$\simeq q(E, \bar{E}) * q(C_1, \bar{D}) \simeq q(F, \bar{x}) * q(G, \bar{E}) * q(F^{-1}, \bar{x}) * q(C_1, \bar{D}) \simeq q(C_1, \bar{H} * \bar{D})$, i.e. the points x, y are in the same class. ∎

I would like to express my gratitude to Zdzisław Dzedzej who associated the proof of the above proposition with the subject of [1].

The Nielsen theory. In 1983, N. Rallis developed a fixed point index theory for symmetric product maps of compact euclidean neighbourhood retracts (ENRs). Using this index we follow S. Masih [2] and define the Nielsen number of $f\colon X \to X_n$ as the number of essential fixed point classes (a class is an essential class if its index does not equal zero).

Now, let X be a compact connected ENR space. It follows from the Proposition that there exists at most one fixed point class of $f\colon X \to X_n$. By [4], the index of the unique fixed point class equals the Lefschetz number of f.

COROLLARY. *The Nielsen number $N(f)$ of the symmetric product map $f\colon X \to X_n$ equals* 0 *if the Lefschetz number equals* 0 *and it equals* 1 *otherwise.* ∎

Remark. In 1988, H. Schirmer [5] and [6] proved that if X is a triangulable manifold of dimension not less than 3, then for any map $f\colon X \to X_2$ there exists a map homotopic to f having exactly $N(f)$ fixed points.

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Normal subgroups of measurable automorphisms

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Abstract. Given $X \subseteq \mathbf{R}$, let $F(X)$ be the group of all Borel-isomorphisms of X onto itself and let $CO(X)$ be the normal subgroup comprising all f with $\{x\colon f(x) \neq x\}$ countable. Then $G(X) = F(X)/CO(X)$ is the "reduced automorphism group" of X. Using a technique of R. D. Anderson (Lemma 2.1), we explore the normal subgroup structure of $G(X)$. For certain sets X, there is a simple relationship between this structure and the poset of Borel-isomorphism types of measurable subsets of X (Proposition 2.6). As a consequence of this result, one sees that $G(X)$ is a simple group when X is a Borel subset of $\mathbf{R}$ (Corollary 2.8).

Whether $G(A)$ can be simple for A analytic and non-Borel is not known. Under analytic determinacy and even weaker assumptions, the answer is no. If there is such as et A, then there is a topologically rigid, zero-dimensional $A' \subseteq \mathbf{R}$ Borel-isomorphic with A (Proposition 3.3). Such rigid sets have been considered by van Engelen, Miller, and Steel.

1. Preliminary notions. We deal exclusively with subsets of the real line, although all our results are easily transferable to the context of any complete, separable, metric space. A subset X of $\mathbf{R}$ is *analytic* if it is a continuous image of the space of irrational numbers (or is empty). A set $X \subseteq \mathbf{R}$ is *co-analytic* if $\mathbf{R} - X$ is analytic. Souslin showed that X is a Borel subset of $\mathbf{R}$ if and only if X is both analytic and co-analytic [7; pp. 485–6]. Given $X \subseteq \mathbf{R}$, define

$$\mathcal{B}(X) = \{B \cap X\colon B \text{ is a Borel subset of } \mathbf{R}\}.$$

Then $\mathcal{B}(X)$ is a σ-field whose elements are the *measurable subsets* of X. A function $f\colon X \to Y$ is a (Borel) *isomorphism* if f is a one-one correspondence of X with Y, and $B \in \mathcal{B}(X)$ if and only if $f(B) \in \mathcal{B}(Y)$. If $X = Y$, then f is a (Borel) *automorphism*. If f is an automorphism of X, define its *support* as

$$\operatorname{supp}(f) = \{x \in X\colon f(x) \neq x\}.$$

The collection $F(X)$ of all automorphisms of X forms a group under composition. Note that if f and g are automorphisms of X, then $\operatorname{supp}(g \circ f \circ g^{-1}) = g(\operatorname{supp}(f))$.

Let $CO(X)$ be the set of all $f \in F(X)$ such that $\operatorname{supp}(f)$ is countable. Then $CO(X)$ is a normal subgroup of $F(X)$, and we call the quotient $G(X) = F(X)/CO(X)$ the *reduced automorphism group* of X. If $f \in F(X)$, then $\hat{f}$ is its coset in $G(X)$. A set X is *measurably rigid* if $G(X)$ is trivial. Uncountable measurably rigid sets exist (ZFC), but they can contain no uncountable Borel subsets of $\boldsymbol{R}$. Since every uncountable analytic set contains such a Borel set [7; p. 444], no uncountable analytic set can be measurably rigid. Whether nature will allow the existence of an uncountable, co-analytic, measurably rigid set is not known. We shall return to this question in Proposition 3.3 *infra*. More detailed information about measurable rigidity and $G(X)$ lies in [2], [5], [10].

