Domination by Borel stopping times and some separation properties

by

Ashok Maitra (Minneapolis), **Victor Pestien** (Coral Gables)
and **S. Ramakrishnan** (Coral Gables)

Abstract. Given stopping times t and s such that $t \leqslant s$, does there exist a Borel-measurable stopping time r such that $t \leqslant r \leqslant s$? If t is upper analytic and s is lower analytic, the answer is affirmative. Moreover, if t is everywhere-finite, r can be chosen to be everywhere-finite. (However, this is not true if t is lower analytic.) These results are used to prove certain separation principles.

1. Introduction. Our interest in the problems studied in this paper arose as a by-product of an investigation of a question in the theory of measurable gambling. However, the results presented here are purely set-theoretic in nature. This paper can be read without any knowledge of the theory of gambling and may be of interest to those interested in stopping times or separation theorems. Applications to the theory of gambling will appear in a subsequent paper.

In the mathematical theory of gambling, developed by Dubins and Savage [4], the *fortune space* F, an arbitrary non-empty set, is equipped with the discrete topology, and the *history space* H, the product of countably many copies of F, is equipped with the Cartesian-product topology. A *stopping time* is a function t from H to the set of positive integers together with ∞ such that if h' and h are histories which agree through the first $t(h)$ coordinates, and $t(h) < \infty$, then $t(h') = t(h)$. If a stopping time t is everywhere-finite, then t is a *stop rule*.

A natural example of a stopping time is the hitting time of A, where $A \subseteq F$, defined by

$$t(h) = \min\{n \geqslant 1\colon \text{ the } n\text{th coordinate of } h \text{ lies in } A\}.$$

(The minimum of the empty set is taken to be $+\infty$). Obviously, if A is a proper subset of F, then t is a stopping time which is not a stop rule. A stop rule important in optimal stopping theory is $\min\{t, k\}$, where t is a stopping time and k is a positive integer.

Key words: stopping time, separation, Borel-measurable.
American Mathematical Society 1980 subject classification. Primary: 03A15, 54H05; Secondary: 60G40.

In the theory of gambling, a stopping time of interest is one induced by a proper open subset V of H. This stopping time, denoted by t_V, is defined by

$$t_V(h) = \min\{n \geq 1\colon \text{ for all } h' \text{ which agree with } h \text{ through the first } n \text{ coordinates, we have } h' \in V\}.$$

Loosely speaking, t_V stops along a path at the first instant where one is guaranteed that any possible continuation of that path remains in V. Since V is open in the product of discrete topologies, it is easy to see that

$$V = [t_V < \infty].$$

In this theory, it is often necessary to associate, with a given non-empty proper clopen subset K of H, the time at which a history's membership in K or K^c is "determined" (see [4], Section 2.7). This stopping time (which is in fact a stop rule) is $\min\{t_K, t_{K^c}\}$.

In the theory of "measurable" gambling (see, for example, Strauch [12] or Dellacherie and Meyer [3]), the fortune space F is given a further topological structure. with F metrizable as a complete, separable metric space (*Polish space*). In such a framework, it is desirable to deal with stopping times which are "Borel-measurable", that is, measurable with respect to the product of Polish topologies.

However, even if a set V, open in the product of discrete topologies, is Borel (in the product of Polish topologies), the stopping time t_V need not be Borel-measurable. Indeed, for each positive integer n,

(1.1) $$[t_V > n] = \{(h_1, h_2, \ldots) \in H\colon (h_1, h_2, \ldots, h_n) \in \pi_n(V^c)\}$$

where π_n denotes projection on the first n coordinates. It is well-known that the projection of a Borel set need not be Borel. A natural question then is whether there is a Borel-measurable stopping time s_V such that

$$V = [s_V < \infty].$$

It can be verified that such a stopping time must necessarily satisfy $s_V \geq t_V$.

In this spirit, we investigate (Section 2) the question of domination by Borel-measurable stopping times. Our main theorem (Theorem 2.1) implies that for any upper analytic stopping time and any larger lower analytic stopping time, there is a Borel-measurable stopping time which lies between the two. (Terminology is defined in Section 2.) It is a consequence of this theorem that the stopping time s_V, as desired above, exists. Further, any upper analytic stop rule is dominated by a Borel-measurable stop rule. However, it is also shown that there exist lower-analytic stop rules which are not dominated by any Borel stop rule. In Section 3, we use the stopping time theorem of Section 2 to obtain separation properties which relate classes of sets associated with the two topologies on the history space H. In Section 4, we analyse the structure of clopen Borel sets and use set-theoretic notation and machinery to establish an "effective" version of a separation principle from Section 3.

