\[ g(F, E) \cdot g(C_1, D) = g(F, 3) \cdot g(G, E) \cdot g(F^{-1}, 3) \cdot g(C_1, H) = g(C_1, H) \cdot D, \]

i.e., the points \( x, y \) are in the same class. 

I would like to express my gratitude to Zdzislaw Dzedzej who associated the proof of the above proposition with the subject of [1].

The Nielsen theory. In 1983, N. Rallis developed a fixed point index theory for symmetric product maps of compact euclidean neighbourhood retracts (ENRs). Using this index we follow S. Masih [2] and define the Nielsen number of \( f : X \to X \) as the number of essential fixed point classes (a class is an essential class if its index does not equal zero).

Now, let \( X \) be a compact connected ENR space. It follows from the Proposition that there exists at most one fixed point class of \( f : X \to X \). By [4], the index of the unique fixed point class equals the Lefschetz number of \( f \).

**Corollary.** The Nielsen number \( N(f) \) of the symmetric product map \( f : X \to X \) equals 0 if the Lefschetz number equals 0 and equals 1 otherwise.

**Remark.** In 1988, H. Schirmer [5] and [6] proved that if \( X \) is a triangulable manifold of dimension not less than 3, then for any map \( f : X \to X \) there exists a map homotopic to \( f \) having exactly \( N(f) \) fixed points.

I would like to express my gratitude to Professor H. Schirmer and to the referee for their suggestions and to Professor L. Górniiewicz for his kind encouragement during the preparation of this paper.

**References**


1. Preliminary notions. We deal exclusively with subsets of the real line, although all our results are easily transferable to the context of any complete, separable, metric space. A subset \( X \) of \( R \) is **analytic** if it is a continuous image of the space of irrational numbers (or empty). A set \( X \subseteq R \) is **co-analytic** if \( X \) is analytic. Souslin showed that \( X \) is a Borel subset of \( R \) if and only if it is both analytic and co-analytic [7].

Given \( X \subseteq R \), define

\[ B(X) = \{ B \cap X : B \text{ is a Borel subset of } R \}. \]

Then \( B(X) \) is a field whose elements are the **measurable subsets** of \( X \). A function \( f : X \to Y \) is a (Borel) **isomorphism** if it is a one-one correspondence of \( X \) with \( Y \), and \( B \in B(X) \) if and only if \( f(B) \in B(Y) \). If \( X = Y \), then \( f \) is a (Borel) **automorphism**.

If \( f \) is an automorphism of \( X \), define its **support** as

\[ \text{supp}(f) = \{ x \in X : f(x) \neq x \}. \]

The collection \( F(X) \) of all automorphisms of \( X \) forms a group under composition. Note that if \( f \) and \( g \) are automorphisms of \( X \), then \( \text{supp}(g \circ f \circ g^{-1}) = \text{supp}(f) \).

Let \( CO(X) \) be the set of all \( f \in F(X) \) such that \( \text{supp}(f) \) is countable. Then \( CO(X) \) is a normal subgroup of \( F(X) \), and we call the quotient \( G(X) = F(X)/CO(X) \) the **reduced automorphism group** of \( X \). If \( f \in F(X) \), then \( f \) is its coset in \( G(X) \). A set \( X \) is **measurably rigid** if \( G(X) \) is trivial. Uncountable measurably rigid sets exist (ZFC),

**Normal subgroups of measurable automorphisms**

by

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Abstract. Given \( X \subseteq R \), let \( F(X) \) be the group of all Borel-isomorphisms of \( X \) onto itself and let \( CO(X) \) be the normal subgroup comprising all \( f \) with \( (x : f(x) \neq x) \) countable. Then \( G(X) = F(X)/CO(X) \) is the "reduced automorphism group" of \( X \). Using a technique of R. D. Anderson (Lemma 2.1), we explore the normal subgroup structure of \( G(X) \). For certain sets \( X \), there is a simple relationship between this structure and the set of Borel-isomorphism types of measurable subsets of \( X \) (Proposition 2.6). As a consequence of this result, one sees that \( G(X) \) is a simple group when \( X \) is a Borel subset of \( R \) (Corollary 2.8).
but they can contain no uncountable Borel subsets of $\mathbb{R}$. Since every uncountable analytic set contains such a Borel set [7; p. 444], no uncountable analytic set can be measurably rigid. Whether nature will allow the existence of an uncountable, coanalytic, measurably rigid set is not known. We shall return to this question in Proposition 3.3 infra. More detailed information about measurable rigidity and $G(X)$ lies in [2], [5], [10].