Let X be a subset of $\boldsymbol{R}$. We define the *isomorphism type* of X by

$$t(X) = \begin{cases} \{Y \subseteq \boldsymbol{R}\colon X \text{ and } Y \text{ isomorphic}\} & \text{if } X \text{ is uncountable} \\ 0 & \text{if } X \text{ is countable.} \end{cases}$$

Put $S = \{t(X)\colon X \subseteq \boldsymbol{R}\}$. Given t_1 and t_2 in S, choose $X_1 \subseteq (0,1)$ and $X_2 \subseteq (1,2)$ such that $t_1 = t(X_1)$ and $t_2 = t(X_2)$. Then define

$$t_1 + t_2 = t(X_1 \cup X_2).$$

It is not hard to check that t_1+t_2 is well-defined. Denumerable sums $t_1+t_2+\ldots$ are defined analogously. We use the notations

$$nt = t+\ldots+t \ (n \text{ times}), \qquad \omega t = t+t+\ldots$$

Define a relation $\leqslant$ on S by putting $s \leqslant t$ whenever $t = s+a$ for some a. Then $t(X) \leqslant t(Y)$ just in case X is isomorphic with a measurable subset of Y. A Schröder–Bernstein argument shows that $\leqslant$ is a partial order on S: see [4; Theorem 2.5]. It seems that Tarski was first to notice that such isomorphism types have an algebraic behaviour quite similar to that of cardinal numbers. In fact, we have

1.1. Lemma. *The system $(S, +, \leqslant)$ is a partially ordered commutative monoid with identity element* 0. *Indeed, S is a cardinal algebra in the sense of Tarski* [11].

Indication. The commutative, associative, and identity element postulates are rather immediate. It remains to check the refinement and remainder axioms:

Refinement. Suppose that a, b, c_n are elements of S such that $a+b = c_0+c_1+\ldots$ We produce elements a_n and b_n such that

$$a = a_0+a_1+\ldots \qquad b = b_0+b_1 \ldots \qquad a_n+b_n = c_n.$$

Let $X \subseteq (0,1)$, $Y \subseteq (1,2)$ and $Z_n \subseteq (n, n+1)$ be sets with $a = t(X)$, $b = t(Y)$, and $c_n = t(Z_n)$. By hypothesis, there is an isomorphism f mapping $X \cup Y$ onto $Z_0 \cup Z_1 \cup \ldots$ Put $a_n = t(Z_n \cap f(X))$ and $b_n = t(Z_n \cap f(Y))$.

Remainder. Suppose that a_n and b_n are elements of S such that $a_n = b_n + a_{n+1}$ for $n = 0, 1, \ldots$ We exhibit an element u such that $a_n = u+b_n+b_{n+1}+\ldots$ for each n. Let X_n and Y_n be subsets of $\boldsymbol{R}$ such that

$$X_n = Y_n \cup X_{n+1}, \qquad Y_n \cap X_{n+1} = \emptyset,$$
$$Y_n \in \mathscr{B}(X_n), \qquad X_{n+1} \in \mathscr{B}(X_n),$$
$$t(X_n) = a_n, \qquad t(Y_n) = b_n.$$

Put $u = t(\bigcap X_n)$. ■

For further details concerning the algebraic structure of S, *vide* [9].

Given $t \in S$, define $\sec(t) = \{s \in S\colon s \leqslant t\}$. There is an analytic subset U of $\boldsymbol{R}$ such that $X \subseteq \boldsymbol{R}$ is analytic if and only if $t(X) \leqslant t(U)$. Such are the so-called "universal" analytic sets [7; p. 459]. Given Borel subsets B_1 and B_2 of $\boldsymbol{R}$, a well-known result of Kuratowski asserts that $t(B_1) = t(B_2)$ if and only if B_1 and B_2 have the same cardinality [7; p. 451]. It follows that $\omega t(B) = t(B)$ for each Borel $B \subseteq \boldsymbol{R}$. Also, $t(B) \leqslant t(A)$ whenever A and B are uncountable with A analytic and B Borel. Thus there are only two types for Borel sets: $t(B) = 0$ for B countable, and $t(B) = t(\boldsymbol{R})$ for B uncountable. No such clear structure result is available for analytic sets in general. The postulate that all analytic games are determined is equivalent to the statement that for each analytic set A, either $t(A) = 0$ (A countable), $t(A) = t(\boldsymbol{R})$ (A uncountable Borel), $t(A) = t(U)$ (universal type). This is true e.g. if measurable cardinals exist [8]. The situation is quite different under constructibility ($V = L$): see [3].