2. Domination by Borel stopping times. A subset A of a Polish space Ω is *analytic* if there exists a Polish space P and a continuous function $f\colon P \to \Omega$ such that $f(P) = A$. A set is *coanalytic* if its complement is analytic. (For an exposition of the theory of analytic sets, see [2].) If A and B are disjoint subsets of Ω, then the set C *separates* A from B if $A \subseteq C$ and $B \cap C = \emptyset$.

Now let F be a Polish space, N be the set of positive integers, and let $H = F^N$. Endow H with the product of Polish topologies and also with the product of discrete topologies. From now on, the term "analytic (coanalytic, Borel) subset of H" refers to the product of Polish topologies on F.

Suppose t is a mapping from H to $N \cup \{\infty\}$. If for each $n \in N$, the set $[t \geq n]$ is analytic, then t is *upper analytic*. If for each $n \in N$, $[t \leq n]$ is analytic, then t is *lower analytic*. Clearly, if t is both lower analytic and upper analytic, then t is Borel-measurable.

THEOREM 2.1. *Suppose t is an upper analytic stopping time, s is a lower analytic stopping time, and that $t \leq s$. Assume further that D is a coanalytic subset of H such that*

$$[t = \infty] \subseteq D.$$

Then there exists a Borel-measurable stopping time r such that

$$t \leq r \leq s$$

and

$$[t = \infty] \subseteq [r = \infty] \subseteq D.$$

To prove Theorem 2.1, we shall use two standard separation results which we state here for ease of reference.

LEMMA 2.2 (Kuratowski [5], page 485). *If A and B are disjoint analytic subsets of a Polish space Ω, then there is a Borel set which separates A from B.*

LEMMA 2.3 (Kuratowski [5], page 511). *If $\{A_n\colon n \geq 1\}$ is a sequence of analytic subsets of a Polish space Ω and if D is a coanalytic subset of Ω such that*

$$\bigcap_{n=1}^{\infty} A_n \subseteq D,$$

then there exists a sequence $\{C_n\colon n \geq 1\}$ of coanalytic subsets of Ω such that for each n,

$$A_n \subseteq C_n$$

and such that

$$\bigcap_{n=1}^{\infty} C_n = D.$$

Proof of Theorem 2.1. For $n \geq 1$, let $T_n = [t > n]$ and $S_n = [s > n]$. Since $t \leq s$, we have $T_n \subseteq S_n$ for each $n \geq 1$. Apply Lemma 2.3 to get a sequence $\{C_n\colon n \geq 1\}$ of coanalytic subsets of H such that

$$(\forall n)(T_n \subseteq C_n) \quad \text{and} \quad \bigcap_{n=1}^{\infty} C_n = D.$$

For each n, let

$$P_n = \{\pi_n[(S_n \cap C_n)^c]\}^c,$$

where π_n denotes projection on the first n coordinates. Then P_n is a coanalytic subset of F^n, and $P_n \times H \subseteq S_n \cap C_n$. The set $P_n \times H$ is identified with a subset of H in the obvious way. (Throughout the paper, we find this abuse of notation convenient.) Since $T_n \subseteq S_n \cap C_n$ for each n, and since T_n depends only on the first n coordinates, we have

$$T_n \subseteq P_n \times H \subseteq S_n \cap C_n.$$

We now construct, inductively, a sequence $\{R_n\colon n \geqslant 1\}$ where for each n, R_n is a Borel subset of F^n, as follows. Begin by using Lemma 2.2 to obtain a Borel subset R_1 of F such that

$$\pi_1(T_1) \subseteq R_1 \subseteq P_1.$$

For each $n > 1$, use Lemma 2.2 to find a Borel subset R_n of F^n such that

$$\pi_n(T_n) \subseteq R_n \subseteq P_n \cap (R_{n-1} \times F).$$

Then $\{R_n \times H\colon n \geqslant 1\}$ is a decreasing sequence of Borel subsets of H, and it is easily checked that for each n,

$$T_n = \pi_n(T_n) \times H \subseteq R_n \times H \subseteq P_n \times H \subseteq S_n \cap C_n, \tag{2.1}$$

and

$$\bigcap_{n=1}^{\infty} T_n \subseteq \bigcap_{n=1}^{\infty} (R_n \times H) \subseteq \bigcap_{n=1}^{\infty} C_n = D. \tag{2.2}$$

Now we can define a Borel stopping time r so that for each n, $[r > n] = R_n \times H$. Then by (2.1), for each n,

$$[t > n] \subseteq [r > n] \subseteq [s > n],$$

and so $t \leqslant r \leqslant s$. Furthermore, by (2.2),

$$[t = \infty] \subseteq [r = \infty] \subseteq D. \ ■$$

COROLLARY 2.4. *If t is an upper analytic stopping time, and $[t = \infty]$ is Borel, then there is a Borel-measurable stopping time r such that $t \leqslant r$ and $[t = \infty] = [r = \infty]$.*

