Universal spaces for locally finite-dimensional and strongly countable-dimensional metrizable spaces

by

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Abstract. In the present paper, we describe concrete and simple examples of universal spaces: namely, the spaces $V(\tau)$, $W(\tau)$ and $U(\tau)$ (for the definitions, see Section 3) — universal for locally finite-dimensional metrizable spaces of weight not larger than τ, and the spaces $E(\tau)$, $F(\tau)$ and $C(\tau)$ (for the definitions, see Section 4) — universal for strongly countable-dimensional metrizable spaces of weight not larger than τ. We give also some information about other universal spaces for the above classes of spaces (see Sections 1 and 5).

1. Introduction. We restrict our considerations to metrizable spaces, and the word "space" means here, except in the definition of partial products, "metrizable space". Our terminology and notation follow the books [4] and [5]. In particular, the symbol $J(\tau)$ denotes the hedgehog of spininess τ with its standard metric d (see [4], Example 4.1.5), and 0 the "origin" of this hedgehog.

Let us first recall the definitions of a locally finite-dimensional space and a strongly countable-dimensional space, and outline the history of the problem.

A space is *locally finite-dimensional* if it has an open cover by finite-dimensional sets.

B. R. Wenner showed in [19] that the subspace J_ω of the Hilbert cube $I^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in I \times I \times \ldots$ which satisfy the conditions:

(1) there exists an $n \in N$ such that $x_n \neq 0$,

(2) there exists a $k \in N$ such that $x_m \leqslant 1/k$ for $m < k$ and $x_m = 0$ for $m \geqslant k$

is universal for locally finite-dimensional spaces of weight $\aleph_0$. A few years later, L. Luxemburg established in [9] the existence of a universal space for locally finite-dimensional spaces of weight not larger than τ, where τ is an arbitrary cardinal number, but he did not indicate any concrete example of such a space. The same result can also be found in L. J. Bobkov's paper [2] and M. G. Charalambous' paper [3]; they have both proved a factorization theorem, and — using Pasynkov's method — deduced from it just the existence of a universal space. In addition, L. J. Bobkov announced in [2] a concrete example (involving the notion of partial product) of a universal space for locally finite-dimensional spaces of weight not larger than τ (see Remark 5.1).

A space is *strongly countable-dimensional* if it is the union of a countable family of finite-dimensional closed sets.

J. M. Smirnov and J. Nagata showed independently in [17] and [11] that the subspace K_ω of the Hilbert cube $I^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in I \times I \times \ldots$ with only finitely many non-zero coordinates is universal for strongly countable-dimensional spaces of weight $\aleph_0$. A few years later, A. Arhangel'skiĭ established in [1] the existence of a universal space for strongly countable-dimensional spaces of weight not larger than τ, where τ is an arbitrary cardinal number. The same result can also be found in B. A. Pasynkov's paper [14]. They have both deduced the existence of a universal space from the corresponding factorization theorem.

The universal spaces for locally finite-dimensional spaces and strongly countable-dimensional spaces we shall construct here follow the pattern of the spaces J_ω and K_ω. However, the following example, due to R. Engelking, shows that the most obvious analogues of the spaces J_ω and K_ω are not universal spaces.

1.1. Example. Let us denote by $J_\omega(\tau)$ the subspace of $[J(\tau)]^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in J(\tau) \times J(\tau) \times \ldots$ which satisfy the conditions:

(3) there exists an $n \in N$ such that $x_n \neq 0$,

(4) there exists a $k \in N$ such that $d(x_m, 0) \leqslant 1/k$ for $m < k$ and $x_m = 0$ for $m \geqslant k$,

and by $K_\omega(\tau)$ the subspace of $[J(\tau)]^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in J(\tau) \times J(\tau) \times \ldots$ with only finitely many non-zero coordinates.

One can easily check that for each natural n, the Baire space $B(\tau)$ of weight $\tau > \aleph_0$ is not homeomorphic to any subspace of $[J(\tau)]^n$. Applying the Baire category theorem, we deduce that neither $J_\omega(\tau)$ nor $K_\omega(\tau)$ contains a copy of $B(\tau)$, whence $J_\omega(\tau)$ is not universal for locally finite-dimensional spaces of weight not larger than τ, and $K_\omega(\tau)$ is not universal for strongly countable-dimensional spaces of weight not larger than τ.

Acknowledgements. The paper contains some of the results from my Master's Thesis, which was supervised by Professor R. Engelking, whom I would like to thank for his comments and improvements. I am also grateful to Professor R. Pol for valuable discussion. He simplified my original proof of the universality of the spaces $E(\tau)$ and $F(\tau)$ (the proof given here is carried out along his indications) and suggested the use of the category method in the proofs of the corollaries from Section 2.

2. Auxiliary results. For any space X, we shall consider on the space $C(X, [J(\tau)]^{\aleph_0})$ the metric d^*, where

$$d^*(f, g) = \sup\{\sum_{i=0}^{\infty} 1/2^i d(f_i(x), g_i(x)) \colon x \in X\}$$

for $f = (f_0, f_1, \ldots), g = (g_0, g_1, \ldots) \in C(X, [J(\tau)]^{\aleph_0})$; the space $C(X, [J(\tau)]^{\aleph_0})$ with the metric d^* is complete (see [4], Theorems 4.3.12 and 4.3.13).