1.2. Lemma. *Let $A \subseteq \boldsymbol{R}$ be analytic and non-Borel. Then $t(A) = t(\boldsymbol{R})+s$ implies that $s = t(A)$.*

Proof. First note [7; p. 444] that every uncountable analytic subset D of $\boldsymbol{R}$ contains a homeomorph B of the Cantor set. Thus

$$\begin{aligned} t(D)+t(\boldsymbol{R}) &= t(D-B)+t(B)+t(\boldsymbol{R}) \\ &= t(D-B)+t(B)+t(B) \\ &= t(D-B)+t(B) = t(D). \end{aligned}$$

Given A analytic and non-Borel with $t(A) = t(\boldsymbol{R})+s$, take $B \in \mathscr{B}(A)$ with $t(B) = t(\boldsymbol{R})$ and $t(A-B) = s$. Then $A-B$ is uncountable and analytic. Thus $t(A) = t(\boldsymbol{R})+s = s$. ■

An isomorphism type $t \in S$ is *countably compact* if whenever $t_1 \leqslant t_2 \leqslant \ldots \leqslant t$ with $\sup_n(t_n) = t$, then $t_N = t$ for some N.

1.3. Lemma. *Let X be a subset of $\boldsymbol{R}$ such that $t(X)$ is countably compact, and suppose that $A_1 \subseteq A_2 \subseteq \ldots$ are elements of $\mathscr{B}(X)$ whose union is X. Then there is some N with $t(A_N) = t(X)$.*

This lemma follows immediately from the definition of countably compact types, and to some degree justifies the use of the term.

1.4. Lemma. *Let $t \in S$ be countably compact. Then $t = \omega t$.*

Proof. Let X be a subset of $(0,1)$ with $t(X) = t$. For countable ordinals α, we define real numbers $a(\alpha)$ and the isomorphism types s_α as follows. Given α, assume that $a(\beta)$ and s_β are defined for $\beta < \alpha$ so that $a(0) = 1$, $s_0 = 0$, and

$$0 < a(\beta) < a(\gamma) \quad \text{for } \gamma < \beta$$
$$s_\beta = t\big((a(\beta), 1) \cap X\big), \qquad \omega s_\beta \leqslant t.$$

These are two cases to consider.

Case 1. $\alpha = \beta+1$, a successor ordinal.

In this case, consider the sequence $b_n = (a(\beta)-1/n) \vee 0$ for $n = 1, 2, 3, \ldots$ For each n, put

$$B_n = [(0, b_n) \cup (a(\beta), 1)] \cap X .$$

These sets form an increasing sequence whose union is (within a singleton) X. From the general theory of cardinal algebras [11: Theorem 3.19], we know that $t(X) = \sup_n t(B_n)$. By Lemma 1.3, there is some n with $b_n > 0$ and $t = t(X) = t(B_n)$. Define $a(\alpha) = b_n$. Then $0 < a(\alpha) < a(\beta)$. Since $t = t(B_n)$, we have

$$t = t+t(X-B_n), \quad t = t+nt(X-B_n),$$

whence it follows that $\omega t(X-B_n) \leqslant t$. If $s_\alpha = t((a(\alpha), 1) \cap X)$, then

$$\omega s_\alpha = \omega t(X-B_n)+\omega s_\beta = [\omega t(X-B_n) \vee \omega s_\beta] \leqslant t$$

as desired. We have used [11; Theorem 4.7].

Case 2. α is a limit ordinal.

In this case, put $a(\alpha) = \inf\{a(\beta)\colon \beta < \alpha\}$, and set

$$s_\alpha = t\big((a(\alpha), \;) \cap X\big) = \sup\{s_\beta\colon \beta < \alpha\} .$$

It follows that $\omega s_\alpha \leqslant t$. Note that if $a(\alpha) = 0$, then $s_\alpha = t$, the construction halts, and the lemma is proved.

In fact, the countable chain condition forces this construction to halt at some countable limit ordinal α. Then $a(\alpha) = 0$, and $t = \omega t$ as desired. ■

1.5. LEMMA. *Let t_1 and t_2 be countably compact types in S. Then their supremum exists and equals $t_1 \vee t_2 = t_1+t_2$.*

Proof. This follows from Theorem 4.7 of [11]. ■

1.6. LEMMA. *Let $t \in S$ be such that* $\sec(t)$ *is finite. Then t is countably compact.*

As mentioned above, the number of analytic types is finite if analytic determinary holds. In this case, every analytic type is countably compact. Under $V = L$, this is not true: $t = \omega t$ does not always obtain [3; Lemma 4.4].