Proof. In Theorem 2.1, let $D = [t = \infty]$ and $s \equiv \infty$. ■

COROLLARY 2.5. *Let t be any upper analytic stop rule. There is a Borel-measurable stop rule r such that $t \leqslant r$. More generally, if s is a lower analytic stopping time such that $t \leqslant s$, then there is a Borel-measurable stop rule r such that $t \leqslant r \leqslant s$.*

Proof. In Theorem 2.1, take D to be the empty set. ■

Remark. In Corollary 2.5, "upper analytic" and "lower analytic" cannot be interchanged. To see this, let F be an uncountable Polish space. Choose disjoint

coanalytic subsets C and D of F such that C cannot be separated from D by a Borel subset of F. (The existence of such C and D was shown by Luzin [9].) Define a lower analytic stop rule t by

$$t(h_1, h_2, \ldots) = \begin{cases} 1 & \text{if } h_1 \notin D, \\ 2 & \text{if } h_1 \in D \end{cases}$$

and define an upper analytic stop rule s by

$$s(h_1, h_2, \ldots) = \begin{cases} & \text{if } h_1 \in C, \\ 2 & \text{if } h_1 \notin C. \end{cases}$$

Since C and D are disjoint, $t \leqslant s$. Towards a contradiction, assume that there is a Borel measurable stop rule r such that $t \leqslant r \leqslant s$. Let A be the projection of the set $[r = 1]$ to the first coordinate. Clearly A is a Borel subset of F. But, since $t \leqslant r \leqslant s$, it follows that $C \subseteq A \subseteq D^c$, which contradicts the assumption that C and D are not Borel separable.

We now apply Corollary 2.5 to obtain a measurable analogue of a result of Purves and Sudderth ([11], Lemma 5.1).

PROPOSITION 2.6. *Suppose $O_1 \subseteq O_2 \subseteq \ldots$ are coanalytic subsets of H which are open in the product of discrete topologies and which satisfy $\bigcup_{k \geqslant 1} O_k = H$. Then there is a Borel-measurable stop rule r such that $O_r = H$, where $O_r = \{h \in H\colon h \in O_{r(h)}\}$.*

Proof. Let

$$s = \inf_{k \geqslant 1} \{\max(t_{O_k}, k)\},$$

where t_{O_k} is as defined in Section 1. Plainly, s is a stopping time and is everywhere finite since $\bigcup_{k \geqslant 1} O_k = H$. Because each O_k is coanalytic, it follows from (1.1) that t_{O_k} is upper analytic. Therefore s is an upper analytic stop rule. By Corollary 2.5, there is a Borel stop rule r such that $r \geqslant s$. It is straightforward to check, using the definitions of t_{O_k} and the increasing nature of the O_k's, that $O_r = H$. ■

We have shown so far in this section that any *upper* analytic stop rule is dominated by some Borel-measurable stop rule. In Theorem 2.7 below, we shall give an example of a *lower*-analytic stop rule which *cannot* be dominated by any Borel-measurable stop rule. The lower analytic stop rule constructed in the proof of Theorem 2.9 also has this property.

THEOREM 2.7. *There is a Polish space F and a lower analytic stop rule τ on $H = F^N$ such that there is no Borel stop rule σ on H with $\tau \leqslant \sigma$.*

Proof. Take F to be any uncountable Polish space and let $A_1 \subseteq A_2 \subseteq \ldots$ be a sequence of analytic sets such that $\bigcup_{i=1}^{\infty} A_i = F$, but for every sequence of Borel sets $B_i \subseteq A_i$, $\bigcup_{i=1}^{\infty} B_i \neq F$. The existence of such a sequence of analytic sets is a classical result of Liapunov [6].

We define a stop rule τ on $H = F^N$ as follows:

$$\tau(h_1, h_2, \ldots) = \text{least } j \text{ such that } h_1 \in A_j.$$

Plainly, τ is a lower analytic stop rule. Towards a contradiction, assume that there is a Borel measurable stop rule $\sigma \geqslant \tau$. Let

$$B_i = \{x \in F\colon \sigma(x, x, x, \ldots) \leqslant i\}, \quad i \geqslant 1.$$

Then B_i is Borel, and $B_i \subseteq A_i$ because $\tau \leqslant \sigma$ and $\bigcup_{i=1}^{\infty} B_i = F$, contradicting the special property of the A_i's. ■

Remarks. We are indebted to the referee for suggesting that we might obtain an example using Liapunov's result. Other help for our example came from an unpublished construction by Roger Purves and William Sudderth. The argument for Theorem 2.7 actually shows that there cannot be *any Borel function* σ from H to N such that $\sigma \geqslant \tau$.

To formulate the final result of this section, we define "index" of a stop rule (cf. Dellacherie [2]) and establish the existence of a lower analytic stop rule of uncountable index. This is to be contrasted with Corollary IV.21 of [2], which states that any *upper* analytic stop rule has countable index.

