A reduction of the Nielsen fixed point theorem for symmetric product maps to the Lefschetz theorem

by

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Abstract. In 1957, C. N. Maxwell [3] defined a Lefschetz number and proved a Lefschetz fixed point theorem for symmetric product maps of compact polyhedra. In 1979, S. Masih [2] defined a fixed point index and a Nielsen number for such maps. The goal of this paper is a proof that the Masih–Nielsen number is 0 or 1 and it equals 1 if and only if the Lefschetz number does not equal zero. Consequently the Nielsen theorem [2] for symmetric product maps reduces to Lefschetz's.

Introduction. Let X be a topological space and $n \geqslant 2$ be an integer. The n-th symmetric group S_n acts on the n-th cartesian product X^n by the formula $s(x_1, \ldots, x_n) = (x_{s(1)}, \ldots, x_{s(n)})$. The n-th symmetric product X_n of X is the orbit space X^n/S_n. Continuous maps from X to X_n are called symmetric product maps.

Let $q\colon X^n \to X_n$ be the projection onto the quotient space and $f\colon X \to X_n$ be a symmetric product map. A point x is said to be a fixed point of f if $f(x)$ is the projection of a point having x as a coordinate. According to [2], fixed points x, y of f are in the same *fixed point class* if there exists a path C in the cartesian product X^n such that the first coordinate C_1 of C is a path from x to y and $f \circ C_1$ is fixed end-point homotopic to $q \circ C$ ($f \circ C_1 \simeq q \circ C$).

The main result. The notion of fixed point classes is trivial in view of the following:

PROPOSITION. *If X is a pathwise connected topological space, then every symmetric product map $f\colon X \to X_n$ has at most one fixed point class.*

Proof. Let x, y be fixed points of f. There are $\bar{x}, \bar{y} \in X^{n-1}$ such that $f(x) = q(x, \bar{x}), f(y) = q(y, \bar{y})$. Let C_1 be a path from x to y and $\bar{D}$ be a path from $\bar{x}$ to $\bar{y}$. The projection $q\colon X^n \to X_n$ induces an epimorphism of fundamental groups [1], p. 91, Cor. 6.3. Since X^n is a pathwise connected space, this result does not depend on the choice of the basepoint of the fundamental group of X^n, in particular there is a loop $(E, \bar{E})$ based in $(x, \bar{x})$ such that $(f \circ C_1) * (q(C_1, \bar{D}))^{-1} \simeq q(E, \bar{E})$. Let (x', x'') be coordinates of $\bar{x}$ in $X \times X^{n-2}$ and $(E', E'') = \bar{E}$ (if $n = 2$ then the second coordinate should be omitted). Let F be a path from x to x', $G = F^{-1} * E * F$, $\bar{H} = (G * E', x'' * E'')$. Then $q(G, \bar{E}) \simeq q((G, \bar{x}) * (x', \bar{E})) = q(x', \bar{H})$ and $f \circ C_1$

$\simeq q(E, \bar{E}) * q(C_1, \bar{D}) \simeq q(F, \bar{x}) * q(G, \bar{E}) * q(F^{-1}, \bar{x}) * q(C_1, \bar{D}) \simeq q(C_1, \bar{H} * \bar{D})$, i.e. the points x, y are in the same class. ■

I would like to express my gratitude to Zdzisław Dzedzej who associated the proof of the above proposition with the subject of [1].

The Nielsen theory. In 1983, N. Rallis developed a fixed point index theory for symmetric product maps of compact euclidean neighbourhood retracts (ENRs). Using this index we follow S. Masih [2] and define the Nielsen number of $f\colon X \to X_n$ as the number of essential fixed point classes (a class is an essential class if its index does not equal zero).

Now, let X be a compact connected ENR space. It follows from the Proposition that there exists at most one fixed point class of $f\colon X \to X_n$. By [4], the index of the unique fixed point class equals the Lefschetz number of f.

COROLLARY. *The Nielsen number $N(f)$ of the symmetric product map $f\colon X \to X_n$ equals* 0 *if the Lefschetz number equals* 0 *and it equals* 1 *otherwise.* ■

Remark. In 1988, H. Schirmer [5] and [6] proved that if X is a triangulable manifold of dimension not less than 3, then for any map $f\colon X \to X_2$ there exists a map homotopic to f having exactly $N(f)$ fixed points.

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Normal subgroups of measurable automorphisms

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Abstract. Given $X \subseteq \boldsymbol{R}$, let $F(X)$ be the group of all Borel-isomorphisms of X onto itself and let $CO(X)$ be the normal subgroup comprising all f with $\{x\colon f(x) \neq x\}$ countable. Then $G(X) = F(X)/CO(X)$ is the "reduced automorphism group" of X. Using a technique of R. D. Anderson (Lemma 2.1), we explore the normal subgroup structure of $G(X)$. For certain sets X, there is a simple relationship between this structure and the poset of Borel-isomorphism types of measurable subsets of X (Proposition 2.6). As a consequence of this result, one sees that $G(X)$ is a simple group when X is a Borel subset of $\boldsymbol{R}$ (Corollary 2.8).

Whether $G(A)$ can be simple for A analytic and non-Borel is not known. Under analytic determinacy and even weaker assumptions, the answer is no. If there is such as et A, then there is a topologically rigid, zero-dimensional $A' \subseteq \boldsymbol{R}$ Borel-isomorphic with A (Proposition 3.3). Such rigid sets have been considered by van Engelen, Miller, and Steel.

1. Preliminary notions. We deal exclusively with subsets of the real line, although all our results are easily transferable to the context of any complete, separable, metric space. A subset X of $\boldsymbol{R}$ is *analytic* if it is a continuous image of the space of irrational numbers (or is empty). A set $X \subseteq \boldsymbol{R}$ is *co-analytic* if $\boldsymbol{R} - X$ is analytic. Souslin showed that X is a Borel subset of $\boldsymbol{R}$ if and only if X is both analytic and co-analytic [7; pp. 485–6]. Given $X \subseteq \boldsymbol{R}$, define

$$\mathscr{B}(X) = \{B \cap X\colon B \text{ is a Borel subset of } \boldsymbol{R}\}.$$

Then $\mathscr{B}(X)$ is a σ-field whose elements are the *measurable subsets* of X. A function $f\colon X \to Y$ is a (Borel) *isomorphism* if f is a one-one correspondence of X with Y, and $B \in \mathscr{B}(X)$ if and only if $f(B) \in \mathscr{B}(Y)$. If $X = Y$, then f is a (Borel) *automorphism*. If f is an automorphism of X, define its *support* as

$$\operatorname{supp}(f) = \{x \in X\colon f(x) \neq x\}.$$

The collection $F(X)$ of all automorphisms of X forms a group under composition. Note that if f and g are automorphisms of X, then $\operatorname{supp}(g \circ f \circ g^{-1}) = g(\operatorname{supp}(f))$.

Let $CO(X)$ be the set of all $f \in F(X)$ such that $\operatorname{supp}(f)$ is countable. Then $CO(X)$ is a normal subgroup of $F(X)$, and we call the quotient $G(X) = F(X)/CO(X)$ the *reduced automorphism group* of X. If $f \in F(X)$, then $\hat{f}$ is its coset in $G(X)$. A set X is *measurably rigid* if $G(X)$ is trivial. Uncountable measurably rigid sets exist (ZFC),