On reduction theorems in the problem of composition of functions

by

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Abstract. The sequence $\langle p_n \rangle$ of non-negative integers is represented by a clone (algebra) $\underset{\sim}{A}$ if the number $p_n(\underset{\sim}{A})$ of essentially n-ary functions (polynomials) in $\underset{\sim}{A}$ is equal to p_n for all $n \geqslant 0$. In his paper *Composition of functions* G. Grätzer proposed to prove the following "reduction theorems":

Each sequence represented by an algebra of type α can be represented by an algebra of type β.

In particular, he asked wether this is true for $\alpha = \langle 2, 2 \rangle$ and $\beta = \langle 2 \rangle$. In spite of many results on representable sequences no reduction theorem was proved so far. In this paper we show that actually, except trivial and certain peculiar cases, *no reduction theorem holds*. In fact, some variants of reduction theorems are considered, and as a by-product, the characterizations of sequences represented by clones of unary functions for all possible types are obtained.

The problem mentioned above, suggested by G. Grätzer in [4], § 10, concerns actually the complexity of generating sets for clones. General problems in this area lie on borderlines between such disciplines as multi-valued logic, universal algebra and synthesis of automata (cf. [1, 2, 3, 8, 10]), our results having natural interpretations in all these disciplines. In this paper we use the terminology of [9] (rather than that of [4]) combined with some universal algebra concepts [4]. In proofs also basic concepts of graph theory are applied.

The paper is divided into five sections. In § 1 definitions and notations are given. § 2 contains a preliminary analysis of the problem showing that some very different cases have to be distinguished here. Then, in § 3, our main results are presented. Lemmas and proofs are given in §§ 4 and 5.

1. Definitions. A *clone* is a composition closed set of finitary functions (on a fixed universe U) containing all projections. For a set of functions F (on U) the least clone containing F is denoted by $[F]$ (thus $[F]$ is the set of polynomials of the algebra $\langle U, F \rangle$). We write $[f_1, \ldots, f_n]$ for $[\{f_1, \ldots, f_n\}]$. Denoting functions we abbreviate $f(x_1, \ldots, x_n)$ by $fx_1 \ldots x_n$ whenever possible.

If the algebra $\langle U, F \rangle$ is of (similarity) type α, then the clone $[F]$ is said to have a *generating set of type* α. In this paper we deal with finitely generated clones: by types we mean *finite* sequences of non-negative integers (of length $n \geqslant 1$). The class of all clones with generating sets of type α (on arbitrary universes of cardinality > 1) is denoted by $T(\alpha)$.

Taking into account a point of view of multi-valued logic and synthesis of automata a special attention is given to clones on finite sets and without nullary operations, i.e. to types without zeros. In this connection, a clone is said to be *essential* if it is not a pointed set, i.e. contains a nonconstant function different from projections.

Given a clone $\underset{\sim}{A}$, for $n \geqslant 1$, by $p_n(\underset{\sim}{A})$ we denote the number of essentially n-ary functions in $\underset{\sim}{A}$, while by $p_0(\underset{\sim}{A})$ — the number of constant unary functions. The sequence $\langle p_n(\underset{\sim}{A})\rangle$ is called the *p_n-sequence* of $\underset{\sim}{A}$ (the sequence represented by $\underset{\sim}{A}$). For results concerning p_n-sequences cf. [1, 2, 4, 6, 7] and [5] for further references. The basic concepts of graph theory are assumed to be familiar for the reader.

2. Preliminary discussion. Following Grätzer [4] we consider the following reduction theorem-schema:

$R(\alpha, \beta)$: For every clone $A \in T(\alpha)$ there exists a clone $\underset{\sim}{B} \in T(\beta)$ such that $p_n(B) = p_n(\underset{\sim}{A})$ for all $n \geqslant 0$.