For a fortune space F, let $\mathscr{T}(F)$ be the collection of all functions t on F^N such that either t is a stopping time or t is identically zero. (Many authors (e.g. Dellacherie [2]) regard the zero function to be a "stopping time". In that terminology, $\mathscr{T}(F)$ is the collection of all "stopping times".) Let $\mathscr{Q}(F)$ denote the set of those elements of $\mathscr{T}(F)$ which are everywhere-finite. Following [2], define for $t \in \mathscr{Q}(F)$,

$$t^* = \inf\{s\colon s \in \mathscr{Q}(F) \text{ and } s \geqslant t-1\}.$$

(i.e. for each $h \in F^N$, $t^*(h) = \inf\{s(h)\colon s \in \mathscr{Q}(F) \text{ and } s \geqslant t-1\}$.)

It is easily checked that $t^* \in \mathscr{Q}(F)$. Let $t^0 = t$. Then the formula

$$t^\xi = [\inf_{\eta<\xi} t^\eta]^*$$

inductively defines t^ξ for every ordinal ξ. Clearly, if $\xi > \eta$, then $t \geqslant t^\eta \geqslant t^\xi$. For $t \in \mathscr{Q}(F)$, the *index* $j(t)$ is defined to be the least ordinal ξ such that $t^\xi \equiv 0$. For $t \in \mathscr{T}(F)$ where t is not everywhere-finite, the *index* $j(t)$ is defined to be the least infinite ordinal of cardinality greater than the cardinal of F.

For $t \in \mathscr{Q}(F)$ and $x \in F$, let $t[x]$ be the element of $\mathscr{Q}(F)$ defined by

$$t[x](h) = 0 \quad \text{for all } h \in F^N \quad \text{if } t \equiv 0,$$
$$t[x](h) = t(xh)-1 \quad \text{for all } h \in F^N \quad \text{if } t \not\equiv 0.$$

The following relationship will be helpful:

LEMMA 2.8. *For each element t of $\mathscr{Q}(F)$, not identically zero,*

$$j(t) = \sup\{j(t[x])+1\colon x \in F\}.$$

Proof. The verification is routine. See discussion in ([2], III. 13). ■

Remark. It can be shown using Lemma 2.8 that the lower analytic stop rule τ constructed in Theorem 2.7 has countable index.

THEOREM 2.9. There is a Polish space F and a lower analytic stop rule τ on $H = F^N$ such that the index $j(\tau)$ is uncountable.

Proof. Let F be the collection $\mathscr{T}(N)$ of "stopping times" associated with the fortune space N. Give $\mathscr{T}(N)$ the topology of pointwise convergence. By Dellacherie ([2], III. 1), $\mathscr{T}(N)$ is a compact, metrizable space. To define the stop rule τ, let

$$\tau(h) = n+1 \Leftrightarrow [j(h_n)<j(h_{n-1})<\ldots<j(h_1)] \,\&\, [j(h_{n+1}) \geqslant j(h_n)],$$

where $h = (h_1, h_2, \ldots)$. Plainly τ is a stop rule because there is no strictly descending sequence with respect to the relation "$<$". Also, τ is lower analytic because $\tau > 1$, and for $m > 1$,

$$\{h\colon \tau(h) > m\} = \{h\colon j(h_m)<j(h_{m-1})<\ldots<j(h_1)\},$$

and the latter set is coanalytic by Theorem III. 20 of Dellacherie [2].

In the second part of the proof, we show that the index $j(\tau)$ is uncountable. To do so, let

$$\alpha = \sup\{j(r[r])+1\colon \tau \in \mathscr{Q}(N)\}.$$

By Lemma 2.8, $j(\tau) \geqslant \alpha$. If α were countable, we could find a sequence $\{r_n\}$ of elements of $\mathscr{Q}(N)$ such that

$$\alpha = \sup\{j(\tau[r_n])+1\colon n \geqslant 1\}.$$

However, given any sequence $\{r_n\}$ of elements of $\mathscr{Q}(N)$, there exists r in $\mathscr{Q}(N)$ such that

$$j(\tau[r])+1 > \sup\{j(\tau[r_n])+1\colon n \geqslant 1\}. \tag{2.3}$$

To verify this, given a sequence $\{r_n\}$ from $\mathscr{Q}(N)$, choose $r \in \mathscr{Q}(N)$ such that $j(r) > j(r_n)$ for each n. It is straightforward to check from the definition of τ that $\tau[r][r_n] = \tau[r_n]$ for all $n \geqslant 1$. Then Lemma 2.8 implies that for each n,

$$j(\tau[r]) \geqslant j(\tau[r][r_n])+1 = j(\tau[r_n])+1.$$

Therefore (2.3) holds and α is uncountable. Thus the index $j(\tau)$ is uncountable. ■