A type $t \in S$ is a *cover of* 0 if $\sec(t)$ has exactly two elements. Otherwise said, $t(X)$ is a cover of 0 if and only if X is uncountable and Borel isomorphic with each of its uncountable measurable subsets. Clearly, $t(R)$ is a cover of 0, and each cover of 0 is countably compact. In [9], a large family of such types was constructed.

These covers of 0 can be used to construct "successors" to a type t. A successor to t is a type $s > t$ such that $s \geqslant u \geqslant t$ implies either $u = s$ or $u = t$. (See [9] for the details.) Every successor type is countably compact (cf. compactness of ordinal spaces).

2. Factorization of automorphisms. The nuts and bolts part of our method is a technique used by R. D. Anderson in [1], whereby he proved that, under certain conditions, a given homeomorphism of a space onto itself could be factored as the composition product of four conjugates of another such homeomorphism (or its inverse). The technique is easily adapted to the measurable context, and it is used to prove

2.1. BASIC LEMMA. *Let f and g be Borel automorphisms of a subset X of $\mathbf{R}$. Suppose that $\ldots A_{-1} A_0 A_1 \ldots$ is a sequence of disjoint, isomorphic, measurable subsets of some $B \in \mathscr{B}(X)$ such that $\operatorname{supp}(f) \subseteq A_0$ and $B \cap g^{-1}(B) = \varnothing$. Then f is a product (under composition) of two conjugates of g with two conjugates of g^{-1}. In particular, $\hat{f}$ belongs to the normal closure of $\hat{g}$ in $G(X)$.*

Proof. Let $r\colon X \to X$ be a Borel automorphism such that $\operatorname{supp}(r) \subseteq B$ and $r(A_n) = A_{n+1}$ for each n. Now $\operatorname{supp}(r^n \circ f \circ r^{-n}) \subseteq r^n(A_0) = A_n$ for $n = 0, 1, \ldots$ Define $k\colon X \to X$ by putting

$$k(x) = \begin{cases} r^n(f(r^{-n}(x))) & \text{for } x \in A_n \ (n \geqslant 0), \\ x & \text{for } x \in X - \bigcup_{n \geqslant 0} A_n . \end{cases}$$

Then k is an automorphism of X with $\operatorname{supp}(k) \subseteq B$. Hence $\operatorname{supp}(r \circ k^{-1} \circ r^{-1}) \subseteq B$, and likewise $\operatorname{supp}(g^{-1} \circ k \circ g) \cup \operatorname{supp}(g^{-1} \circ k^{-1} \circ g) \subseteq g^{-1}(B)$. Thus, each of the pairs

$$r \text{ and } g^{-1} \circ k \circ g \qquad r \circ k^{-1} \circ r^{-1} \text{ and } g^{-1} \circ k^{-1} \circ g$$

commutes. We calculate

$$\begin{aligned}(rg^{-1}kgk^{-1}r^{-1})(g^{-1}k^{-1}gk) &= (rg^{-1}kgr^{-1})(r^{-1}k^{-1}r^{-1})(g^{-1}k^{-1}g)k \\ &= (rg^{-1}kgr^{-1})(g^{-1}k^{-1}g)(rk^{-1}r^{-1})k \\ &= (rk^{-1}r^{-1})k = f .\end{aligned}$$

Then

$$\begin{aligned} f &= (rg^{-1}kgk^{-1}r^{-1})(g^{-1}k^{-1}gk) \\ &= (rg^{-1}r^{-1})(rkgk^{-1}r^{-1})(g^{-1})(k^{-1}gk) ,\end{aligned}$$

the product of four conjugates. ■

2.2. LEMMA. *Suppose that f and g are Borel automorphisms of a subset X of $\mathbf{R}$ such that*

(1) $\omega t(\operatorname{supp}(f)) \leqslant t(X-\operatorname{supp}(f))$;

(2) *there is some $B \in \mathscr{B}(X)$ with $g(B) \cap B = \varnothing$ and $t(B) \geqslant \omega t(\operatorname{supp}(f))$.*

Then $\hat{f}$ is in the normal closure of $\hat{g}$ in $G(X)$.

Proof. Using condition 2, choose disjoint measurable subsets $\ldots A_{-1} A_0 A_1 A_2 \ldots$ of B such that $t(A_n) = t(\operatorname{supp}(f))$ for each n. By condition 1, there are disjoint measurable sets $C_1 C_2 \ldots$ of $X-\operatorname{supp}(f)$ with $t(C_n) = t(\operatorname{supp}(f))$ for each n. Then

$$\begin{aligned} t(X-\operatorname{supp}(f)) &= t(\textstyle\bigcup C_n)+t(X-\operatorname{supp}(f)-\bigcup C_n) \\ &= \omega t(\operatorname{supp}(f))+t(X-\operatorname{supp}(f)-\textstyle\bigcup C_n) \\ &= t(\operatorname{supp}(f) \cup \textstyle\bigcup C_n)+t(X-\operatorname{supp}(f)-\bigcup C_n) \\ &= t(X) = t(\textstyle\bigcup A_n)+t(X-\bigcup A_n) \\ &= \omega t(\operatorname{supp}(f))+t(X-\textstyle\bigcup A_n) \\ &= t(\textstyle\bigcup_{n \neq 0} A_n)+t(X-\bigcup A_n) = t(X-A_0) .\end{aligned}$$