At first note that if we ask e.g. whether $R(\alpha, \beta)$ is true for $\alpha = \langle 2, 2\rangle$ and $\beta = \langle 2\rangle$, the answer is "no" because of rather a trivial reason: for the clone $\underset{\sim}{A} = [f_1, f_2]$, where $f_1 x = c_1$ and $f_2 x = c_2$ are constant unary functions, the p_n-sequence of $\underset{\sim}{A}$ is $\langle 2, 1, 0, 0, \ldots\rangle$ (just two constant unary functions and the unary projection), while for each clone $\underset{\sim}{B} \in T(\langle 2\rangle)$ we have either $p_0(\underset{\sim}{B}) \leqslant 1$ (whenever B is not essential) or $p_1(\underset{\sim}{B}) > 1$ or $p_k(\underset{\sim}{B}) > 0$ for some $k \geqslant 2$ (whenever $\underset{\sim}{B}$ is essential). Of course, we wish for such trivial examples not to affect our solution and that is why we consider simultaneously the following weakened version of reduction theorems (for types $\alpha \neq \langle 0, 0, \ldots, 0\rangle$):

$RE(\alpha, \beta)$: For every essential clone $\underset{\sim}{A} \in T(\alpha)$ there exists a clone $\underset{\sim}{B} \in T(\beta)$ such that $p_n(\underset{\sim}{B}) = p_n(\underset{\sim}{A})$ for all $n \geqslant 0$.

Another special case is when α is a type without zeros, while β contains 0 among its elements. For any α with the property above there exists a clone $\underset{\sim}{A}$ in $T(\alpha)$ without constant functions, while for any clone $\underset{\sim}{B}$ in $T(\beta)$ it has to be necessarily $p_0(\underset{\sim}{B}) > 0$. Hence, for such α and β neither $R(\alpha, \beta)$ nor $RE(\alpha, \beta)$ holds. Moreover, $T(\alpha) \subseteq T(\beta)$ does not hold either. (Note, that $T(\alpha) \subseteq T(\beta)$ means that each clone generated by a set of type α has a generating set of type β, and so, can be considered as the strongest version of the reduction theorem.) Simply, 0 in a type, unlike other numbers, forces existence of a non-projection function. That is why types without zeros and those with zeros must be treated separately.

On the other hand, observe that if e.g., $\alpha = \langle 2\rangle$ and $\beta = \langle 3, 3\rangle$, then both $R(\alpha, \beta)$ and $RE(\alpha, \beta)$ trivially hold, since each clone in $T(\alpha)$ belongs also to $T(\beta)$. (If e.g., $f = fxy$, then putting $gxyz = fxy$ and $hxyz = x$ (a projection), we have obviously $[f] = [g, h]$). In order to describe all such trivial cases we introduce the following:

DEFINITION. For two types $\alpha = \langle a_1, \ldots, a_n\rangle$ and $\beta = \langle b_1, \ldots, b_m\rangle$ we write $\alpha \leqslant \beta$ whenever there exists a 1-1 mapping φ of $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$ such that $a_i \leqslant b_{\varphi i}$ for all $i = 1, \ldots, n$.

Using arguments as that above we obtain easily

LEMMA 1. *If $\alpha \leqslant \beta$, then $T(\alpha) \subseteq T(\beta)$, provided that if α is without zeros, then so is β.*

In particular, in such a case both $R(\alpha, \beta)$ and $RE(\alpha, \beta)$ trivially hold.

3. Results. Our result presented below concern arbitrary clones, i.e. also those on infinite universes with infinite p_n's. It should be mentioned however that all these results hold as well when merely clones on finite universes are taken into account (this can be checked easily by examining our proofs and is left to the reader).

Our main result is the following (we write α^2 for the type obtained from $\alpha = \langle \alpha_1, \ldots, a_n\rangle$ by deleting all those a_i with $a_i < 2$):

THEOREM 1. *Let α be a type other than $\langle 0, 0, \ldots, 0\rangle$. If* (i) *$\alpha^2 \leqslant \beta$ does not hold, or* (ii) *the length of β is less than that of α, then there exists an essential clone $\underset{\sim}{A}$ in $T(\alpha)$, on a finite universe, such that the p_n-sequence of $\underset{\sim}{A}$ is different from any p_n-sequence of a clone $\underset{\sim}{B}$ in $T(\beta)$.*

Using this result we obtain the following:

THEOREM 2. *For any two types α and β, $T(\alpha) \subseteq T(\beta)$ iff the following conditions hold*:

(i) $\alpha \leqslant \beta$,

(ii) *if α is without zeros, then so is β.*

THEOREM 3. *For α and β without zeros the following are equivalent*:

(i) $T(\alpha) \subseteq T(\beta)$,

(ii) $R(\alpha, \beta)$,

(iii) $RE(\alpha, \beta)$,

(iv) $\alpha \leqslant \beta$.