3. Some separation properties. Define the following classes of subsets of the history space H:

$$\mathscr{B} = \{A\colon A \text{ is Borel in the product of Polish topologies}\},$$
$$\Sigma_0 = \{A\colon A \text{ is clopen in the product of discrete topologies}\},$$
$$\Sigma_0^* = \Sigma_0 \cap \mathscr{B},$$
$$\Sigma_1 = \{A\colon A \text{ is a countable union of elements of } \Sigma_0\},$$
$$\Sigma_1^* = \{A\colon A \text{ is a countable union of elements of } \Sigma_0^*\},$$
$$\Pi_1 = \{A\colon \text{the complement of } A \text{ belongs to } \Sigma_1\}.$$

Remark. Σ_1 is actually the collection of all open sets in the product of discrete topologies, because for any such set V,

$$V = \bigcup_{n=1}^{\infty} [t_V = n].$$

THEOREM 3.1. *Let A and B be disjoint analytic subsets of H, and assume there is a set in Σ_1 which separates A from B. Then there is a set in Σ_1^* which separates A from B.*

Proof. Suppose $G \in \Sigma_1$ and that G separates A from B. That is, $A \subseteq G \subseteq B^c$. Notice that

$$\bigcap_{n=1}^{\infty} ([\pi_n(B)] \times H)$$

is the closure of B in the product of discrete topologies. Since G^c is closed in this topology and since $G^c \supseteq B$, we have

$$A^c \supseteq G^c \supseteq \bigcap_{n=1}^{\infty} ([\pi_n(B)] \times H) \supseteq B. \tag{3.1}$$

Define a stopping time t such that for each n,

$$[t > n] = [\pi_n(B)] \times H.$$

Then t is upper analytic, and by (3.1), $A^c \supseteq [t = \infty]$. By Theorem 2.1, there is a Borel-measurable stopping time r such that $t \leqslant r$ and

$$B \subseteq [t = \infty] \subseteq [r = \infty] \subseteq A^c.$$

Let

$$C = [r < \infty] = \bigcup_{n=1}^{\infty} [r \leqslant n].$$

Then $C \in \Sigma_1^*$ and C separates A from B. ■

COROLLARY 3.2. $\Sigma_1 \cap \mathscr{B} = \Sigma_1^*$.

Proof. It is immediate that $\Sigma_1 \cap \mathscr{B} \supseteq \Sigma_1^*$. For the opposite inclusion, let $A \in \Sigma_1 \cap \mathscr{B}$ and, in the statement of Theorem 3.1, take B to be the complement of A. Then $A \in \Sigma_1^*$. ■

Because of Corollary 3.2 and the remark preceding Theorem 3.1, Σ_1^* is precisely the collection of open sets in the product of discrete topologies which are Borel in the product of Polish topologies.

COROLLARY 3.3 (reduction property for $\Sigma_1 \cap \mathscr{B}$). *If A and B belong to $\Sigma_1 \cap \mathscr{B}$, then there exist disjoint sets A' and B' in $\Sigma_1 \cap \mathscr{B}$ such that $A' \subseteq A$, $B' \subseteq B$, and $A \cup B = A' \cup B'$.*

Proof. By Corollary 3.2, $\Sigma_1 \cap \mathscr{B} = \Sigma_1^*$, and so any $A \in \Sigma_1 \cap \mathscr{B}$ and any $B \in \Sigma_1 \cap \mathscr{B}$ can be written in the form

$$A = \bigcup_{k=1}^{\infty} A_k \quad \text{and} \quad B = \bigcup_{k=1}^{\infty} B_k,$$

where $\{A_k\}$ and $\{B_k\}$ are sequences of elements from the field Σ_0^*. Let

$$A_1' = A_1, \quad B_1' = B_1 \cap A_1^c,$$

and for $n > 1$,

$$A_n' = A_n \cap \left(\bigcap_{k=1}^{n-1} A_k^c\right) \cap \left(\bigcap_{j=1}^{n-1} B_j^c\right)$$

and

$$B_n' = B_n \cap \left(\bigcap_{k=1}^{n} A_k^c\right) \cap \left(\bigcap_{j=1}^{n-1} B_j^c\right).$$

Now let

$$A' = \bigcup_{k=1}^{\infty} A_k' \quad \text{and} \quad B' = \bigcup_{k=1}^{\infty} B_k',$$

and check that A' and B' have the required properties. ■

COROLLARY 3.4 (separation property for $\Pi_1 \cap \mathscr{B}$). *If A and B belong to $\Pi_1 \cap \mathscr{B}$ and A and B are disjoint, then there is a set C in Σ_0^* which separates A from B.*

Proof. Take complements in Corollary 3.3. ■

THEOREM 3.5. *Let A and B be disjoint analytic subsets of H. Assume there is a set in Σ_0 which separates A from B. Then there is a set in Σ_0^* which separates A from B.*