We shall also consider the subspace $K_n(\tau)$ of $[J(\tau)]^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in J(\tau) \times J(\tau) \times \ldots$ such that the set $\{k \in N\colon d(0, x_k)$ is a positive rational number$\}$ has at most n elements; one can prove (see [12], Theorem VI. 10) that $\dim K_n(\tau) = n$.

Let us recall that a subset of a space X is *residual* if it contains a dense G_δ-set of X.

E. Pol has proved in [16] the following theorem.

2.1. Theorem. *If X is an n-dimensional space of weight not larger than τ, then the set*

$$\mathscr{H} = \{h \in C(X, [J(\tau)]^{\aleph_0})\colon h \text{ is an embedding and } \operatorname{cl} h(X) \subseteq K_n(\tau)\}$$

is residual in $C(X, [J(\tau)]^{\aleph_0})$.

J. Nagata has established in [9] the following theorem (see also [18]).

2.2. Theorem. *For each n-dimensional space X of weight not larger than τ, there exists a 1-dimensional space Z of weight not larger than τ such that X is homeomorphic to a subspace of Z^{n+1}.*

From the above theorems, we shall deduce in the sequel a few corollaries, but first let us formulate two lemmas.

2.3. Lemma. *If X is a space of weight not larger than τ, then the set of all homeomorphic embeddings of X into $[J(\tau)]^{\aleph_0}$ is residual in $C(X, [J(\tau)]^{\aleph_0})$.*

2.4. Lemma. *Let X be a space of weight not larger than τ, and F its closed subset. If f is a continuous map of X to $J(\tau)$, and g is a continuous map of F to $J(\tau)$ such that $d(f(x), g(x)) \leqslant \varepsilon$ for $x \in F$, then the map g is continuously extendable to a map g^* of X such that $d(f(x), g^*(x)) \leqslant 2\varepsilon$ for $x \in X$.*

The reader can find a proof of Lemma 2.3 in E. Pol's paper [16], and the proof of Lemma 2.4 is straightforward.

We shall denote by $\mathbf{0}$ the point of $[J(\tau)]^{\aleph_0}$ with all coordinates equal to 0; the same symbol will denote the corresponding point of $([J(\tau)]^{\aleph_0})^{n+1}$, which is canonically homeomorphic to $[J(\tau)]^{\aleph_0}$.

For any space X and any point $x_0 \in X$, by $C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))$ we shall denote the subspace of $C(X, [J(\tau)]^{\aleph_0})$ consisting of all maps f such that $f(x_0) = \mathbf{0}$; obviously, the space $C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))$ is complete as a closed subspace of $C(X, [J(\tau)]^{\aleph_0})$.

Let us state now the corollaries announced above. The first of them will be applied only in our discussion of locally finite-dimensional spaces (see Theorem 3.2), whereas the third will be used only in the discussion of strongly countable-dimensional spaces (see Theorem 4.1).

2.5. Corollary. *For each n-dimensional space X of weight not larger than τ and any point $x_0 \in X$, the set*

$$\mathscr{P} = \{h \in C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))\colon h \text{ is an embedding and } \operatorname{cl} h(X) \subseteq K_n(\tau)\}$$

is residual in $C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))$.

Proof. Let F be a closed subset of X such that $x_0 \notin F$. We shall show that

(1) the set $\mathscr{K}_F = \{f \in C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0})): f|F$ is an embedding and $\operatorname{cl} f(F) \subseteq K_n(\tau)\}$ is residual in $C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))$.

Indeed, by virtue of Theorem 2.1, the set $\mathscr{N}_F = \{g \in C(F, [J(\tau)]^{\aleph_0}): g$ is an embedding and $\operatorname{cl} g(F) \subseteq K_n(\tau)\}$ is residual in $C(F, [J(\tau)]^{\aleph_0})$, and by Lemma 2.4 the operation of restriction T is an open map of $C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))$ onto $C(F, [J(\tau)]^{\aleph_0})$. Hence the set $\mathscr{K}_F = T^{-1}(\mathscr{N}_F)$ is also residual.

It is easy to verify that

(2) the set $\mathscr{L}_F = \{f \in C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0})): \mathbf{0} \notin \operatorname{cl} f(F)\}$ is open and dense in $C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))$.

Let σ be an arbitrary metric on X, and $F_n = \{x \in X: \sigma(x_0, x) \geqslant 1/n\}$ for $n = 1, 2, \ldots$ By virtue of (1) and (2), the set $\mathscr{M} = \bigcap_{n=1}^{\infty} \mathscr{K}_{F_n} \cap \bigcap_{n=1}^{\infty} \mathscr{L}_{F_n}$ is residual in $C((X, x_0), ([J(\tau)]^{\aleph_0}, \mathbf{0}))$. Since $\mathscr{M} \subseteq \mathscr{P}$, the set $\mathscr{P}$ is also residual.