These results mean that actually no interesting reduction theorem holds, and generally, the characterization of p_n-sequences for algebras of a given type cannot be reduced to a simpler type (thus shedding a light on complexity of generating sets for clones).

The equivalences in Theorem 3 do not hold in general case, as the following general result shows:

THEOREM 4. *Let $\alpha = \langle a_1, \ldots, a_n\rangle$, where $a_i = 0$ or 1 for all i. Then, the sequence $\langle p_0, p_1, p_2, \ldots\rangle$ is represented by a clone belonging to $T(\alpha)$ iff the following conditions are satisfied*:

(i) *$p_n = 0$ for all $n \geqslant 2$,*

(ii) $0 < p_1 \leqslant \aleph_0$,

(iii) *$p_1 = 1$ whenever $a_i = 0$ for all i,*

(iv) *$p_0 > 0$ whenever $a_i = 0$ for some i,*

and

(v) $p_0 \leqslant (n-1)p_1 + 1$.

Now, if e.g., $\alpha = \langle 1, 1, 0\rangle$ and $\beta = \langle 1, 0, 0\rangle$, then by Theorem 4 $R(\alpha, \beta)$ holds, while $\alpha \leqslant \beta$ is obviously not satisfied.

For natural interpretations of our results in multi-valued logic, synthesis of automata, and equational logic (in terms of free spectra of varieties) cf. [3, 8].

4. Lemmas. At first, some general remarks should be made.

Since in the definition of $p_n(\underset{\sim}{A})$ we do not exclude the unary projection (like in [3], but unlike in [4]), for any clone $\underset{\sim}{A}$ (on a universe of cardinality $\geqslant 1$) $p_1(\underset{\sim}{A}) \geqslant 1$. Thus, $p_1(\underset{\sim}{A}) = 1$ means that the only essentially unary function in $\underset{\sim}{A}$ is just the unary projection.

For simplicity, constant unary functions are called briefly *constants* and denoted like nullary functions just by their values. Note, that if c is a value of a nullary function in $\underset{\sim}{A}$ or of an n-ary constant function for any n, then c is a constant in $\underset{\sim}{A}$, i.e. the value of a constant unary function.

We note also that since we deal with finitely generated clones, $p_n(\underset{\sim}{A})$ is countable for any n. In proofs, at first, finite values of p_n's are taken into account, remarks concerning infinite values to be given in conclusions.

LEMMA 2. *Let* $\underset{\sim}{A} = [f_1, \dots, f_m]$ *be a clone generated by unary functions. Then,*

$$p_0(\underset{\sim}{A}) \leqslant (m-1)p_1(\underset{\sim}{A})+1$$

Proof. For the clone $\underset{\sim}{A}$ we form an m-ary tree $T(\underset{\sim}{A})$ ($m \geqslant 1$) with nodes labeled by unary functions of $\underset{\sim}{A}$ in the following way:

(1) The root node of $T(\underset{\sim}{A})$ is labeled by the unary projection x.

(2) If a node is labeled by an essentially unary function gx of $\underset{\sim}{A}$, then the branches emanating from gx go to m nodes labeled successively by $gf_1x, gf_2x, \dots, gf_mx$, unless $gf_ix = hx$ for some function hx, which is already used as a label — then the node is terminal and is left unlabeled.

(3) If node is labeled by a constant, then it is terminal.

As an example, Figure 1 is the tree $T(\underset{\sim}{A})$ for the clone $\underset{\sim}{A} = [f, g]$ where f and g are unary functions on the set $S = \{0, 1, \dots, 5\}$ defined as follows: $fx = \min\{x+1, 5\}$, $gx = \min\{x+3, 5\}$.