Thus there is an automorphism h of X such that $h(\mathrm{supp}(f)) = A_0$. Then $\mathrm{supp}(h \circ f \circ h^{-1}) = h(\mathrm{supp}(f)) = A_0$. Without loss of generality, therefore, we may and do assume that $\mathrm{supp}(f) \subseteq A_0$. Now apply Lemma 2.1. ■

In the next group of lemmas, we explore conditions that facilitate the application of these basic results.

2.3. Lemma. *Let g be an automorphism of a subset X of $\mathbf{R}$ and let s be such that $t(\mathrm{supp}(g)) \geqslant s$, where every $r \leqslant s$ is countably compact. Then there is some $B \in \mathscr{B}(X)$ such that $t(B) \geqslant s$ and $g(B) \cap B = \emptyset$.*

Proof. Let S be a measurable subset of $\mathrm{supp}(g)$ with $t(S) = s$. Define $S_n = \{x \in S\colon |g(x)-x| > 1/n\}$ for $n = 1, 2, \ldots$ Then S is the ascending union of the sets S_n, so that by Lemma 1.5, $t(S_N) = s$ for some N. Define

$$A_k = [k/N, (k+1)/N) \cap S_N \text{ for } k = 0, \pm 1, \pm 2, \ldots,$$

noting that $A_k \cap g(A_k) = \emptyset$. Re-index the sets A_k as $B_1 B_2 \ldots$ Define $C_1 = B_1$ and

$$C_{n+1} = (C_n \cup B_{n+1}) - g(C_n) - g^{-1}(C_n)$$

for every $n \geqslant 1$. Then $C_1 \subseteq C_2 \subseteq \ldots$ and $g(C_n) \cap C_n = \emptyset$.

Claim. *For each n, we have $t(C_n) = t(B_1 \cup \ldots \cup B_n)$.*

Proof of claim. We induct on n. The case for $n = 1$ being trivial, we assume the result for n and establish it for $n+1$. Clearly $t(C_{n+1}) \leqslant t(B_1 \cup \ldots \cup B_n)$. Also $C_{n+1} \cup g(C_n) \cup g^{-1}(C_n) \supseteq C_n \cup B_{n+1}$, so that

$$t(C_{n+1}) + t(g(C_n)) + t(g^{-1}(C_n)) \geqslant t(C_n \cup B_{n+1})$$
$$t(C_{n+1}) + t(C_n) + t(C_n) \geqslant t(C_n) + (B_{n+1}) = t(B_1 \cup \ldots \cup B_n \cup B_{n+1})$$
$$t(C_{n+1}) + t(C_{n+1}) + t(C_{n+1}) \geqslant t(B_1 \cup \ldots \cup B_{n+1}).$$

Since $r = \omega r$ for each $r \leqslant s$ (Lemma 1.4), we see that $3t(C_{n+1}) = t(C_{n+1}) \geqslant t(B_1 \cup \ldots \cup B_{n+1})$, proving the claim.

Theorem 3.19 in [11] implies that $t(\bigcup C_n) = t(B_1) + \ldots = t(S_N) = s$. Put $B = \bigcup C_n$. ■

2.4. Lemma. *Let f be an automorphism of a subset X of $\mathbf{R}$ and suppose that every $r \leqslant t(\mathrm{supp}(f))$ is countably compact. Then $f = f_1 \circ f_2 \circ f_3$, where $\mathrm{supp}(f_n) \subseteq \mathrm{supp}(f)$ and $t(\mathrm{supp}(f_n)) \leqslant t(X - \mathrm{supp}(f_n))$ for $n = 1, 2, 3$.*

Proof. Apply Lemma 2.3 to find a measurable subset B of X such that $t(B) = t(\mathrm{supp}(f))$ and $f(B) \cap B = \emptyset$. Define an automorphism h of X by

$$h(x) = \begin{cases} f(x) & x \in B, \\ f^{-1}(x) & x \in f(B), \\ x & x \in X - (B \cup f(B)), \end{cases}$$

setting $f_1 = f \circ h$. Then $\mathrm{supp}(f_1) \subseteq \mathrm{supp}(f) - f(B)$, so that

$$t(\mathrm{supp}(f_1)) \leqslant t(\mathrm{supp}(f)) = t(B) = t(f(B)) \leqslant t(X - \mathrm{supp}(f_1)).$$