2.6. Corollary. *For each n-dimensional space X of weight not larger than τ and any point $x_0 \in X$, there exists a homeomorphic embedding f of X into $[K_1(\tau)]^{n+1}$ such that $f(x_0) = \mathbf{0}$.*

Proof. Let $g = (g_1, g_2, \ldots, g_{n+1})$ be a homeomorphic embedding of X into Z^{n+1}, where Z is a 1-dimensional space of weight not larger than τ (see Theorem 2.2). Further, for $i = 1, 2, \ldots, n+1$, let f_i be a homeomorphic embedding of Z into $K_1(\tau)$ such that $f_i(g_i(x_0)) = \mathbf{0}$ (see Corollary 2.5).

Then the map $f = (f_1 \circ g_1, f_2 \circ g_2, \ldots, f_{n+1} \circ g_{n+1})$ is a homeomorphic embedding and $f(x_0) = \mathbf{0}$.

2.7. Corollary. *For each space X of weight not larger than τ and any sequence $\{F_i: i = 1, 2, \ldots\}$ of its finite-dimensional closed subsets, where $\dim F_i = \alpha_i$ for $i = 1, 2, \ldots$, the set $\mathscr{R} = \{h \in C(X, [J(\tau)]^{\aleph_0}): h$ is an embedding and $\operatorname{cl} h(F_i) \subseteq K_{\alpha_i}(\tau)$ for $i = 1, 2, \ldots\}$ is residual in $C(X, [J(\tau)]^{\aleph_0})$.*

Proof. By virtue of Theorem 2.1, the set $\mathscr{K}_i = \{h \in C(F_i, [J(\tau)]^{\aleph_0}): h$ is an embedding and $\operatorname{cl} h(F_i) \subseteq K_{\alpha_i}(\tau)\}$ is residual in $C(F_i, [J(\tau)]^{\aleph_0})$ for $i = 1, 2, \ldots$ From Lemma 2.3 it follows that the set $\mathscr{K}_0 = \{h \in C(X, [J(\tau)]^{\aleph_0}): h$ is an embedding$\}$ is residual, and Lemma 2.4 implies that the operation of restriction T_i is an open map of $C(X, [J(\tau)]^{\aleph_0})$ onto $C(F_i, [J(\tau)]^{\aleph_0})$.

Hence the set $\mathscr{K} = \mathscr{K}_0 \cap \bigcap_{i=1}^{\infty} T_i^{-1}(\mathscr{K}_i)$ is residual in $C(X, [J(\tau)]^{\aleph_0})$. Since $\mathscr{K} \subseteq \mathscr{R}$, the set $\mathscr{R}$ is residual, too.

2.8. Remark. In the sequel, we shall need only the fact that for each n-dimensional space X of weight not larger than τ and any point $x_0 \in X$, there exists an embedding $h: X \to [J(\tau)]^{\aleph_0}$ such that $h(X) \subseteq K_n(\tau)$ and $h(x_0) = \mathbf{0}$, and that for each space X of weight not larger than τ and any sequence $\{F_i: i = 1, 2, \ldots\}$ of its closed sets with $\dim F_i = \alpha_i < \infty$ for $i = 1, 2, \ldots$, there exists an embedding $h: X \to [J(\tau)]^{\aleph_0}$ such that $h(F_i) \subseteq K_{\alpha_i}(\tau)$ for $i = 1, 2, \ldots$ This can be deduced by a standard argument

from Remark 5.1 in [15], which contains the basic lemma for E. Pol's simple proof of the universality of $K_n(\tau)$ for n-dimensional spaces of weight not larger than τ. Instead of the lemma, we can also use the original method of J. Nagata, i.e. Proposition VI.2.A from [12].

Now, we shall state a technical lemma that will be applied in the proofs of Theorems 3.2 and 4.2. This lemma is a consequence of Hausdorff's theorem on extending mapping (see Problem 4.5.20 in [4]), but it can also be proved in a more direct way.

2.9. Lemma. *For each space X of weight not larger than τ and any its closed subset E, there exist a space Y of weight not larger than τ, a point $y \in Y$ and a continuous map q of X to Y such that $q^{-1}(y) = E$ and $q|X - E$ is a homeomorphism onto $Y - \{y\}$.*

The remaining part of this section is devoted to some notions and results which will be used in the proof of Theorem 4.2.

We say that a continuous map f of a metric space X to a metric space Y is *uniformly 0-dimensional on a compact $Z \subseteq Y$* if for each point $z \in Z$ and each real number $\varepsilon > 0$, there exists a neighbourhood W of z in Y such that the set $f^{-1}(W)$ is the union of a family of its open pairwise disjoint subsets of diameter less than ε. It is easy to observe that if Y is compact, then our notion of uniformly 0-dimensional map $f: X \to Y$ on the compact Y is equivalent to the notion of uniformly 0-dimensional map introduced by M. Katětov in [7].

In [6], M. Katětov has proved the following theorem (see also [7] and [12], Section III.8).

2.10. Theorem. *For every n-dimensional metric space X, there exists a uniformly 0-dimensional map f of X to the n-dimensional cube I^n.*

The next theorem is a generalization of Theorem 2.10. In this theorem, we consider the n-dimensional cube I^n as the lower face of the $(n+1)$-dimensional cube I^{n+1}.