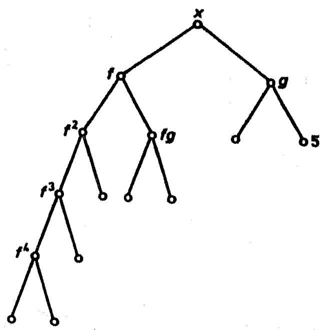

Fig. 1

Here, $m = 2$ and therefore the tree is binary. Since $fgx = gfx = \min\{x+4, 5\}$, the node corresponding to gfx is left unlabeled (according to (2)). Similarly, $g^2x = ggx = 5$, and also $f^2gx = fgfx = 5$, etc., corresponding nodes being left unlabeled. Note that $p_0(\underset{\sim}{A}) = 1$ and $p_1(\underset{\sim}{A}) = 7$. This can be checked directly, but also can be seen in Fig. 1.

In general case, the following three observations are easy to make:

(i) each unary function in $\underset{\sim}{A}$ is a label for one and only one node in $T(\underset{\sim}{A})$,

(ii) terminal nodes are labeled by constants or are unlabeled, while each inner node is labeled by an essentially unary function in $\underset{\sim}{A}$,

(iii) for $t(\underset{\sim}{A})$ — the number of terminal nodes in $T(\underset{\sim}{A})$, and $i(\underset{\sim}{A})$ — the number of inner nodes in $T(\underset{\sim}{A})$, we have the equality

$$t(\underset{\sim}{A}) = (m-1)i(\underset{\sim}{A})+1$$

(since $T(\underset{\sim}{A})$ is an m-ary tree).

By (i)–(iii) the lemma easily follows.

We note that for $p_1(\underset{\sim}{A}) = \aleph_0$ the result is trivial. The same concerns the result in Lemma 3 below, which is a generalization of (and is based on) that just proved.

LEMMA 3. *If* $\underset{\sim}{B} = [f_1, \dots, f_n]$ *is a nullary-unary clone, i.e. each* f_i *is a nullary or unary function, then*

$$p_0(\underset{\sim}{B}) \leqslant (n-1)p_1(\underset{\sim}{B})+1 .$$

Proof. Let $\underset{\sim}{B} = [c_1, \dots, c_k, f_1, \dots, f_m]$, where $c_1, \dots, c_k$ are nullary functions, and $f_1, \dots, f_m$ unary ones, $k+m = n$. If $m = 0$, then $\underset{\sim}{B}$ is a pointed set and the lemma is trivial. So, assume that $m \geqslant 1$, and denote $\underset{\sim}{A} = [f_1, \dots, f_m]$, $p_0(\underset{\sim}{A}) = r$, and $p_1(\underset{\sim}{A}) = s$. Let $d_1, \dots, d_r$ be the constants of $\underset{\sim}{A}$, and $g_1, \dots, g_s$ — essentially unary functions of $\underset{\sim}{A}$. One of g_i, say, g_1, is the unary projection, i.e. $g_1x = x$. Finally, denote $c_j^i = g_ic_j$ for $i = 1, \dots, s$, $j = 1, \dots, k$ (then $c_j^1 = c_j$). We claim that each constant in $\underset{\sim}{B}$ is equal to one of c_j^i or d_j.

Indeed, let e be a constant in $\underset{\sim}{B}$. There are two possibilities: either e can be obtained by composition of unary functions $f_1, \dots, f_m$, or $e = fc_j$ for some j and an essentially unary function f of $\underset{\sim}{B}$. In the former, $e = d_i$ for some i, as required. In the latter, we note that since obviously essentially unary functions of $\underset{\sim}{B}$ are those of $\underset{\sim}{A}$, $f = g_i$ for some i, and consequently $e = g_ic_j = c_j^i$, as required. Also, in particular, $p_1(\underset{\sim}{A}) = p_1(\underset{\sim}{B})$.

Now, it follows that $p_0(\underset{\sim}{B}) \leqslant ks+r = kp_1(\underset{\sim}{B})+p_0(\underset{\sim}{A})$. But, in view of Lemma 2, $p_0(\underset{\sim}{A}) \leqslant (m-1)p_1(\underset{\sim}{A})+1$, and since $p_1(\underset{\sim}{A}) = p_1(\underset{\sim}{B})$, we obtain finally,

$$p_0(\underset{\sim}{B}) \leqslant (k+m-1)p_1(\underset{\sim}{A})+1 ,$$

which completes the proof.