Let $X$ be a subset of $\mathbb{R}$. We define the isomorphism type of $X$ by

$$t(X) = \begin{cases} Y \subseteq X \text{ such that } Y \text{ is uncountable and } t(Y) \text{ is countable} \\ 10 \end{cases}$$

Put $S = \{ t(X) \colon X \subseteq \mathbb{R} \}$. Given $t_1$ and $t_2$ in $S$, choose $X_1 \subseteq (0, 1)$ and $X_2 \subseteq (1, 2)$ such that $t_1 = t(X_1)$ and $t_2 = t(X_2)$. Then define

$$t_1 + t_2 = t(X_1 \cup X_2).$$

It is not hard to check that $t_1 + t_2$ is well-defined. Denumerable sums $t_1 + t_2 + \ldots$ are defined analogously. We use the notations

$$nt = t + \ldots + t \text{ (n times)}, \quad a nt = t + t + \ldots$$

Define a relation $\leq$ on $S$ by putting $s \leq t$ whenever $t = s + a$ for some $a$. Then $t(X) \leq t(Y)$ just in case $X$ is isomorphic with a measurable subset of $Y$. A Schröder-Bernstein argument shows that $\leq$ is a partial order on $S$: see [4; Theorem 2.5]. It seems that Tarski was first to notice that such isomorphism types have an algebraic behaviour quite similar to that of cardinal numbers. In fact, we have

1.1. Lemma. The system $(S, +, \leq)$ is a partially ordered commutative monoid with identity element 0. Indeed, $S$ is a cardinal algebra in the sense of Tarski [11].

Indication. The commutative, associative, and identity element postulates are rather immediate. It remains to check the refinement and remainder axioms:

Refinement. Suppose that $a$, $b$, $c$, are elements of $S$ such that $a + b = c_0 + c_1 + \ldots$. We produce elements $a_n$ and $b_n$ such that

$$a = a_0 + a_1 + \ldots, \quad b = b_0 + b_1 + \ldots, \quad a_n + b_n = c_n.$$}

Let $X \subseteq (0, 1)$, $Y \subseteq (1, 2)$ and $Z \subseteq (n, n+1)$ be sets with $a = t(X)$, $b = t(Y)$, and $c = t(Z)$. By hypothesis, there is an isomorphism $f$ mapping $X \cup Y$ onto $Z \cup Z \cup \ldots$. Put $a_n = t(Z_n \cap f(X))$ and $b_n = t(Z_n \cap f(Y))$.

Remainder. Suppose that $a$ and $b$ are elements of $S$ such that $a = b_n + a_{n+1}$ for $n = 0, 1, \ldots$. We exhibit an element $s$ such that $a_s = a + b_s + b_{s+1} + \ldots$ for each $s$.

Let $X_s$ and $Y_s$ be subsets of $\mathbb{R}$ such that

$$X_s = Y_s \cup X_{s+1}, \quad Y_s \cap X_{s+1} = \emptyset,$$

$$X_s \in \mathcal{B}(X_s), \quad X_{s+1} \in \mathcal{B}(X_s),$$

$$t(X_s) = a_s, \quad t(X_{s+1}) = b_s.$$}

Put $s = t(\cap X_s)$. ■

For further details concerning the algebraic structure of $S$, vide [9].

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Given $t \in S$, define $\sec(t) = \{ s \in S \colon s \leq t \}$. There is an analytic subset $U$ of $\mathbb{R}$ such that $X \subseteq R$ is analytic if and only if $t(X) \leq t(U)$. Such are the so-called "universal" analytic sets [7; p. 459]. Given Borel subsets $B_1$ and $B_2$ of $\mathbb{R}$, a well-known result of Kuratowski asserts that $t(B_1) = t(B_2)$ if and only if $B_1$ and $B_2$ have the same cardinality [7; p. 451]. It follows that $\omega t(B) = t(B)$ for each Borel $B \subseteq \mathbb{R}$. Also, $t(B) \leq t(A)$ whenever $A$ and $B$ are uncountable with $A$ analytic and $B$ Borel. Thus there are only two types for Borel sets: $t(B) = 0$ for $B$ countable, and $t(B) = r(B)$ for $B$ uncountable. No such clear structure result is available for analytic sets in general. The postulate that all analytic games are determined is equivalent to the statement that for each analytic set $A$, either $t(A) = 0$ ( $A$ countable), $t(A) = t(R)$ ( $A$ uncountable Borel), $t(A) = t(U)$ (universal type). This is true e.g. if measurable cardinals exist [8]. The situation is quite different under constructibility ($V = L$): see [3].