Since $t(B)$ is countably compact, we may write (Lemma 1.4) $B = B_1 \cup B_2$, where B_1 and B_2 are disjoint measurable sets with $t(B_1) = t(B_2) = t(B)$. Define automorphisms f_2 and f_3 of X by putting

$$f_2(x) = \begin{cases} f(x) & x \in B_1, \\ f^{-1}(x) & x \in f(B_1), \\ x & \text{otherwise}, \end{cases}$$

$$f_3(x) = \begin{cases} f(x) & x \in B_2, \\ f^{-1}(x) & x \in f(B_2), \\ x & \text{otherwise}. \end{cases}$$

Then $\mathrm{supp}(f_2) = B_1 \cup f(B_1)$ and $\mathrm{supp}(f_2) = B_2 \cup f(B_2)$. Also,

$$t(\mathrm{supp}(f_2)) = t(B_1 \cup f(B_1)) = t(B_2 \cup f(B_2)) \leqslant t(X - \mathrm{supp}(f_2)),$$

and likewise $t(\mathrm{supp}(f_3)) \leqslant t(X - \mathrm{supp}(f_3))$. Then $f = f_1 \circ f_2 \circ f_3$ as desired. ■

2.5. Lemma. *Let X be a subset of $\mathbf{R}$ with the property that every type $s \leqslant t(X)$ is countably compact. Let f and g be automorphisms of X. Then $\hat{f}$ belongs to the normal closure of $\hat{g}$ if and only if $t(\mathrm{supp}(f)) \leqslant t(\mathrm{supp}(g))$.*

Proof. Suppose first that $t(\mathrm{supp}(f)) \leqslant t(\mathrm{supp}(g))$. Lemma 2.4 allows us to assume that $t(\mathrm{supp}(f)) \leqslant t(X - \mathrm{supp}(f))$. Also Lemma 2.3 guarantees the existence of a set $B \in \mathscr{B}(X)$ with $t(B) \geqslant t(\mathrm{supp}(f))$ and $g(B) \cap B = \emptyset$. Noting that $s = \omega s$ for all $s \leqslant t(X)$, we apply Lemma 2.2 to make the desired conclusion.

Conversely, suppose that $\hat{f}$ belongs to the normal closure of $\hat{g}$ in $G(X)$. Every conjugate of g has a support of type $t(\mathrm{supp}(g))$, and the composition of two automorphisms whose supports are of types bounded by $t(\mathrm{supp}(g))$ will have the same property. The result follows. ■

Let t be an isomorphism type in S. A set $I \subseteq \mathrm{sec}(t)$ is an *ideal* in $\mathrm{sec}(t)$ if (1) I is non-empty (2) $s \in I$ and $r \leqslant s$ imply $r \in I$ (3) if s and s' are elements of I, so, too, is $s + s'$. An ideal is *principal* if it is of the form $I = \mathrm{sec}(s)$ for some s with $s = \omega s$.

2.6. Proposition. *Let X be a subset of $\mathbf{R}$ such that for each $t \leqslant t_0 = t(X)$, there is some type s with $t = 2s$. For each ideal I in $\mathrm{sec}(t_0)$, define*

$$H(I) = \{\hat{f} \in G(X)\colon t(\mathrm{supp}(f)) \in I\}.$$

Then the mapping $I \to H(I)$ is one-one and order preserving from ideals of $\mathrm{sec}(t_0)$ to normal subgroups of $G(X)$.

If each $s \leqslant t_0$ is countably compact, then $I \to H(I)$ is surjective.

Proof. If I is an ideal in $\mathrm{sec}(t_0)$, and f, g are automorphisms of X, then the relations

$$\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g), \quad \mathrm{supp}(gfg^{-1}) = g(\mathrm{supp}(f))$$

show that $H(I)$ is a normal subgroup of $G(X)$. Clearly, $I_1 \subseteq I_2$ implies that $H(I_1) \subseteq H(I_2)$.

Now suppose that I_1 and I_2 are distinct ideals of $\sec(t_0)$. Suppose, say, that $s \in I_1 - I_2$. By hypothesis, there are disjoint sets B_1 and B_2 in $\mathscr{B}(X)$ with $s = t(B_1 \cup B_2)$, $t(B_1) = t(B_2)$. Let $f\colon B_1 \to B_2$ be a Borel isomorphism and define $g\colon X \to X$ by

$$g(x) = \begin{cases} f(x) & x \in B_1, \\ f^{-1}(x) & x \in B_2, \\ x & x \in X - (B_1 \cup B_2). \end{cases}$$

Then g is an automorphism with $t(\operatorname{supp}(g)) = s$, and $\hat{g} \in H(I_1) - H(I_2)$. Thus the mapping $I \to H(I)$ is one-one.