Proof. First observe that if a subset A of H is analytic, then its closure $\bar{A}$ in the product of discrete topologies is also analytic because

$$\bar{A} = \bigcap_{n=1}^{\infty} ([\pi_n(A)] \times H).$$

Suppose, now, that A and B are analytic subsets of H and A can separated from B by a set in Σ_0. It follows that $\bar{A}$ and $\bar{B}$ are disjoint. So, by Theorem 3.1, there exists A_1 in $\Pi_1 \cap \mathscr{B}$ such that $\bar{A} \subseteq A_1$ and A_1 and $\bar{B}$ are disjoint. Again using Theorem 3.1, there exists B_1 in $\Pi_1 \cap \mathscr{B}$ such that $\bar{B} \subseteq B_1$ and A_1 and B_1 are disjoint. Hence, by Corollary 3.4, there exists $K \in \Sigma_0^*$ such that $A_1 \subseteq K \subseteq B_1^c$. Consequently, K separates A from B. ■

4. Structure, separation, and the effective theory. In Dubins and Savage [4], the notions of "finitary function" and of "structure" of a finitary function have been defined. A function from F^N into a set R is *finitary* if it is continuous when both F and R are given the discrete topology. It is a straightforward consequence of the discussion in Section 2.7 of [4] that for any non-constant finitary function g and $x \in F$, if g^x is the finitary function defined by

$$g^x(h) = g(xh),$$

then the structure of g satisfies

$$str(g) = \sup\{str(g^x)+1 \colon x \in F\}. \tag{4.1}$$

The structure of a constant function is defined to be 0. If t is a stop rule, not identically zero, let π_t denote the finitary function defined by

$$\pi_t(h) = \text{the } t(h)\text{th coordinate of } h .$$

The following theorem establishes the relationship between index and structure.

Proposition 4.1. *If the fortune space F contains at least two points, then for every stop rule t, not identically zero,*

$$j(t) = str(\pi_t).$$

Proof. First note that the result is true for constant t. In view of (4.1) and Lemma 2.7, it is enough to prove that for each $x \in F$,

$$[\pi_t]^x = \pi_{t[x]},$$

for then the result will follow by transfinite induction. Now for $h \in H$,

$$\begin{aligned}[\pi_t]^x(h) &= \pi_t(xh)\\ &= \text{the } t(xh)\text{th coordinate of } xh\\ &= \text{the } t[x](h)\text{th coordinate of } h\\ &= \pi_{t[x]}(h). \blacksquare\end{aligned}$$

We are grateful to William Sudderth for pointing out that it is necessary in Proposition 4.1 to assume F has at least two points.

If K belongs to Σ_0 (notation as in Section 3), it is easy to see that the indicator of K is a finitary function. Thus we define the *structure* of K as the structure of its indicator function. Proposition 4.1 can be combined with a result of Dellacherie regarding the index of upper analytic stop rules to show the following result about structure of sets in Σ_0^* (notation as in Section 3):

Proposition 4.2. *Let F be a Polish space. If K belongs to Σ_0^*, then the structure of K is countable.*

Proof. It is easily seen that the structure of K is the same as the structure of the function π_s, where s is the stop rule given by

$$s = \min\{t_K, t_{K^c}\}.$$

(The notation "t_K, t_{K^c}" was introduced in Section 1.) Since K is Borel, the stop rule s is upper analytic, and by Dellacherie ([2], Corollary IV.21), any upper analytic stop rule has countable index. Therefore, with the aid of Proposition 4.1, we have

$$str(K) = str(\pi_s) = j(s) < \omega_1,$$

where ω_1 is the first uncountable ordinal. ■

Let ω be the set of non-negative integers with the discrete topology, and let $F = \omega^\omega$, the set of all infinite sequences of non-negative integers, with the product topology. It can be seen that F is a Polish space which is homeomorphic to the

set of irrational numbers, considered as a subspace of the reals. (For details, see [10]). Also, let $H = F^N$. As before, a subset of H will be called "clopen" if it is clopen in the product of discrete topologies on F, and "Borel" if it is Borel in the product of the usual Polish topologies of F. In this section, the notations Σ_1^1, Π_1^1, and Δ_1^1 (cf. Moschovakis [10]) will mean these concepts in the product of the usual Polish topologies on F. The symbols $\alpha, \beta, \gamma, \delta$ will denote elements of F, while $x, y, z, \ldots$, will denote elements of H. If $x \in H$, $y \in H$, and $n \in \omega$, we write $x \equiv_n y$ if $x_i = y_i$ for all i, $1 \leqslant i \leqslant n$. For a discussion of "effective" methods in set theory, see Moschovakis [10]. The approach in this section was used by Louveau in [7].

