Model theory for infinite quantifier languages

by

Tapani Hyttinen (Helsinki)

Abstract. In this paper we shall prove theorems for infinite quantifier languages (e.g. for $L_{\varkappa^+\varkappa}$, $\varkappa > \omega$) that are similar to those which are familiar to us from the model theory of e.g. $L_{\omega_1\omega}$ and $L_{\infty\omega}$. We do this by using games and tree constructions. The main result of this paper is Theorem 4.14 and its corollary, which states that if $\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa} = \varkappa$ then one cannot say in $L_{\varkappa^+\varkappa}$ that an ordering contains no descending sequence of length $\varkappa$. This generalizes the undefinability of well-order in $L_{\omega_1\omega}$, due to M. Morley and E. Lopez-Escobar ([Mo] and [Lo]).

This work continues the work started in [Hy] and the main results of this paper are improvements of those in [Hy]. This paper can be read independently of [Hy]. This work also parallels [Oi2], which uses proof-theoretic methods to derive related results.

1. Standard games. Although we are going to deal with many different games, the central technical ideas we use are common to all of them. And so we begin by studying games in general.

All the games in this paper are played by two players, which we call A and E. It is practical to assume that A is male (he) and E is female (she). All the moves in these games are made by choosing elements from certain sets. Most of the games in this paper are what we will call *standard a-games* or *standard e-games*:

1.1. Definition. Let α be an ordinal, let $I = \{I_\beta\colon \beta < \alpha\}$ and $J = \{J_\beta\colon \beta < \alpha\}$ be families of sets and let $W \subseteq (\Pi_{\beta<\alpha} I_\beta) \times (\Pi_{\beta<\alpha} J_\beta)$. Then $G = A(I, J, W)$ is a *standard a-game* (*sa*-game in short) of length α. It is played by A and E. In every move $\beta < \alpha$, first A chooses an element $x_\beta \in I_\beta$ and then E chooses an element $y_\beta \in J_\beta$. We say that E *wins the game* if the pair of sequences $(x, y) = ((x_\beta)_{\beta<\alpha}, (y_\beta)_{\beta<\alpha})$ chosen during the game belongs to W.

We define a *standard e-game* (*se-game* in short) $G = E(I, J, W)$ as we defined *sa*-game above, except that now on every move E chooses first from I and then A chooses from J.

For any cardinals $\varkappa$ and λ we say that the *sa*-game $A(I, J, W)$ is a $\lambda, \varkappa$-*a-game* if it is of length $\varkappa$ and for all $\alpha < \varkappa$ $|I_\alpha|, |J_\alpha| < \lambda$. The *sa*-game G is an $\infty, \varkappa$-*a-game* if for some λ, G is a $\lambda, \varkappa$-*a*-game. We define λ, ∞-*a*-games and ∞, ∞-*a*-games in the same way.

Similarly we define $\lambda, \varkappa$-*e*-games.

With every game we associate the concept "a strategy of A for the game" and "a strategy of E for the game". In the case of the s-games we do this as follows:

1.2. Definition. A *strategy* of A (E) for the sa-game $A(I, J, W)$ of length α is a set $F = \{f_\beta\colon \beta < \alpha\}$ of functions $f_\beta\colon \Pi_{\gamma<\beta} J_\gamma \to I$ ($f_\beta\colon \Pi_{\gamma\leqslant\beta} I_\gamma \to J_\beta$).

A *strategy* of $E(A)$ for the se-game $E(I, J, W)$ of length α is a set $F = \{f_\beta\colon \beta < \alpha\}$ of functions $f_\beta\colon \Pi_{\gamma<\beta} J_\gamma \to I_\beta$ ($f_\beta\colon \Pi_{\gamma\leqslant\beta} I_\gamma \to J_\beta$).

Usually we are not interested in strategies in general but in winning strategies.