1.2. Lemma. Let $A \subseteq R$ be analytic and non-Borel. Then $t(A) = t(R) + s$ implies that $s = t(A)$.

Proof. First note [7; p. 444] that every uncountable analytic subset $D$ of $\mathbb{R}$ contains a homeomorph $B$ of the Cantor set. Thus

$$t(D) + t(R) = (t(D-B) + t(B)) + t(R) = t(D-B) + t(B) + t(R) = t(D-B) + t(B) = t(D) .$$

Given $A$ analytic and non-Borel with $t(A) = t(R) + s$, take $B \in \mathcal{B}(A)$ with $t(B) = t(R)$ and $t(A-B) = s$. Then $A-B$ is uncountable and analytic. Thus $t(A) = t(R) + s$.

An isomorphism type $t \in S$ is countably compact if whenever $t_1 \leq t_2 \leq \ldots \leq t$ with $\sup_{t_1} = t$, then $t_1 = t$ for some $N$.

1.3. Lemma. Let $X$ be a subset of $\mathbb{R}$ such that $t(X)$ is countably compact, and suppose that $A_i \subseteq A_j \subseteq \ldots$ are elements of $\mathcal{B}(X)$ whose union is $X$. Then there is some $N$ with $t(A_N) = t(X)$.

This lemma follows immediately from the definition of countably compact types, and to some degree justifies the use of the term.

1.4. Lemma. Let $t \in S$ be countably compact. Then $t = \omega t$.

Proof. Let $X$ be a subset of $\mathbb{R}$ with $t(X) = t$. For countable ordinals $\alpha$, we define real numbers $a(\alpha)$ and the isomorphism types $a_\alpha$ as follows. Given $\alpha$, assume that $a(\beta)$ and $a_\beta$ are defined for $\beta < \alpha$ so that $a(0) = 1$, $a_\alpha = 0$, and

$$0 < a(\beta) < a(\gamma) \quad \text{for} \quad \gamma < \beta.$$}

$$a_\alpha = t((a(\beta), 1) \cap X), \quad a_{\alpha+\gamma} \leq t .$$

These are two cases to consider.

Case 1. $\alpha = \beta + 1$, a successor ordinal.

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In this case, consider the sequence $b_n = (a(\beta) - \frac{1}{n}) \cdot \forall n = 1, 2, 3, \ldots$

For each $n$, put

$$B_n = [0, b_n] \cup [a(\beta), 1) \cdot \forall X.$$  

These sets form an increasing sequence whose union is (within a singleton) $X$. From the general theory of cardinal algebras [11; Theorem 3.19], we know that $r(X) = \sup(\tau(B_n))$. By Lemma 3.3, there is some $n$ with $b_n > 0$ and $t = r(X) = \tau(B_n)$.

Define $a(x) = b_n. 0 < a(x) < a(\beta).$ Since $t = \tau(B_n)$, we have

$$t = t + \tau(X - B_n),$$

whence it follows that $\omega(X - B_n) \leq t$. If $x = \tau((a(x), 1) \cdot \forall X)$, then

$$\omega_x = \omega(\tau(X - B_n) + \omega_x) = [\omega(X - B_n) \cup \omega_x] \leq t$$

as desired. We have used [11; Theorem 4.7].

Case 2. $a$ is a limit ordinal.

In this case, put $a(a) = \inf(a(a): \beta < a)$, and set

$$x_{a} = \tau((a(a), 1) \cdot \forall X) = \sup\{z_{a}: \beta < a\}.$$  

It follows that $\omega_x \leq t$. Note that if $a(a) = 0$, then $x_{a} = t$, the construction halts, and the lemma is proved.

In fact, the countable chain condition forces this construction to halt at some countable limit ordinal $a$. Then $a(a) = 0$, and $t = a$ as desired. ■

1.5. Lemma. Let $t_1$ and $t_2$ be countably compact types in $S$. Then their supremum exists and equals $t_1 \cup t_2 = t_1 + t_2$.