Let (W, C) be a *coding* of Borel subsets of H; that is, W and C satisfy the following properties:

(a) W is a Π_1^1 subset of $\omega^\omega \times \omega$;
(b) C is a Π_1^1 subset of $\omega^\omega \times \omega \times H$ whose projection on $\omega^\omega \times \omega$ is W;
(c) the set $\{(\alpha, n, x) \in \omega^\omega \times \omega \times H\colon (\alpha, n) \in W \,\&\, (\alpha, n, x) \notin C\}$ is Π_1^1;
(d) for fixed $(\alpha, n) \in \omega^\omega \times \omega$, the section $C_{\alpha,n} = \{x \in H\colon (\alpha, n, x) \in C\}$ is $\Delta_1^1(\alpha)$;
(e) if $P \subseteq H$ is $\Delta_1^1(\alpha)$, then there is n such that $(\alpha, n) \in W$ and $P = C_{\alpha,n}$.

Define $W^* \subseteq \omega^\omega \times \omega$ as follows:

$$\begin{aligned}(\alpha, n) \in W^* \leftrightarrow (\alpha, n) \in W \,\&\\ (\forall x)[(\alpha, n, x) \notin C \vee (\exists m)(\forall y)((y \equiv_m x) \rightarrow (\alpha, n, y) \in C)] \,\&\\ (\forall x)[(\alpha, n, x) \in C \vee (\exists m)(\forall y)((y \equiv_m x) \rightarrow (\alpha, n, y) \notin C)].\end{aligned}$$

Plainly W^* is Π_1^1 and if $(\alpha, n) \in W$, then $(\alpha, n) \in W^*$ if and only if $C_{\alpha,n}$ is clopen. Set

$$C^* = C \cap (W^* \times H).$$

Then C^* is Π_1^1, and we observe that the set $(W^* \times H) \cap (C^*)^c$ is also Π_1^1. Let

$$W_\xi^* = \{(\alpha, n) \in W^*\colon str(C_{\alpha,n}^*) \leqslant \xi\}.$$

We wish to show that these sets W_ξ^* have pleasant definability properties as well. Towards this, define an inductive operator φ on the power set of $\omega^\omega \times \omega$ thus:

$$\begin{aligned}(\alpha, n) \in \varphi(S) \leftrightarrow (\alpha, n) \in W^* \,\&\\ \big[C_{\alpha,n}^* = H \vee C_{\alpha,n}^* = \emptyset \vee (\forall \beta)(\exists m)\big((\langle\alpha, \beta\rangle, m) \in S \,\&\\ (\forall y)((\alpha, n, \beta y) \in C^* \leftrightarrow (\langle\alpha, \beta\rangle, m, y) \in C^*)\big)\big]\end{aligned}$$

where $\langle\alpha, \beta\rangle\colon \omega \rightarrow \omega$ is defined by

$$\langle\alpha, \beta\rangle(m) = 2^{\alpha(m)}(2\beta(m)+1) -$$

Define the iterates of φ by transfinite induction:

$$\varphi^\xi = \varphi\Big(\bigcup_{\eta<\xi} \varphi^\eta\Big).$$

As in Louveau [7], a straightforward induction on ξ establishes the following:

LEMMA 4.3. *For each ordinal* ξ, $\varphi^{\xi} = W_{\xi}^{*}$.

It follows immediately from the definition that, regarded as a relation in n, α, and S, φ is a positive Π_1^1 set relation. Consequently, the set

$$Q = \{(\alpha, \gamma, n) \in \omega^{\omega} \times \omega^{\omega} \times \omega\colon \gamma \in \mathrm{WO} \,\&\, (\alpha, n) \in \varphi^{|\gamma|}\}$$

is Π_1^1. Here WO is the set of ordinal codes, and if $\delta \in \mathrm{WO}$, then $|\delta|$ is the ordinal coded by δ.

Now assume $\xi < \omega_1^{\beta}$(= the least non-recursive in β ordinal). Let $\delta \in \mathrm{WO}$ be such that $|\delta| = \xi$ and δ is recursive in β. Then it follows from Lemma 4.3 that

$$(\alpha, n) \in W_{\xi}^{*} \leftrightarrow (\alpha, \delta, n) \in Q$$

and

$$(\alpha, n) \in \bigcup_{\eta<\xi} W_{\eta}^{*} \leftrightarrow (\exists n)((\alpha, n, \psi(\delta, n)) \in Q)$$

where ψ is a recursive function such that if $\gamma \in \mathrm{WO}$, then $\psi(\gamma, m)$ codes the initial segment of $|\gamma|$ determined by m.

The following lemma is now immediate.

LEMMA 4.4. If $\xi < \omega_1^{\beta}$, *then* W_{ξ}^{*} *and* $\bigcup_{\eta<\xi} W_{\eta}^{*}$ *are* $\Pi_1^1(\beta)$ *sets.*

The preliminaries being complete, we can state the main result of this section.