2.11. Theorem. *If F is a closed n-dimensional subset of a metric space X, then there exists a continuous map $f: X \to I^{n+1}$ such that f is uniformly 0-dimensional on I^n and $f^{-1}(I^n) = F$.*

Proof. Let g be a uniformly 0-dimensional map of F to I^n (see Theorem 2.10), and $g_*: X \to I^n$ an extension of the map g over X. By virtue of Proposition III.8.D from [12], for every $m \in N$, there exists an open subset U_m of X containing F and a real number $\delta_m > 0$ such that

(3) if $W \subseteq I^n$ is an open set of diameter less than δ_m, then the set $g_*^{-1}(W) \cap U_m$ is the union of a family of pairwise disjoint open subsets of X of diameters less than $1/m$.

Obviously, we can assume that $\operatorname{cl} U_{m+1} \subseteq U_m$ for each $m \in N$ and $F = \bigcap_{m=1}^{\infty} U_m$. Let h be a continuous map of X to the unit interval I such that $h^{-1}([0, 1/m)) = U_m$ for each $m \in N$.

We shall verify that the map $f = g_* \triangle h\colon X \to I^{n+1}$ has the required properties.

Let us take a point $z \in I^n$ and a real number $\varepsilon > 0$, and next a number $m \in N$ such that $1/m < \varepsilon$, and a neighbourhood $W \subseteq I^n$ of z of diameter less than δ_m. By virtue of (3), the set

$$f^{-1}(W \times [0,1/m)) = g_*^{-1}(W) \cap h^{-1}([0,1/m)) = g_*^{-1}(W) \cap U_m$$

is the union of a family of pairwise disjoint open sets of diameters less than $1/m$, and therefore the set $V = W \times [0,1/m)$ is a neighbourhood of z in I^{n+1} the inverse image of which is the union of a family of pairwise disjoint open sets of diameters less than ε. Thus f is uniformly 0-dimensional on I^n. Since $F = \bigcap_{m=1}^{\infty} U_m$, we obtain $f^{-1}(I^n) = F$.

Let us recall the basic information on Pasynkov's partial products. For a topological space X, its open subset U, and a topological space Z, the formula

$$q(y) = \begin{cases} y & \text{if } y \in X - U, \\ x & \text{if } y = (x, z) \in U \times Z \end{cases}$$

defines a map of a set $Y = (X-U) \cup (U \times Z)$ to X; by the *partial product* $P(X, U, Z)$ we mean the set Y with the topology generated by the family of sets $\{q^{-1}(V)\colon V \text{ is an open subset of } X\} \cup \{V \times W\colon V \text{ is an open subset of } U, \text{ and } W \text{ is an open subset of } Z\}$. Obviously, q is a continuous map of $P(X, U, Z)$ to X.

Consider now a topological space X, a family $\{U_\alpha\colon \alpha \in A\}$ of its open subsets, and a family $\{Z_\alpha\colon \alpha \in A\}$ of topological spaces. We know how to define for each $\alpha \in A$ the partial product $P(X, U_\alpha, Z_\alpha)$ and the map q_α of $P(X, U_\alpha, Z_\alpha)$ to X. By *the partial product* $P(X, \{U_\alpha\colon \alpha \in A\}, \{Z_\alpha\colon \alpha \in A\})$ we mean the set of all points $\{y_\alpha\colon \alpha \in A\}$ of the Cartesian product $\mathbf{P}_{\alpha \in A} P(X, U_\alpha, Z_\alpha)$ such that $q_\alpha(y_\alpha) = q_\beta(y_\beta)$ for every $\alpha, \beta \in A$, endowed with the coarsest topology for which all the projections p_β (where $p_\beta(\{y_\alpha\colon \alpha \in A\}) = y_\beta$ for $\beta \in A$) are continuous. If $Z_\alpha = Z$ for each $\alpha \in A$, then we write $P(X, \{U_\alpha\colon \alpha \in A\}, Z)$ rather than

$$P(X, \{U_\alpha\colon \alpha \in A\}, \{Z_\alpha\colon \alpha \in A\}).$$

The notion of the partial product is thoroughly investigated in B. A. Pasynkov's paper [13]. In particular, one can find there the proofs of the following three properties of the partial product $P(X, \{U_i\colon i \in N\}, D(\tau))$, where $D(\tau)$ is a discrete space of cardinality $\tau \geqslant \aleph_0$ (see [13], (9) p. 177, (12) p. 175 and (8) p. 183).

A. If X is metrizable, then $P(X, \{U_i\colon i \in N\}, D(\tau))$ is also metrizable.

B. If the weight of X is not larger than τ, then the weight of

$$P(X, \{U_i\colon i \in N\}, D(\tau))$$

is not larger than τ.

C. If X is n-dimensional, then $P(X, \{U_i\colon i \in N\}, D(\tau))$ is also n-dimensional.

In the proof of the next theorem, we shall apply the same argument as that used by B. A. Pasynkov in the proof of Theorem 8.1 of [13].

2.12. THEOREM. *For each space X of weight not larger than τ and any its n-dimensional closed subset F, there exist an $(n+1)$-dimensional space Z of weight not larger than τ and a continuous map $g\colon X \to Z$ separating points of F and closed subsets of X.*

Proof. Let us consider a map $f\colon X \to I^{n+1}$ such that f is uniformly 0-dimensional on I^n and $f^{-1}(I^n) = F$. Then, for each $x \in F$ and each $\varepsilon > 0$, there exists a neighbourhood V of $f(x)$ in I^{n+1} such that

(4) the set $f^{-1}(V)$ is the union of a family of pairwise disjoint open sets of diameters less than ε.

