinality of the continuum is $\varkappa$ and Martin Axiom holds, by clause 2.

Let us suppose that $P(\omega_1)$ is partitioner-representable on a m.a.d. E and derive a contradiction. First, let us fix partitioners C_ξ corresponding to sets $\{\xi\} \times \omega_1$, for $\xi < \omega_1$. By Lemma 1 each family

$$E(C_\xi) = \{e \in E\colon e \subseteq_* C_\xi\}$$

has a subfamily E_ξ, with $\mathrm{card}\, E_\xi = \omega_1$.

If $D = \bigcup \{E_\xi\colon \xi < \omega_1\}$, then both $\{C_\xi\colon \xi < \omega_1\}$ and D belong to a submodel $V[G_\alpha] \subseteq V[G]$ and w.l.o.g. we may assume that $V[G_\alpha] = V$.

For each $\gamma < \varkappa$, with $\mathrm{cf}(\gamma) = \omega_1$, we have a cofinal Cohen set $X_\gamma \in V[G_\xi]$ and the corresponding chain $k(X_\gamma) = \langle \{x_\alpha(\gamma)\};\ \{y_\beta(\gamma)\}\rangle$. Let B and B_γ denote subalgebras of $P(\omega_1 \times \omega_1)$ generated by $\{k(X_\gamma)\colon \gamma < \varkappa\}$ (i.e. by $x_\alpha(\gamma)$, $y_\beta(\gamma)$ for $\alpha, \beta < \omega_1$ and $\gamma < \varkappa$), and $\{k(X_\alpha)\colon \alpha \leqslant \gamma\}$, respectively. Thus, $B_\gamma = B \cap V[G_\gamma]$. Note that $\mathrm{card}\, B_\gamma = \omega_1$, for $\gamma < \omega_2$. Define a function $r\colon B \to P(\omega)$ so that $r(x)$ is a partitioner of E corresponding to x and $r(\{\xi\} \times \omega_1) = C_\xi$, for $\xi < \omega_1$, and choose a nice name $\underline{r} \subseteq \mathrm{HC}(\varkappa)$ for r. If $r_\gamma = r \cap V[G_\gamma]$, then standard reasonings show that the set

$$N_1 = \{\gamma < \varkappa\colon \mathrm{cf}(\gamma) = \omega_1 \text{ and } r_\gamma \in V[G_\gamma]\}$$

is normal, i.e. it is unbounded in $\varkappa$ and closed under limits of cofinality ω_1.

Similarly, if $\underline{r}|\gamma$ is the part of $\underline{r}$ in $V^{(P_\gamma)}$, then the set

$$N_2 = \{\gamma < \varkappa\colon \mathrm{cf}(\gamma) = \omega_1 \text{ and } \underline{r}|\gamma = \underline{r} \cap \mathrm{HC}_\gamma\}$$

is normal, and hence so is $N = N_1 \cap N_2$.

By Diamond, N intersects the set $\{\gamma < \varkappa\colon \mathrm{cf}(\gamma) = \gamma \text{ and } \underline{r} \cap \mathrm{HC}(\varkappa) = T_\gamma\}$, and hence there are arbitrarily large $\gamma < \varkappa$, with $\mathrm{cf}(\gamma) = \omega_1$, for which T_γ is a P_γ-name for $r_\gamma \in V[G_\gamma]$ which is a D-representation of $k(X_\gamma)$ in $V[G_\gamma]$. By Lemma 4, there is a D-unfilled, regular gap H contained in $K(X_\gamma)$ and we have forced with Q_H^*, associated with such an H, at this stage of iteration. By Lemma 3 and since P has the c.c.c., each $E_H(g)$ has an uncountable antichain in $V[G]$.

On the other hand, Corollary of Section 3 shows that there is a function $f \in V[G]$ and a set S which separates $H(f)$ in $V[G]$. Since P has the c.c.c., there is a $g \in V$, with $g(\alpha) \geqslant f(\alpha)$, for each $\alpha < \omega_1$. Now, $H(g)$ is even smaller, and hence can be separated as well. By Lemma 2, $E_H(g)$ has the c.c.c., a contradiction.

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