Proof. This follows from Theorem 4.7 of [11]. ■

1.6. Lemma. Let $t \in S$ be such that $\sec(t)$ is finite. Then $t$ is countably compact.

As mentioned above, the number of analytic types is finite if analytic determinacy holds. In this case, every analytic type is countably compact. Under $\text{AD}$, this is not true: $t = a$ does not always obtain [5; Lemma 4.4].

A type $t \in S$ is a cover of $0$ if $\sec(t)$ has exactly two elements. Otherwise said, $\tau(X)$ is a cover of $0$ if and only if $X$ is uncountable and Borel isomorphic with each of its uncountable measurable subsets. Clearly, $\tau(B)$ is a cover of $0$, and each cover of $0$ is countably compact. In [9], a large family of such types was constructed.

These covers of $0$ can be used to construct “successors” to a type $t$. A successor to $t$ is a type $s > t$ such that $s \supseteq u \supseteq t$ implies either $u = s$ or $u = t$. (See [9] for the details.) Every successor type is countably compact (cf. compactness of ordinal spaces).

2. Factorization of automorphisms. The nuts and bolts part of our method is a technique used by R. D. Anderson in [1], whereby he proved that, under certain conditions, a given homeomorphism of a space onto itself could be factored as the composition product of four conjugates of another such homeomorphism (or its inverse). The technique is adapted to the measurable context, and it is used to prove

2.1. Basic Lemma. Let $f$ and $g$ be Borel automorphisms of a subset $X$ of $R$. Suppose that $A_{\alpha_1}, A_{\alpha_2}, \ldots$ is a sequence of disjoint, isomorphic, measurable subsets of some $B \in \mathcal{B}(X)$ such that $\tau(f) X B$ and $r(B) = 0$. Then $f$ is a product (under composition) of two conjugates of $g$ with two conjugates of $g^{-1}$. In particular, $f$ belongs to the normal closure of $g$ in $G(X)$.

Proof. Let $r: X \to X$ be a Borel automorphism such that $\tau(r) X B$ and $r(A_\alpha) = A_{\alpha + 1}$ for each $\alpha$. Now $\tau(s) X B$ implies $r(B) = A_\alpha$ for $\alpha = 0, 1, \ldots$

Define $k: X \to X$ by putting

$$k(x) = \begin{cases} r^{-1}(f(r^{-n}(x))) & \text{for } x \in A_\alpha, \quad (n \geq 0), \\ x & \text{for } x \in X \setminus \bigcup_{n \geq 0} A_\alpha. \end{cases}$$

Then $k$ is an automorphism of $X$ with $\tau(k) \subseteq B$. Hence $\tau(r^{-1} s^{-1} r^{-1}) \subseteq B$, and likewise $\tau(g^{-1} k^{-1} s) \subseteq B$. Thus, each of the pairs

$$r \text{ and } g^{-1} k^{-1} s \text{ and } g^{-1} k^{-1} s^{-1} \text{ and } g^{-1} k^{-1}$$

commutes. We calculate

$$(r g^{-1} k g^{-1} r^{-1})(g^{-1} k^{-1} g) = (r g^{-1} k g^{-1} r^{-1})(g^{-1} k^{-1} g) k = (r g^{-1} k g^{-1} r^{-1})(g^{-1} k^{-1} g) = (r k^{-1} r^{-1}) k = f.$$  

Then

$$f = (r g^{-1} k g^{-1} r^{-1})(g^{-1} k^{-1} g) = (r g^{-1} r^{-1})(g^{-1} k^{-1} g),$$

the product of four conjugates. ■

2.2. Lemma. Suppose that $f$ and $g$ are Borel automorphisms of a subset $X$ of $R$ such that

1. $\tau(\sup(s)) \subseteq (X - \sup(s));$

2. there is some $B \in \mathcal{B}(X)$ with $g(B) \cap B = \emptyset$ and $\tau(B) \supseteq \tau(\sup(s)).$

Then $f$ is in the normal closure of $g$ in $G(X)$.