Now suppose that each $s \leqslant t_0$ is countably compact and that H is a normal subgroup of $G(X)$. Define

$$I = \{t(\operatorname{supp}(g))\colon \hat{g} \in H\}.$$

We show that I is an ideal and that $H(I) = H$. Suppose that $s \leqslant t(\operatorname{supp}(g))$ for some $\hat{g} \in H$. As above, there is some $\hat{f} \in G(X)$ with $t(\operatorname{supp}(f)) = s$. Apply Lemma 2.5 and conclude that $\hat{f}$ is in the normal closure of $\hat{g}$; hence, $\hat{f} \in H$ and $s \in I$. We have proved that I is order-hereditary. Next, given $\hat{f}$ and $\hat{g}$ in H with $s_1 = t(\operatorname{supp}(f))$ and $s_2 = t(\operatorname{supp}(g))$, let h be an automorphism of X with $\operatorname{supp}(h) = \operatorname{supp}(g) - \operatorname{supp}(f)$. Again Lemma 2.5 implies that $\hat{h} \in H$. Then $\operatorname{supp}(fh) = \operatorname{supp}(f) \cup \operatorname{supp}(g)$, which is (Lemma 1.4) of isomorphism type $s_1 + s_2 = s_1 \vee s_2$. I is an ideal. Obviously, $H \subseteq H(I)$. Given $\hat{f} \in H(I)$, we know that there is some $\hat{g} \in H$ with $t(\operatorname{supp}(f)) = t(\operatorname{supp}(g))$. Another application of Lemma 2.5 shows that $\hat{f} \in H$. Thus $H = H(I)$. ■

2.7. Corollary. *Let X be a subset of* **R** *such that* $\sec(t(X))$ *is finite, and* $\mathscr{N}$ *be the poset of normal subgroups of* $G(X)$. *Then* $\mathscr{N}$ *and* $\sec(t(X))$ *are isomorphic complete, distributive lattices.*

Proof. Apply the proposition, noting that every ideal in $\sec(t(X))$ is principal. Then use [11; 3.30, 3.32] and the fact that the normal subgroups of a group form a lattice. ■

As mentioned before, under analytic determinacy, the set of analytic types is

$$\sec(t(U)) = \{0, t(\mathbf{R}), t(U)\}$$

for $U \subseteq \mathbf{R}$ universal analytic. Corollary 2.7 shows that in this case, $G(U)$ has exactly three normal subgroups.

2.8. Corollary. *Let X be a subset of* **R** *with $t(X)$ a cover of* 0. *Then $G(X)$ is simple. In particular, $G(X)$ is simple for X a Borel subset of* **R**.

3. Automorphisms of analytic sets. A subset X of $\mathbf{R}$ is *topologically rigid* if the only homeomorphism of X onto itself is the identity map. In [6], it is was shown that there are no zero-dimensional, topologically rigid Borel subsets of $\mathbf{R}$ other than singletons. However, it was also shown that, under $V = L$, there is an uncountable, zero-dimensional, topologically rigid, analytic subset of $\mathbf{R}$. As we shall see, this

matter is related to the question of whether $G(A)$ can be simple for A analytic, non-Borel. It is also related to the problem of whether an uncountable co-analytic set can be measurably rigid.

3.1. Lemma. *Let A be an analytic subset of* **R** *and let $N(A)$ be the subgroup of $G(A)$ comprising all $\hat{f}$ for which* $\operatorname{supp}(f)$ *is a Borel subset of* **R**. *Then $N(A)$ is a simple group.*

Proof. The case where A is countable is trivial; for A uncountable and Borel, Corollary 2.8 pertains. So suppose A analytic and non-Borel. Let f and g be automorphisms of A with $\operatorname{supp}(g)$ uncountable and $\operatorname{supp}(f)$ Borel in $\mathbf{R}$. Then:

$$t(\operatorname{supp}(f)) = \omega t(\operatorname{supp}(f)), \quad t(\operatorname{supp}(f)) \leqslant t(\operatorname{supp}(g)),$$
$$t(\operatorname{supp}(f)) \leqslant t(A - \operatorname{supp}(f)) = t(A).$$

Since each $r \leqslant t(\operatorname{supp}(f))$ is countably compact, there is, by Lemma 2.3, some $B \in \mathscr{B}(A)$ such that $t(B) \geqslant t(\operatorname{supp}(f))$ and $g(B) \cap B = \emptyset$. An application of Lemma 2.2 shows that the normal closure of $\hat{g}$ contains $\hat{f}$. There follows the simplicity of $N(A)$. ■

3.2. Corollary. *Let $A \subseteq \mathbf{R}$ be analytic. Then $G(A)$ is simple if and only if $G(A) = N(A)$.*

Proof. We may assume that A is uncountable. It is then easily checked that $N(A)$ is a normal subgroup of $G(A)$ with more than one element. If $G(A)$ is simple, then $N(A) = G(A)$. The proposition supplies the converse. ■

Remark. The earlier result (Corollary 2.8), that $G(A)$ is simple for A Borel, may be obtained from this proposition as well.