LEMMA 4. *Let* $\alpha = \langle 1, \dots, 1, 0, \dots, 0\rangle$, *where* 1 *occurs* $m \geqslant 1$ *times, and* 0 *occurs* $k \geqslant 0$ *times. Then, for any positive integers* p_0, p_1 *satisfying* $p_0 \leqslant (k+m-1)p_1+1$ *there exists a clone* $\underset{\sim}{A}$ *in* $T(\alpha)$ *on a finite universe such that* $p_0(\underset{\sim}{A}) = p_0$ *and* $p_1(\underset{\sim}{A}) = p_1$.

Proof. Denote $n = k+m$ (the length of α), $p_0 = s$, and $p_1 = p$. Let T be a directed tree with $s+p$ nodes labeled by elements of the set $S = \{0, 1, \ldots, s+p-1\}$ in the following way: The root node is labeled by 0. Coming out of the root are $r \leqslant n$ paths of the length not exceeding $p+1$ with terminal nodes labeled by $1, 2, \ldots, r$, the length of the path connecting r and 0 to be just $p+1$. The remaining nodes are labeled by $r+1, \ldots, s+p-1$. The orientation of all edges is towards the root, i.e. the outdegree of 0 is equal to 0, while for any node $x \neq 0$, the outdegree is equal to 1. It is easy to check that for any $s \leqslant (n-1)p+1$ $(s \geqslant 1)$ there exists a tree T, which for some $r \leqslant n$ satisfies the above conditions. (Figure 2 is such a tree for $n \geqslant 3$, $p = 4$, and $s = 6$.)

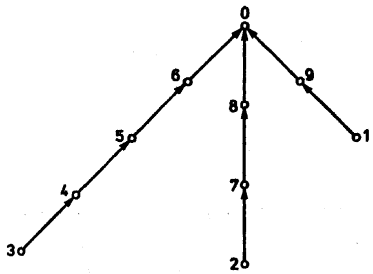

Fig. 2

Now, we define an unary function f on S as follows: $f0 = 0$, while if $x \neq 0$ and (x, y) is the (unique) edge of T directed from x towards y, then $fx = y$. Since the path from r towards 0 is of the length $p+1$, the functions $x, fx, f^2x, \ldots, f^{p-1}x$ are essentially unary and pairwise distinct. Moreover, $f^px = 0$ identically.

We use the function f and its properties established above to construct the clone $\underset{\sim}{A}$ with a generating set of type α whose existence is to be proved. Namely, let $\underset{\sim}{A} = [f_1, \ldots, f_m, c_1, \ldots, c_k]$, where $f_1 = f$, while $f_2, \ldots, f_m$ are constant unary functions, and $c_1, \ldots, c_k$ — nullary ones, the values of $f_2, \ldots, f_m, c_1, \ldots, c_k$ to be choosen from the set $\{0, 1, \ldots, r-1\}$ so that each of the numbers $1, \ldots, r-1$ is the value of at least one of the functions (this is possible, since $r-1 \leqslant n-1 = k+m-1$). It follows that each of nodes in the subgraph of T consisting of the paths $(1, 0), (2, 0), \ldots, (r-1, 0)$ is a constant in $\underset{\sim}{A}$, but none of the p nodes of the path $(r, 0)$ different from 0. Now, since the cardinality of S is $p+s$, $p_0(\underset{\sim}{A}) = s$. In turn, by properties of the function f, $p_1(\underset{\sim}{A}) = p$, which completes the proof.

Remark. The lemma also holds for $p_1 = \aleph_0$ except that the universe of the clone $\underset{\sim}{A}$ is now infinite. The proof is essentially the same, but the tree T has to be infinite.

LEMMA 5. *Given an integer $k \geqslant 2$, let $\underset{\sim}{B} = [f_1, \ldots, f_s]$ be a clone such that $p_1(\underset{\sim}{B}) = 1$ and for all $n > 1$ other than k, $p_n(\underset{\sim}{B}) = 0$. If m is the number of functions in the set $\{f_1, \ldots, f_s\}$ depending on not less than k variables, then $p_k(\underset{\sim}{B}) \leqslant mk!$.*

Proof. Let $f_1, \ldots, f_m$ be all those functions among $f_1, \ldots, f_s$ that depend on not less than k variables. Each of these functions depends actually on exactly k variables, since $p_n(\underset{\sim}{B}) = 0$ for all $n > k$. On the other hand, by further properties assumed for $p_n(\underset{\sim}{B})$, each of the remaining functions $f_{m+1}, \ldots, f_s$ (if $s > m$) is either a projection or a constant function. We leave to the reader checking that by composition one can obtain merely projections, constant functions, or functions obtained from $f_i x_1 \ldots x_k$ by permuting of variables $(i = 1, \ldots, m)$. From this the lemma easily follows.