Proof. Using condition 2, choose disjoint measurable subsets $A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}$ of $B$ such that $\tau(A_\alpha) = \tau(\sup(s))$ for each $\alpha$. By condition 1, there are disjoint measurable sets $C_1, C_2, \ldots$ of $X - \sup(s)$ with $\tau(C_\alpha) = \tau(\sup(s))$ for each $\alpha$. Then

$$\tau(X - \sup(s)) = \tau(\bigcup C_\alpha) + \tau((X - \sup(s)) - \bigcup C_\alpha)$$

$$= \sup(\tau(s)) + \tau((X - \sup(s)) - \bigcup C_\alpha)$$

$$= \tau(s) \cup C_\alpha + \tau((X - \sup(s)) - \bigcup C_\alpha)$$

$$= \tau(C_\alpha) + \tau((X - \sup(s)) - \bigcup C_\alpha)$$

$$= \bigcup A_\alpha + (X - \bigcup A_\alpha) = (X - A_\alpha) .$$
Thus there is an automorphism $h$ of $X$ such that $h(\text{supp}(f)) = A_0$. Then $\text{supp}(h \circ f \circ h^{-1}) = h(\text{supp}(f)) = A_0$. Without loss of generality, therefore, we may and do assume that $\text{supp}(f) \subseteq A_0$. Now apply Lemma 2.1.

In the next group of lemmas, we explore conditions that facilitate the application of these basic results.

2.3. Lemma. Let $g$ be an automorphism of a subset $X$ of $R$ and let $s$ be such that $\text{supp}(g) \supseteq s$, where every $r < s$ is countably compact. Then there is some $B \in \mathcal{B}(X)$ such that $t(B) \ni s$ and $g(B) \cap B = \emptyset$.

Proof. Let $S$ be a measurable subset of $\text{supp}(g)$ with $t(S) = s$. Define $S_n = \{x \in S: |g(x) - x| > 1/n\}$. Then $S$ is the ascending union of the sets $S_n$, so that by Lemma 1.5, $t(S_n) = s$ for some $N$. Define $A_k = [k(N, (k+1)/N) \cap S_n]$ for $k = 0, \pm 1, \pm 2, \ldots$.

noting that $A_k \cap g(A_k) = \emptyset$. Re-index the sets $A_k$ as $B_1, B_2, \ldots$. Define $C_1 = B_1$ and $C_{n+1} = (C_n \cup B_{n+1}) - g(C_n) - g^{-1}(C_n)$ for every $n \geq 1$. Then $C_1 \subseteq C_2 \subseteq \ldots$ and $g(C_1) \cap C_2 = \emptyset$.

Claim. For each $n$, we have $t(C_n) = t(B_1 \cup \ldots \cup B_n)$.

Proof of claim. We induct on $n$. The case for $n = 1$ being trivial, we assume the result for $n$ and establish it for $n+1$. Clearly $t(C_n) \subseteq t(B_1 \cup \ldots \cup B_n)$. Also $C_{n+1} \cup g(C_n) \cup g^{-1}(C_n) = C_n \cup B_{n+1}$, so that $t(C_n) + t(C_n) = t(C_n \cup B_{n+1})$.

Since $r = \omega$ for each $r < s$ (Lemma 1.4), we see that $3t(C_n) = t(C_n) \ni t(B_1 \cup \ldots \cup B_n)$, proving the claim.

Theorem 3.19 in [11] implies that $t(C_n) = t(B_1) + \ldots + t(S_n) = s$. Put $B = \cup C_n$.

2.4. Lemma. Let $f$ be an automorphism of a subset $X$ of $R$ and suppose that every $r < t(\text{supp}(f))$ is countably compact. Then $f = f_1 * f_2 * f_3$, where $\text{supp}(f_n) \subseteq \text{supp}(f)$ and $t(\text{supp}(f_n)) = t(X - \text{supp}(f_n))$ for $n = 1, 2, 3$.

Proof. Apply Lemma 2.3 to find a measurable subset $B$ of $X$ such that $t(B) = t(\text{supp}(f))$ and $B \cap b = \emptyset$. Define an automorphism $h$ of $X$ by

$$h(x) = \begin{cases} f(x) & x \in B, \\ f^{-1}(x) & x \in f(B), \\ x & x \in X - (B \cup f(B)), \end{cases}$$

setting $f_1 = f \circ h$. Then $\text{supp}(f_1) \subseteq \text{supp}(f) - f(B)$, so that $t(\text{supp}(f_1)) = t(\text{supp}(f)) - t(B) = t(f(B)) \leq t(X - \text{supp}(f_1))$.