Let $\{V_k\colon k \in N\}$ be a countable family of open subsets of I^{n+1} containing for each $x \in F$ and each $\varepsilon > 0$ a neighbourhood V_k of $f(x)$ satisfying condition (4), and let $Z = P(I^{n+1}, \{V_k\colon k \in N\}, D(\tau))$. From the properties A, B, and C of partial products it follows that Z is an $(n+1)$-dimensional space of weight not larger than τ.

We are going to define maps $g_k\colon X \to P(I^{n+1}, V_k, D(\tau))$ which will satisfy the condition.

(5) $q_k \circ g_k(x) = q_l \circ g_l(x)$ for $x \in X$ and $k, l \in N$, where $q_k\colon P(I^{n+1}, V_k, D(\tau)) \to I^{n+1}$ are the maps appearing in the definition of partial products.

We can assume that each member of the family $\{f^{-1}(V_k)\colon k \in N\}$ is the union of a family $\{W_k^\lambda\colon \lambda \in \Lambda\}$ of cardinality τ of pairwise disjoint open subsets of X such that for each $x \in F$ and each $\varepsilon > 0$, there exists a neighbourhood V_k of $f(x)$ such that all members of the family $\{W_k^\lambda\colon \lambda \in \Lambda\}$ are of diameter less than ε. For any $x \in X$, let

$$g_k(x) = \begin{cases} (f(x), h(\lambda)) & \text{if } x \in W_k^\lambda, \\ f(x) & \text{if } x \notin \bigcup_{\lambda \in \Lambda} W_k^\lambda, \end{cases}$$

where h is an injection of the set Λ into the space $D(\tau)$. From the definition of $P(I^{n+1}, V_k, D(\tau))$ it follows immediately that g_k is a continuous map of X to $P(I^{n+1}, V_k, D(\tau))$. Further, by the definition of q_k and g_k we obtain $(q_k \circ g_k)(x) = f(x) = (q_l \circ g_l)(x)$ for $x \in X$ and $k, l \in N$.

Equality (5) guarantees that the family of maps $\{g_k\colon k \in N\}$ determines the map g of X to $P(I^{n+1}, \{V_k\colon k \in N\}, D(\tau))$. Since $p_k \circ g = g_k$ for $k \in N$, where $p_k\colon P(I^{n+1}, \{V_k\colon k \in N\}, D(\tau)) \to P(I^{n+1}, V_k, D(\tau))$ is the map appearing in the definition of partial products, g is continuous.

It remains to check that the map g separates points of F and closed subsets of X. Let us take a point $x \in F$ and closed subset $E \subseteq X$ such that $x \notin E$, and let us consider a neighbourhood V_k of $f(x)$ such that the diameters of the members of the family $\{W_k^\lambda\colon \lambda \in \Lambda\}$ are less than the distance between the point x and the set E. Then there exists a $\lambda \in \Lambda$ such that the set $p_k^{-1}(V_k \times \{h(\lambda)\})$ is a neighbourhood of $g(x)$ disjoint from $g(E)$, which shows that g separates x from E.

3. Locally finite-dimensional spaces. Let us denote by $V(\tau)$, where τ is an infinite cardinal number, the subspace of $[J(\tau)]^{\aleph_0} = ([J(\tau)]^{\aleph_0})^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in [J(\tau)]^{\aleph_0} \times [J(\tau)]^{\aleph_0} \times \ldots$ which satisfy the conditions:

(1) there exists an $n \in N$ such that $x_n \neq \mathbf{0}$,

(2) there exists a $k \in N$ such that $x_m \leqslant 1/k$ for $m < k$ and $x_m = \mathbf{0}$ for $m \geqslant k$,

(3) $x_n \in K_1(\tau)$ for each $n \in N$,

where the inequality $x_m \leqslant 1/k$ means that the distance between each coordinate of $x_m \in [J(\tau)]^{\aleph_0}$ and $0 \in J(\tau)$ does not exceed $1/k$.

The space $W(\tau)$ is defined similarly, except that condition (3) is replaced by

(3′) $x_n \in K_n(\tau)$ for each $n \in N$.

Before we formulate the main theorem of this section, we have to establish a certain property of locally finite-dimensional spaces.

3.1. THEOREM. *Each locally finite-dimensional space X has an open cover $\{U_i\colon i = 1, 2, \ldots\}$ such that* $\dim U_i \leqslant i$ *for* $i = 1, 2, \ldots$ *and* $U_i \cap U_j = \emptyset$, *whenever* $|i-j| > 1$.