LEMMA 6. *For any $\alpha = \langle k, \ldots, k \rangle$, where $k \geqslant 2$ occurs $m \geqslant 1$ times, there exists a clone $\underset{\sim}{A}$ in $T(\alpha)$, on a finite universe, such that $p_0(\underset{\sim}{A}) = p_1(\underset{\sim}{A}) = 1$, $p_k(\underset{\sim}{A}) = mk!$, and $p_n(\underset{\sim}{A}) = 0$, otherwise.*

Proof. Put $U = \{0, 1, \ldots, k+m\}$, $f_i(1, 2, \ldots, k) = k+i$ (for $i = 1, 2, \ldots, m$), and $f_i x_1 \ldots x_k = 0$, otherwise. We show that the clone $\underset{\sim}{A} = [f_1, \ldots, f_m]$ is as required.

Indeed, since $1, 2, \ldots, k$ are not among images under the functions f_i, any substitution, and also obviously, any identification of variables, yields a constant function equal to 0. The only nonconstant functions other than projections are the functions $f_i x_1 \ldots x_k$ for $i = 1, \ldots, m$ and those obtained by permuting of variables. Moreover, all these functions, obviously, are pairwise distinct. From these facts the lemma easily follows.

5. Proofs of theorems. Now we prove our result stated in § 3.

Proof of Theorem 1. Let $\alpha = \langle a_1, \ldots, a_n \rangle$ and $\beta = \langle b_1, \ldots, b_s \rangle$. Without loss of generality we can assume that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$ and $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_s$.

(i) If $\alpha^2 \leqslant \beta$ does not hold, then there exists m such that $a_m > b_m$ and $a_m \geqslant 2$ $(m \leqslant \min\{n, s\})$. Denote $a_m = k$, and let $\gamma = \langle k, \ldots, k \rangle$ where k occurs m times. Since $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_m = k$, $\gamma \leqslant \alpha$. By Lemma 6 there exists a clone $\underset{\sim}{A}$ in $T(\gamma)$, on a finite universe, such that $p_0(\underset{\sim}{A}) = p_1(\underset{\sim}{A}) = 1$, $p_k(\underset{\sim}{A}) = mk!$, and $p_i(\underset{\sim}{A}) = 0$, otherwise. Since $m \geqslant 1$, $k \geqslant 2$, the clone $\underset{\sim}{A}$ is essential. By Lemma 1, also $\underset{\sim}{A}$ belongs to $T(\alpha)$. We show that there exists no clone $\underset{\sim}{B}$ in $T(\beta)$ with the same p_n-sequence.

Assume the contrary, i.e. that there is $\underset{\sim}{B} = [f_1, \ldots, f_s]$, where $f_1, \ldots, f_s$ are b_1-, ..., b_s-ary, respectively, and $p_i(\underset{\sim}{B}) = p_i(\underset{\sim}{A})$ for all $i \geqslant 0$. Since $k = a_m > b_m \geqslant \ldots \geqslant b_s$, the number m_0 of functions in the set $f_1, \ldots, f_s$ depending on not less than k variables is less than m. Whence, by Lemma 5, $p_k(\underset{\sim}{B}) \leqslant m_0 k! < mk! = p_k(\underset{\sim}{A})$, a contradiction.

(ii) By the assumption $s < n$. Since $\alpha \neq \langle 0, \ldots, 0 \rangle$, there exists $\gamma = \langle 1, \ldots, 1, 0, \ldots, 0 \rangle$, where 1 occurs $m \geqslant 1$ and 0 occurs $k \geqslant 0$ times, and $m+k = n$, such that $\gamma \leqslant \alpha$. By Lemma 4 there exists a clone $\underset{\sim}{A}$ in $T(\gamma)$, on a finite universe, such that $p_1(\underset{\sim}{A}) = 2$, and $p_0(\underset{\sim}{A}) = 2(n-1)+1$. Since $p_1(\underset{\sim}{A}) > 1$, $\underset{\sim}{A}$ is essential. By Lemma 1, $\underset{\sim}{A}$ belongs to $T(\alpha)$. Now, similarly as in the previous case, we show that there exists no clone $\underset{\sim}{B}$ in $T(\beta)$ with the same p_n-sequence. Here, we use the fact that the length of β is less than n, and apply Lemma 3. This completes the proof.