Since $t(B)$ is countably compact, we may write (Lemma 1.4) $B = B_1 \cup B_2$, where $B_1$ and $B_2$ are disjoint measurable sets with $t(B_1) = t(B_2) = t(B)$. Define automorphisms $f_2$ and $f_3$ of $X$ by putting

$$f_2(x) = \begin{cases} f(x) & x \in B_1, \\ f^{-1}(x) & x \in f(B_1), \\ x & \text{otherwise}, \end{cases}$$

$$f_3(x) = \begin{cases} f(x) & x \in B_2, \\ f^{-1}(x) & x \in f(B_2), \\ x & \text{otherwise}. \end{cases}$$

Then $\text{supp}(f_2) = B_1 \cup f(B_2)$ and $\text{supp}(f_3) = B_2 \cup f(B_2)$. Also,

$$t(\text{supp}(f_2)) = t(B_1) \cup t(f(B_1)) = t(B_1) \cup t(B_2) = t(X - \text{supp}(f_2)),$$

and likewise $t(\text{supp}(f_3)) = t(X - \text{supp}(f_3))$. Then $f = f_1 * f_2 * f_3$ as desired.

2.5. Lemma. Let $X$ be a subset of $R$ with the property that every type $s \subseteq t(X)$ is countably compact. Let $f$ and $g$ be automorphisms of $X$. Then $f \circ g$ belongs to the normal closure of $g$ if and only if $t(\text{supp}(f)) = t(\text{supp}(g))$.

Proof. Suppose first that $t(\text{supp}(f)) \leq t(\text{supp}(g))$. Lemma 2.4 allows us to assume that $t(\text{supp}(f)) \leq t(X - \text{supp}(f))$. Also Lemma 2.3 guarantees the existence of a set $B \in \mathcal{B}(X)$ with $t(B) = t(\text{supp}(f))$ and $g(B) \cap B = \emptyset$. Noting that $s = \omega s$ for all $s \subseteq t(X)$, we apply Lemma 2.2 to make the desired conclusion.

Conversely, suppose that $f \circ g$ belongs to the normal closure of $g$. Every conjugate of $g$ has a support of type $t(\text{supp}(g))$, and the composition of two automorphisms whose supports are of types bounded by $t(\text{supp}(g))$ will have the same property. The result follows.

Let $t$ be an isomorphism type in $S$. A set $I \subseteq \text{sec}(t)$ is an ideal in $\text{sec}(t)$ if $(1)$ is non-empty, $(2) s \in I$ and $t \leq \omega s$ imply $r \in I$ if $s$ and $s'$ are elements of $I$, so, too, is $s + s'$. An ideal is principal if it is of the form $I = \text{sec}(s)$ for some $s$ with $s = \omega s$.

2.6. Proposition. Let $X$ be a subset of $R$ such that for each $t \subseteq t_0 = t(X)$, there is some type $s$ with $t = 2s$. For each ideal $I$ in $\text{sec}(t_0)$, define

$$H(I) = \{ f \in G(X): t(\text{supp}(f)) \in I \}.$$ Then the mapping $I \mapsto H(I)$ is one-one and order preserving from ideals of $\text{sec}(t_0)$ to normal subgroups of $G(X)$.

If each $s \subseteq t_0$ is countably compact, then $I \mapsto H(I)$ is surjective.

Proof. If $I$ is an ideal in $\text{sec}(t_0)$ and $f, g$ are automorphisms of $X$, then the relations

$$\text{supp}(fg) \subseteq \text{supp}(f) \cup \text{supp}(g), \quad \text{supp}(gfg^{-1}) = g(\text{supp}(f))$$

show that $H(I)$ is a normal subgroup of $G(X)$. Clearly, $I_1 \subseteq I_2$ implies that $H(I_1) \subseteq H(I_2)$. 

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Now suppose that $I_1$ and $I_2$ are distinct ideals of $\text{sec}(\mathcal{A})$. Suppose, say, that $s \in I_1 - I_2$. By hypothesis, there are disjoint sets $B_1$ and $B_2$ in $\mathcal{A}(X)$ with $s = \tau(B_1) \cup \tau(B_2)$, $\tau(B_1) = \tau(B_2)$. Let $f: B_1 \to B_2$ be a Borel isomorphism and define $g: X \to X$ by

$$
g(x) = \begin{cases} 
  f(x) & x \in B_1, \\
  f^{-1}(x) & x \in B_2, \\
  x & x \in X - (B_1 \cup B_2).
\end{cases}
$$

Then $g$ is an automorphism with $\tau(\text{supp}(g)) = s$, and $g \in H(I_1) - H(I_2)$. Thus the mapping $I \to H(I)$ is one-one.