Proof. Clearly, $X = \bigcup_{i=1}^{\infty} V_i$, where $V_1 \subseteq V_2 \subseteq \ldots$ and V_i is an at most i-dimensional open subset of X for $i = 1, 2, \ldots$ Let us consider, for $i = 1, 2, \ldots$, a continuous function $f_i\colon X \to I$ which satisfies the condition: $f_i^{-1}(0) = X - V_i$, and define $U_i = f^{-1}((1/i+2, 1/i))$, where

$$f(x) = \sum_{k=1}^{\infty} 1/2^{k+1} f_k(x) \quad \text{for } x \in X.$$

For each $x \in X$, we have $0 < f(x) < 1$, whence

$$X = f^{-1}((0,1)) = \bigcup_{i=1}^{\infty} f^{-1}((1/i+2, 1/i)) = \bigcup_{i=1}^{\infty} U_i.$$

Further, for any $x \in U_i - V_i$ we would have

$$1/i+2 < f(x) = \sum_{k=1}^{\infty} 1/2^{k+1} f_k(x) = \sum_{k=i+1}^{\infty} 1/2^{k+1} f_k(x) \leqslant \sum_{k=i+1}^{\infty} 1/2^{k+1} = 1/2^{i+1},$$

which is impossible. Hence $U_i \subseteq V_i$, and thus $\dim U_i \leqslant \dim V_i \leqslant i$. From the definition of the sets U_i it follows directly that $|i-j| > 1$ implies $U_i \cap U_j = \emptyset$.

3.2. THEOREM. *The spaces $V(\tau)$ and $W(\tau)$ are universal for locally finite-dimensional spaces whose weight is not larger than τ.*

Proof. Obviously, $V(\tau)$ is a subspace of $W(\tau)$, but the proof of the universality of the latter space is more elementary (for it does not involve Theorem 2.2, which is indispensable to the proof of the universality of $V(\tau)$). Hence we shall first prove the universality of $W(\tau)$, and then indicate what modifications of the respective proof are needed in order to show the universality of $V(\tau)$.

First of all, we are going to show that the space $W(\tau)$ is locally finite-dimensional.

Let us consider a point $x = (x_0, x_1, \ldots) \in W(\tau) \subseteq [J(\tau)]^{\aleph_0} \times [J(\tau)]^{\aleph_0} \times \ldots$, where $x_j = (x_{j,0}, x_{j,1}, \ldots) \in J(\tau) \times J(\tau) \times \ldots$ for $j = 0, 1, \ldots$ From (1) it follows that $d(x_{j,i}, 0) > 1/k$ for some $i, j, k \in N$. The set U of all points satisfying this inequality is a neighbourhood of x in $W(\tau)$. From (2) and (3′) it follows directly that $U \subseteq [\mathbf{P}_{j=0}^{k-1} K_j(\tau)] \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \ldots$, and therefore $W(\tau)$ is locally finite-dimensional.

Since $W(\tau)$ is obviously a space of weight τ, in order to establish its universality it remains to show that each locally finite-dimensional space X of weight not larger than τ is homeomorphic to a subspace of $W(\tau)$.

Let $\{U_i\colon i = 1, 2, \ldots\}$ be a sequence of open subsets of X with the properties stated in Theorem 3.1. By virtue of Lemma 2.9 and Corollary 2.5, there exists a map f_i of X to $K_i(\tau)$ such that

(4) $f_i|U_i$ is an embedding,

(5) $f_i^{-1}(\mathbf{0}) = X - U_i$.

Obviously, we can assume that

(6) $f_i(x) \leqslant 1/i+2$ for $x \in X$.

Let $h = \triangle_{i=0}^{\infty} f_i$, where $f_0(x) = \mathbf{0}$ for $x \in X$. From (4) and (5) it follows that the map f_i separates points of U_i from closed subsets of X. Thus the family $\{f_i\colon i = 0, 1, \ldots\}$ separates points and closed sets of X, and therefore the diagonal h is a homeomorphic embedding of X into $\mathbf{P}_{i=0}^{\infty} K_i(\tau)$.

In particular, for each point $x \in X$, the point $h(x) = (x_0, x_1, \ldots)$ satisfies (3′). Now, let us consider an $x \in X$, and take the smallest $i \in N$ such that $x \in U_i$. From (5) it follows that $f_i(x) \neq \mathbf{0}$ and $f_j(x) = \mathbf{0}$ for $j \neq i, i+1$. Further, (6) implies that $f_j(x) \leqslant 1/i+2$ for $x \in X$ and $j = i, i+1$. Hence the point $h(x) = (x_0, x_1, \ldots)$ satisfies also (1) and (2), which concludes the proof of the universality of $W(\tau)$.

Since $V(\tau)$ is a subspace of $W(\tau)$, then $V(\tau)$ is a locally finite-dimensional space of weight not larger than τ. To show that each locally finite-dimensional space X of weight not larger than τ is homeomorphic to a subspace of $V(\tau)$, we have to consider, for $i = 1, 2, \ldots$, a map $f_i = (h_{\gamma(i-1)+1}, h_{\gamma(i-1)+2}, \ldots, h_{\gamma(i)})$, where $\gamma(0) = 0$ and $\gamma(j) = (1+1)+(2+1)+\ldots+(j+1) = j(j+3)/2$ for $j = 1, 2, \ldots$, of X to $[K_1(\tau)]^{i+1}$ satisfying (4) and (5), and such that $h_k(x) \leqslant 1/\gamma(i+1)+1$ for $x \in X$ and $k = \gamma(i-1)+1, \gamma(i-1)+2, \ldots, \gamma(i)$ (such a map exists by virtue of Corollary 2.6).

3.3. Remark. From the last proof it follows that the space

$$U(\tau) = \bigcup_{i=0}^{\infty} \{x = (x_0, x_1, \ldots) \in \mathbf{P}_{j=0}^{\infty} K_j(\tau)\colon \emptyset \neq \{j\colon x_j \neq \mathbf{0}\} \subseteq \{i, i+1\}\}$$

is also universal for locally finite-dimensional spaces of weight not larger than τ.