Proof of Theorem 2. If (i) and (ii) are satisfied, then $T(\alpha) \subseteq T(\beta)$ by Lemma 1.

If (i) does not hold, then $T(\alpha) \subseteq T(\beta)$ does not hold either (cf. § 2). It remains to show that if (ii) does not hold, then neither $T(\alpha) \subseteq T(\beta)$ holds.

To this end, let α and β be as in the proof of Theorem 1. First, we consider the case when $\alpha \neq \langle 0, 0, \ldots, 0\rangle$. Then, in view of Theorem 1 we can assume that $\alpha^2 \leqslant \beta$ and $n \leqslant s$. From the fact that (ii) (i.e. $\alpha \leqslant \beta$) does not hold, it follows that there exists $m \leqslant n$ such that $a_m > b_m$ and $a_m < 2$ (otherwise, $\alpha^2 \leqslant \beta$ does not hold). Consequently, $a_m = 1$, and $b_m = b_{m+1} = \ldots = b_s = 0$.

We show that there exists a clone $\underset{\sim}{A}$ in $T(\alpha)$, not in $T(\beta)$. Let for $i = 1, \ldots, m$, f_i be an unary function on the set $\{0, 1, \ldots, m+1\}$ defined as follows: $f_i i = m+1$, and $f_i x = 0$, otherwise. Then, for any $i, j, f_i f_j x = 0$. Now, the clone $\underset{\sim}{A} = [f_1, \ldots, f_m]$, by Lemma 1, is in $T(\alpha)$, but not in $T(\beta)$, since $p_1(\underset{\sim}{A}) = m+1$ and $\underset{\sim}{A}$ cannot be generated by less than m nonconstant functions. Consequently, $T(\alpha) \subseteq T(\beta)$ does not hold, as required.

It remains to consider the case when $a_1 = \ldots = a_n = 0$. Then, (ii) is equivalent to $s < n$. Obviously, if $\underset{\sim}{A}$ is now any clone generated by n distinct constants, then $\underset{\sim}{A}$ is in $T(\alpha)$, but not in $T(\beta)$. This completes the proof.

Proof of Theorem 3. The implications (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii), hold obviously by definitions, for any α and β. By Theorem 1, if $RE(\alpha, \beta)$ holds, then the length of β is not less than that of α, and $\alpha^2 \leqslant \beta$. Since β is without zeros, this means that actually $\alpha \leqslant \beta$, thus proving (iii)$\Rightarrow$(iv). Finally, for β without zeros the condition (ii) in Theorem 2 trivially holds, and consequently (iv)$\Rightarrow$(i), completing the proof of Theorem 3.

Proof of Theorem 4. It is easy to check that the conditions (i$-$v) are necessary for representability of the sequence $\langle p_0, p_1, p_2, \ldots\rangle$ by a clone $\underset{\sim}{A}$ belonging to $T(\alpha)$: (i) for $\underset{\sim}{A}$ is an unary-nullary clone, (ii) for the unary projection is in $\underset{\sim}{A}$, and $\underset{\sim}{A}$ is generated by a finite set of functions, (iii) for if $a_i = 0$ for all i, then $\underset{\sim}{A}$ is a nullary clone, (iv) obviously, and (v) by virtue of Lemma 3.

Conversely, if (i$-$v) are satisfied and $a_i = 1$ for some i, then $\langle p_0, p_1, p_2, \ldots\rangle$ is represented by a clone in $T(\alpha)$ in view of Lemma 4 (and Remark following it). To complete the proof, observe that if $a_i = 0$ for all i, i.e. α is the sequence of n zeros, then obviously each sequence $\langle p_0, 1, 0, 0, \ldots\rangle$ with $p_0 \leqslant n$ is represented by a clone in $T(\alpha)$.

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