Now suppose that each $s \in I_0$ is countably compact and that $H$ is a normal subgroup of $G(X)$. Define

$$
I = \{ \tau(\text{supp}(g)): g \in H \}.
$$

We show that $I$ is an ideal and that $H(I) = H$. Suppose that $s \subseteq \tau(\text{supp}(g))$ for some $g \in H$. As above, there is some $f \in G(X)$ with $\tau(\text{supp}(f)) = s$. Apply Lemma 2.5 and conclude that $f$ is in the normal closure of $g$; hence, $f \in H$ and $s \subseteq I$. We have proved that $I$ is order-hereditary. Next, given $f$ and $g$ in $H$ with $s_1 = \tau(\text{supp}(f))$ and $s_2 = \tau(\text{supp}(g))$, let $h$ be an automorphism of $X$ with $\tau(\text{supp}(h)) = \tau(\text{supp}(g)) - \tau(\text{supp}(f))$. Again Lemma 2.5 implies that $h \in H$. Then $\tau(\text{supp}(f)) = \tau(\text{supp}(h)) \cup \tau(\text{supp}(g))$, which is (Lemma 1.4) of isomorphism type $s_1 + s_2 = s_1 \cup s_2$. $I$ is an ideal. Obviously, $H \subseteq H(I)$. Given $f \in H(I)$, we know that there is some $g \in H$ with $\tau(\text{supp}(g)) = \tau(\text{supp}(f))$. Another application of Lemma 2.5 shows that $f \in H$. Thus $H = H(I)$.

2.7. COROLLARY. Let $X$ be a subset of $R$ such that $\text{sec}(\text{t}(X))$ is finite, and $\mathcal{N}$ be the poset of normal subgroups of $G(X)$. Then $\mathcal{N}$ and $\text{sec}(\text{t}(X))$ are isomorphic complete, distributive lattices.

Proof. Apply the proposition, noting that every ideal in $\text{sec}(\text{t}(X))$ is principal. Then use [11; 3.30, 3.32] and the fact that the normal subgroups of a group form a lattice.

As mentioned before, under analytic determinacy, the set of analytic types is $\text{sec}(\text{t}(U)) = \{0, \tau(R), \tau(U)\}$ for $U \subseteq R$ universal analytic. Corollary 2.7 shows that in this case, $G(U)$ has exactly three normal subgroups.

2.8. COROLLARY. Let $X$ be a subset of $R$ with $\text{t}(X)$ a cover of $0$. Then $G(X)$ is simple. In particular, $G(X)$ is simple for $X$ a Borel subset of $R$.

3. Automorphisms of analytic sets. A subset $X$ of $R$ is topologically rigid if the only homeomorphism of $X$ onto itself is the identity map. In [6], it is was shown that there are no zero-dimensional, topologically rigid Borel subsets of $R$ other than singletons. However, it was also shown that, under $\text{t} = \text{L}$, there is an uncountable, zero-dimensional, topologically rigid, analytic subset of $R$. As we shall see, this matter is related to the question of whether $G(A)$ can be simple for $A$ analytic, non-Borel. It is also related to the problem of whether an uncountable co-analytic set can be measurably rigid.

3.1. LEMMA. Let $A$ be an analytic subset of $R$ and let $N(A)$ be the subgroup of $G(A)$ comprising all $f$ for which $\tau(\text{supp}(f))$ is a Borel subset of $R$. Then $N(A)$ is a simple group.

Proof. The case where $A$ is countable is trivial; for $A$ uncountable and Borel, Corollary 2.8 pertains. So suppose analytic and non-Borel. Let $f$ and $g$ be automorphisms of $A$ with $\tau(\text{supp}(g))$ uncountable and $\tau(\text{supp}(f))$ Borel in $R$. Then:

$$
\tau(\text{supp}(g)) = \varnothing(\text{supp}(f)) \lor \tau(\text{supp}(g)) = \tau(\text{supp}(f)) \lor \tau(\text{supp}(f)) = \tau(A - \text{supp}(f)) = \tau(A).
$$

Since each $\tau(\text{supp}(f))$ is countably compact, there is, by Lemma 2.3, some $B \in \mathcal{A}(A)$ such that $\tau(B) = \tau(\text{supp}(f))$ and $\tau(B) \cap \varnothing = \varnothing$. An application of Lemma 2.2 shows that the normal closure of $g$ contains $f$. There follows the simplicity of $N(A)$.