4. Strongly countable-dimensional spaces. Let us denote by $E(\tau)$ (where τ is an infinite cardinal number) the subspace of $[K_1(\tau)]^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in K_1(\tau) \times K_1(\tau) \times \ldots$ with only finitely many coordinates distinct from $\mathbf{0}$.

The space $F(\tau)$ is defined similarly — the only exception is the assumption that $x_n \in K_n(\tau)$ for $n \in N$.

The main result of this section is the universality of the spaces $E(\tau)$ and $F(\tau)$ (see Theorem 4.2). However, first we are going to give another example of a universal space.

Let us consider the space $C(\tau) = \bigcup_{n=1}^{\infty} C_n(\tau)$, where $C_n(\tau)$ is the subspace of $[J(\tau)]^{\aleph_0} \times I^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots) \in [J(\tau)]^{\aleph_0} \times I \times I \ldots$ which satisfy the conditions:

(1) $x_0 \in K_n(\tau)$,

(2) $x_k \neq 0$ for $k = 1, 2, \ldots, n$,

(3) $x_k = 0$ for $k > n$.

The space $C(\tau)$ is perhaps not as nice as the spaces $E(\tau)$ and $F(\tau)$, but the proof of its universality is very simple.

4.1. Theorem. *The space $C(\tau)$ is universal for strongly countable-dimensional spaces of weight not larger than τ.*

Proof. By virtue of Theorems 4.1.7, 4.1.18 and 4.1.27 of [5], we have

$$\dim\Big(\bigcup_{n=1}^{l} C_n(\tau)\Big) \leqslant \Big(\sum_{n=1}^{l} \dim C_n(\tau)\Big) + l \leqslant \Big(\sum_{n=1}^{l} 2n\Big) + l < \infty .$$

The $(l+1)$th coordinate of each point of $C(\tau) - \bigcup_{n=1}^{l} C_n(\tau)$ is distinct from 0, and the $(l+1)$th coordinate of each point of $\bigcup_{n=1}^{l} C_n(\tau)$ is equal to 0. Hence $\bigcup_{n=1}^{l} C_n(\tau)$ is a closed subset of $C(\tau)$. Thus $C(\tau)$ is strongly countable-dimensional.

Since, obviously, $C(\tau)$ is a space of weight τ, it remains to show that each strongly countable-dimensional space X of weight not larger than τ is homeomorphic to a subspace of $C(\tau)$.

From the definition of strong countable-dimensionality it immediately follows that $X = \bigcup_{n=0}^{\infty} F_n$, where $\emptyset = F_0 \subseteq F_1 \subseteq \ldots$, and F_n is a closed subset of X such that $\dim F_n \leqslant n$ for $n = 0, 1, \ldots$ By virtue of Corollary 2.7, there exists a homeomorphic embedding f_0 of X into $[J(\tau)]^{\aleph_0}$ which satisfies the condition:

(4) $f_0(x) \in K_n(\tau)$ for $x \in F_n$.

Let $f_{n+1}\colon X \to I$ be continuous function such that

(5) $f_{n+1}^{-1}(0) = F_n$.

Obviously, the diagonal $f = \triangle_{=}^{\infty} f_k$ is a homeomorphic embedding of X into $[J(\tau)]^{\aleph_0} \times I^{\aleph_0}$. We shall show that $f(x) \in C(\tau)$ for $x \in X$.

Let us take a point $x \in X$ and a number $n \in N$ such that $x \in F_{n+1} - F_n$. From (4) it follows that $f_0(x) \in K_{n+1}(\tau)$, and from (5) that $f_k(x) \neq 0$ for $k = 1, 2, \ldots, n+1$ and $f_k(x) = 0$ for $k > n+1$. Hence $f(x) \in C_{n+1}(\tau) \subseteq C(\tau)$.

4.2. Theorem. *The spaces $E(\tau)$ and $F(\tau)$ are universal for strongly countable-dimensional spaces whose weight is not larger than τ.*

Proof. One readily sees that $E(\tau)$ and $F(\tau)$ are strongly countable-dimensional spaces of weight not larger than τ.

Obviously, $E(\tau)$ is a subspace of $F(\tau)$, but first we are going to prove that each strongly countable-dimensional space X of weight not larger than τ is embeddable into $F(\tau)$. In such a way we shall show that the universality of $F(\tau)$ can be established without using Theorem 2.2.