3.3. Proposition. *Consider the following six statements*:

(1) *Any two analytic, non-Borel subsets of* **R** *are Borel-isomorphic.*

(2) *The isomorphism types of analytic sets are linearly ordered.*

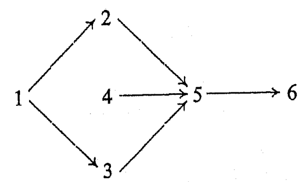
(3) *The partial ordering of analytic isomorphism types is well-founded.*

(4) *Other than singleton sets, there are no topologically rigid, zero-dimensional analytic subsets of* **R**.

(5) *If $A \subseteq \mathbf{R}$ is analytic, then $G(A)$ is simple if and only if A is Borel.*

(6) *There is no uncountable, measurably rigid, co-analytic subset of* **R**.

These implications obtain:



Proof. $1 \Rightarrow 2$ and $1 \Rightarrow 3$. Obvious.

$2 \Rightarrow 5$ and $3 \Rightarrow 5$ and $4 \Rightarrow 5$. It was noted in Corollary 2.8 that $G(A)$ is simple if A is Borel. Now assume that there is some analytic, non-Borel set A with $G(A)$ simple. By Corollary 3.2, $G(A) = N(A)$. We show that conditions 2, 3 and 4 must fail.

Removing a countable dense subset of $\boldsymbol{R}$ if necessary, we may assume that A is zero-dimensional. Let $\mathscr{V}$ be the collection of all open subsets V of $\boldsymbol{R}$ such that $V \cap A$ is Borel in $\boldsymbol{R}$. Put $A_0 = A - \bigcup \mathscr{V}$. By the Lindelof theorem, we see that $A \cap (\bigcup \mathscr{V})$ is Borel in $\boldsymbol{R}$, so that A_0 is not. In fact, A_0 and A Borel-isomorphic. Thus $G(A_0) = N(A_0)$. We stake the

CLAIM. *Suppose that V_1 and V_2 are open subsets of $\boldsymbol{R}$ with $U_1 = V_1 \cap A_0$ and $U_2 = V_2 \cap A_0$ non-void and disjoint. Then U_1 [resp. U_2] is not Borel-isomorphic with any set in $\mathscr{B}(U_2)$ [resp. $\mathscr{B}(U_1)$].*

Proof of claim. Suppose rather that f is a Borel-isomorphism mapping U_1 onto some set $B \in \mathscr{B}(U_2)$. Define $g\colon A_0 \to A_0$ by

$$g(x) = \begin{cases} f(x) & x \in U_1, \\ f^{-1}(x) & x \in B, \\ x & x \in A_0 - (U_1 \cup B). \end{cases}$$

Then g is a Borel-isomorphism of A_0 whose support contains U_1 and is thus not Borel in $\boldsymbol{R}$. This would contradict $G(A_0) = N(A_0)$.

From the claim it follows that A_0 is a topologically rigid set: condition 4 fails. Also, with U_1 and U_2 as above, we see that the types $t(U_1)$ and $t(U_2)$ are not comparable: condition 2 fails. Finally, let $V_1 V_2 \ldots$ be a sequence of open subsets of $\boldsymbol{R}$ such that the sets $U_n = V_n \cap A_0$ are disjoint and non-empty. Then the sets $A_n = V_n \cup V_{n+1} \cup \ldots$ are analytic, and $t(A_1) > t(A_2) > \ldots$: condition 3 fails.

$5 \Rightarrow 6$. We prove the contrapositive. Let C be an uncountable, measurably rigid, co-analytic subset of $\boldsymbol{R}$. Of necessity, C and $A = \boldsymbol{R} - C$ are non-Borel. Now suppose that f is a Borel-automorphism of A. Then f extends to an automorphism g of $\boldsymbol{R}$, and the restriction $h\colon C \to C$ of g to C is again an automorphism. Since C is measurably rigid, it must be that $\operatorname{supp}(h)$ is countable, whence it follows that $\operatorname{supp}(f)$ is Borel in $\boldsymbol{R}$. We have shown that $G(A) = N(A)$. From Corollary 3.2, we see that $G(A)$ is simple. Thus condition 5 fails. ■

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