3.2. COROLLARY. Let $A \subseteq \mathcal{R}$ be analytic. Then $G(A)$ is simple if and only if $G(A) = N(A)$.

Proof. We may assume that $A$ is uncountable. It is then easily checked that $N(A)$ is a normal subgroup of $G(A)$ with more than one element. If $G(A)$ is simple, then $N(A) = G(A)$. The proposition supplies the converse.

Remark. The earlier result (Corollary 2.8), that $G(A)$ is simple for $A$ Borel, may be obtained from this proposition as well.

3.3. PROPOSITION. Consider the following six statements:

1. Any two analytic, non-Borel subsets of $R$ are Borel-isomorphic.
2. The isomorphism types of analytic sets are linearly ordered.
3. The partial ordering of analytic isomorphism types is well-founded.
4. Other than singleton sets, there are no topologically rigid, zero-dimensional analytic subsets of $R$.
5. If $A \subseteq \mathcal{R}$ is analytic, then $G(A)$ is simple if and only if $A$ is Borel.
6. There is no uncountable, measurably rigid, co-analytic subset of $R$.

These implications obtain:

1 \Rightarrow 2 \Rightarrow 3. Obvious.
2 \Rightarrow 5 and 3 \Rightarrow 5 and 4 \Rightarrow 5. It was noted in Corollary 2.8 that $G(A)$ is simple.
if $A$ is Borel. Now assume that there is some analytic, non-Borel set $A$ with $G(A)$ simple. By Corollary 3.2, $G(A) = N(A)$. We show that conditions 2, 3 and 4 must fail.

Removing a countable dense subset of $R$ if necessary, we may assume that $A$ is zero-dimensional. Let $\mathcal{A}$ be the collection of all open subets $V$ of $R$ such that $V \cap A$ is Borel in $R$. Put $A_0 = A - \bigcup \mathcal{A}$. By the Lindelöf theorem, we see that $A = (\bigcup \mathcal{A})$ is Borel in $R$, so that $A_0$ is not. In fact, $A_0$ and $A$ are Borel-isomorphic. Thus $G(A_0) = N(A_0)$. We state the

Claim. Suppose that $V_1$ and $V_2$ are open subsets of $R$ with $U_1 = V_1 \cap A_0$ and $U_2 = V_2 \cap A_0$ non-void and disjoint. Then $U_1$ [resp. $U_2$] is not Borel-isomorphic with any set in $\mathcal{B}(U_2)$ [resp. $\mathcal{B}(U_1)$].

Proof of Claim. Suppose rather that $f$ is a Borel-isomorphism mapping $U_1$ onto some set $B \in \mathcal{B}(U_2)$. Define $g: A_0 \to A_0$ by

$$g(x) = \begin{cases} f(x) & x \in U_1, \\ f^{-1}(x) & x \in B, \\ x & x \in A_0 \setminus (U_1 \cup B). \end{cases}$$

Then $g$ is a Borel-isomorphism of $A_0$ whose support contains $U_1$ and is thus not Borel in $R$. This would contradict $G(A_0) = N(A_0)$.

From the claim it follows that $A_0$ is a topologically rigid set: condition 4 fails. Also, with $U_1$ and $U_2$ as above, we see that the types $t(U_1)$ and $t(U_2)$ are not comparable: condition 2 fails. Finally, let $V_1, V_2, \ldots$ be a sequence of open subsets of $R$ such that the sets $U_n = V_n \cap A_0$ are disjoint and non-empty. Then the sets $A_n = V_n \cup A_0$ are analytic, and $t(A_n) \succ t(A_n) > \ldots$: condition 3 fails.

5 $\Rightarrow$ 6. We prove the contrapositive. Let $C$ be an uncountable, measurable rigid, co-analytic subset of $R$. Of necessity, $C$ and $A = R - C$ are non-Borel. Now suppose that $f$ is a Borel-automorphism of $A$. Then $f$ extends to an automorphism $g$ of $R$, and the restriction $h: C \to C$ of $g$ to $C$ is again an automorphism. Since $C$ is measurable rigid, it must be that supp$(h)$ is countable, whence it follows that supp$(f)$ is Borel in $R$. We have shown that $G(A) = N(A)$. From Corollary 3.2, we see that $G(A)$ is simple. Thus condition 5 fails.

The author wishes to thank Manfred Droste and the referee, who intercepted some earlier mistakes and whose suggestions and encouragement are most appreciated.

References


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Received 22 August 1988;
in revised form 2 March 1989