Clearly, $X = \bigcup_{n=0}^{\infty} E_n$, where $\emptyset = E_0 \subseteq E_1 \subseteq \ldots$ and for each n the set E_n is closed and $\dim E_n \leqslant n$. Let Y_{n+1} be a space, and q_{n+1} a map with the properties described in Lemma 2.9, corresponding to the space X and its closed subset E_n. By virtue of Theorem 2.12, there exists a map g_{n+1} of Y_{n+1} to an $(n+2)$-dimensional space Z_{n+1} of weight not larger than τ which separates points of the set $q_{n+1}(E_{n+1})$ and closed subsets of the space Y_{n+1}. Further, by Corollary 2.5, there exists a homeomorphic embedding p_{n+1} of Z_{n+1} into $K_{n+2}(\tau)$ such that

$$(p_{n+1} \circ g_{n+1} \circ q_{n+1})(E_n) \subseteq \{\mathbf{0}\} .$$

It is easy to verify that the map $h_{n+2} = p_{n+1} \circ g_{n+1} \circ q_{n+1}$ separates points of the set $E_{n+1} - E_n$ and closed subsets of the space X. Thus the diagonal $h = \triangle_{n=0}^{\infty} h_n$, where $h_n(x) = \mathbf{0}$ for $x \in X$ and $n = 0, 1$, is an embedding of X to $\mathrm{P}_{=0}^{\infty} K_n(\tau)$. It remains to show that $h(X) \subseteq F(\tau)$. Consider a point $x \in X$. Then there exists a number $n \in N$ such that $x \in E_n$. Clearly, $x \in E_m$ for $m \geqslant n$. Hence for $m \geqslant n+2$ we have

$$h_m(x) = p_{m-1} \circ g_m \quad q_{m-1}(x) \in p_{m-1} \circ g_{m-1} \circ q_{m-1}(E_{m-2}) \subseteq \{\mathbf{0}\}$$

which yields the relation $h(X) \subseteq F(\tau)$.

To prove that each strongly countable-dimensional space of weight not larger than τ is embeddable into $E(\tau)$ it suffices to observe that the map $(\mathrm{P}_{n=0}^{\infty} f_n)|F(\tau)$, where f_n is an embedding of $K_n(\tau)$ into $[K_1(\tau)]^{n+1}$ satisfying the condition $f_n(\mathbf{0}) = \mathbf{0}$ (which exists by virtue of Corollary 2.6), is a homeomorphic embedding of $F(\tau)$ into $E(\tau)$.

5. Other examples of universal spaces. To conclude the paper, we shall describe, using the Pasynkov partial products and the Lipscomb spaces, other examples of universal spaces for locally finite-dimensional and strongly countable-dimensional spaces.

5.1. Remark. In [2], L. J. Bobkov announced that the partial product $P(J_\omega, \{U_i\colon i \in N\}, D(\tau))$, where $\{U_i\colon i \in N\}$ is an arbitrary countable base for J_ω, is a universal space for locally finite-dimensional spaces of weight not larger than τ.

There exists a similar universal space for strongly countable-dimensional spaces of weight not larger than τ. Namely, the partial product

$$P(K_\omega, \{U_i\colon\ i\in N\}, D(\tau)),$$

where $\{U_i\colon\ i\in N\}$ is an arbitrary countable base for K_ω, is such a universal space.

Both these results can be obtained by defining an embedding of $V(\tau)$ or $W(\tau)$ into $P(J_\omega, \{U_i\colon\ i\in N\}, D(\tau))$, and an embedding of $C(\tau)$, $E(\tau)$ or $F(\tau)$ into $P(K_\omega, \{U_i\colon\ i\in N\}, D(\tau))$. It is easy to verify that the respective partial products belong to the corresponding classes of spaces.

5.2. Remark. Let us consider the Baire space $B(\tau)$ of weight τ and the relation $\sim$ on the set $B(\tau)$, where $(x_1, x_2, \ldots)\sim(y_1, y_2, \ldots)$ if $x_i = y_i$ for each $i\in N$ or else there is an $i\in N$ such that $x_j = y_j$ for $j<i$, $x_{i+k} = y_i$ for $k\geqslant 1$ and $y_{i+k} = x_i$ for $k\geqslant 1$. The quotient space $L(\tau) = B(\tau)/\sim$ is a 1-dimensional space of weight τ.

The space $L(\tau)$ was defined by S. L. Lipscomb, who also discovered its basic properties. In particular, he proved in [8] that each n-dimensional space of weight not larger than τ is homeomorphic to a subspace of $[L(\tau)]^{n+1}$. By modifying the Lipscomb proof of this result, it is possible to obtain proofs of universality of the two spaces defined below.

Let $d_1, d_2, \ldots$ be a sequence of distinct points of the discrete space $D(\tau)$ of weight $\tau\geqslant\aleph_0$, and let 0 be the class of the point $(d_1, d_2, \ldots)\in B(\tau)$. For an arbitrary metric ϱ on $L(\tau)$, the subspace of $[L(\tau)]^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots)\in L(\tau)\times L(\tau)\times\ldots$ which satisfy the conditions:

(1) there exists an $n\in N$ such that $x_n\neq 0$,

(2) there exists a $k\in N$ such that $\varrho(x_m, 0)\leqslant 1/k$ for $m<k$ and $x_m = 0$ for $m\geqslant k$

is universal for locally finite-dimensional spaces of weight not larger than τ, and the subspace of $[L(\tau)]^{\aleph_0}$ consisting of all points $x = (x_0, x_1, \ldots)\in L(\tau)\times L(\tau)\times\ldots$ with only finitely many coordinates distinct from 0 is universal for strongly countable-dimensional spaces of weight not larger than τ.

Added in proof. The author has been told by R. Pol that Theorem 2.12 was announced in I. M. Kozlovskiĭ's paper *Две теоремы о метрических пространствах*, Д. А. Н. СССР 204 (1972), 784–787.

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Received 24 October 1988;
in revised form 24 November 1988