THEOREM 4.5. *Let* $\xi < \omega_1^{\alpha}$. *Suppose A and B are* $\Sigma_1^1(\alpha)$ *subsets of H such that A can be separated from B by a clopen set K satisfying* $str(K) \leqslant \xi$. *Then A can be separated from B by a* $\Delta_1^1(\alpha)$ *clopen set* K^{*} *with* $str(K^{*}) \leqslant \xi$.

Proof. We prove the theorem by induction on ξ. If $\xi = 0$, then $K = H$ or $K = \emptyset$ and we can take $K^{*} = K$. Suppose $\xi > 0$ and the result is true for all $\eta < \xi$. By induction hypothesis, for each β, the section $A\beta$ can be separated from $B\beta$ by a $\Delta_1^1(\langle\alpha, \beta\rangle)$ set of structure less than ξ. Define $R \subseteq \omega^{\omega} \times \omega$ as follows:

$$(\beta, n) \in R \leftrightarrow (\langle\alpha, \beta\rangle, n) \in \bigcup_{\eta<\xi} W_{\eta}^{*} \,\&\, [A\beta \subseteq C_{\langle\alpha\beta\rangle n}^{*}] \,\&\, [C_{\langle\alpha,\beta\rangle n}^{*} \cap B\beta = \emptyset]\,.$$

It is easily verified by using Lemma 4.4 that R is a $\Pi_1^1(\alpha)$ set. Moreover, the sentence before the definition of R says that

$$(\forall\beta)(\exists n)((\beta, n) \in R)\,.$$

So by the Easy Uniformization Theorem ([10], Theorem 4B.4), there is a $\Delta_1^1(\alpha)$-recursive function $f\colon \omega^{\omega} \to \omega$ such that $(\beta, f(\beta)) \in R$ for each β. Define

$$K^{*} = \{x \in H\colon (\langle\alpha, x_1\rangle, f(x_1), x^{*}) \in C^{*}\}\,,$$

where $x^{*} = (x_2, x_3, \ldots)$. Then K^{*} has the desired properties. ■

By combining Theorem 4.5 with Theorem 3.5 and Proposition 4.2, we get a refinement of Theorem 3.5 as follows:

COROLLARY 4.6. *Suppose A and B are analytic subsets of H such that A can be separated from B by a clopen set K. Then A can be separated from B by a Borel, clopen set* K^{*} *such that* $str(K^{*}) \leqslant str(K)$.

Remarks. Corollary 4.6 was proved for the case $F = \omega^{\omega}$. However, using the fact that any uncountable Polish space is Borel isomorphic to ω^{ω} and the fact that the Borel isomorphism can be used to induce a homeomorphism between the history spaces when equipped with the product of copies of the discrete topology, it is easy to transfer Corollary 4.6 to the case where F is an arbitrary uncountable Polish space.

The case $F = \omega$ of Theorem 4.5 was treated by Louveau [8] and by Barua [1].

References

[1] R. Barua, *Studies in set-theoretic hierarchies: from Borel sets to R-sets*, Ph. D. dissertation, Indian Statistical Institute, Calcutta 1986.

[2] C. Dellacherie, *Un cours sur les ensembles analytiques*, in *Analytic Sets*, edited by C. A. Rogers *et al.*, 184–316, Academic Press, London 1980.

[3] C. Dellacherie and P.-A. Meyer, *Probabilités et potentiel, chapitres IX à XI*, édition entièrement refondue, Hermann, Paris 1983.

[4] L. Dubins and L. Savage, *Inequalities for Stochastic Processes* (*How to Gamble if You Must*), Dover, New York 1976.

[5] K. Kuratowski, *Topology*, v. 1, Academic Press, New York 1966.

[6] A. A. Liapunov, *Contribution à l'étude de la séparabilité multiple*, Mat. Sb. 1 (43) (1936), 503–510.

[7] A. Louveau, *Ensembles analytiques et boréliens dans les espaces produits*, Astérisque 78 (1980), 1–87.

[8] — *Some results in the Wadge hierarchy of Borel sets*, in Kechris, Martin and Moschovakis, Cabal Seminar 1979–81 (Proceedings Cal. Tech-UCLA Logic Seminar), Springer Lecture Notes in Math. 1019 (1983).

[9] N. Luzin, *Leçons sur les ensembles analytiques et leurs applications*, Gauthier-Villars, Paris 1930.

[10] Y. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam 1980.

[11] R. Purves and W. Sudderth, *Some finitely additive probability*, Ann. Prob. 4 (1976), 259–276.

[12] R. Strauch, *Measurable gambling houses*, Trans. Amer. Math. Soc. 126 (1967), 64–72.

SCHOOL OF STATISTICS
UNIVERSITY OF MINNESOTA
Minneapolis, MN 55455, U.S.A.

DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
UNIVERSITY OF MIAMI
Coral Gables, FL 33124, U.S.A.

Received 22 September 1988;
in revised form 10 May 1989