1.3. Definition. A strategy of a player for a game is *winning* if the player can always win the game by playing according to this strategy. (We say, e.g., that E *plays according to her strategy* $F = \{f_\beta\colon \beta < \alpha\}$ for an sa-game if in every move $\beta < \alpha$ she chooses $f_\beta(x_0, \ldots, x_\beta)$, where $x_0, \ldots, x_\beta$ are the previous choices of A.)

We say that $\bar{G} = E(I, J, (\Pi_{\beta<\alpha} I_\beta \times \Pi_{\beta<\alpha} J_\beta) \setminus W)$ is the *dual* of $G = A(I, J, W)$, and vice versa. Notice that E has a winning strategy for G if and only if A has a winning strategy for $\bar{G}$ and A has a winning strategy for G if and only if E has a winning strategy for $\bar{G}$.

Our first major goal in this paper is to give in Chapter 2 the Approximation Theorems, which are the main tools in Chapter 4. They do not hold for all sa-games and se-games, but they do for the closed and open ones.

1.4. Definition. An sa-game $A(I, J, W)$ of length α is *closed* if there are sets $W_\beta \subseteq (\Pi_{\gamma<\beta} I_\gamma) \times (\Pi_{\gamma<\beta} J_\gamma)$, $\beta < \alpha$, such that

$$\big((x_\gamma)_{\gamma<\alpha}, (y_\gamma)_{\gamma<\alpha}\big) \in W$$

if and only if for all $\beta < \alpha$

$$\big((x_\gamma)_{\gamma<\beta}, (y_\gamma)_{\gamma<\beta}\big) \in W_\beta .$$

An se-game $E(I, J, W)$ of length α is *open* if there are sets

$$W_\beta \subseteq (\Pi_{\gamma<\beta} I_\gamma) \times (\Pi_{\gamma<\beta} J_\gamma) ,$$

$\beta < \alpha$, such that

$$\big((x_\gamma)_{\gamma<\alpha}, (y_\gamma)_{\gamma<\alpha}\big) \in W$$

if and only if for some $\beta < \alpha$

$$\big((x_\gamma)_{\gamma<\beta}, (y_\gamma)_{\gamma<\beta}\big) \in W_\beta .$$

We conclude this chapter by defining approximations for the sa-games and se-games.

We say that a well-founded tree $T = (T, <)$ is *neat* if for any two different points x and y without immediate predecessor, the sets $\{z \in T\colon z < x\}$ and $\{z \in T\colon z < y\}$ are not the same. For any cardinals $\varkappa$ and λ we say that $T = (T, <)$ is a $\lambda, \varkappa$*-tree* if it is a neat, well-founded tree and no point in T has $\geqslant \lambda$ immediate successors and there are no branches of length $\subset \varkappa$ in T. As with the games, we say that T is an $\infty, \varkappa$*-tree* if it is a $\lambda, \varkappa$-tree for some λ. Similarly we define λ, ∞-trees and ∞, ∞-trees.

1.5. Definition. Let $G = A(I, J, W)$ ($G = E(I, J, W)$) be a closed sa-game (open se-game) of length $\varkappa$ for some cardinal $\varkappa$ and let T be an $\infty, \varkappa$-tree. The T*-approximation* G^T of G is the following game played by A and E. They play G^T as they play G except that during the game they go up the tree T as chosen by $A(E)$. The game is over when they cannot go up the tree any more. So the rules are the following: for each move α, first $A(E)$ chooses elements $t_\alpha \in T$ and $x_\alpha \in I_\alpha$ so that

1. if α is a successor, $\alpha = \beta+1$, then t_α is an immediate successor of t_β;
2. if α is a limit, then $t_\alpha = \sup_{\beta<\alpha} t_\beta$.

After this, the player $E(A)$ chooses $y_\alpha \in J_\alpha$. The game continues only as long as $A(E)$ can choose t_α satisfying 1 and 2. When the game is over, the players have made α moves for some $\alpha < \varkappa$ and they have chosen sequences $x = (x_\beta)_{\beta<\alpha}$, $y = (y_\beta)_{\beta<\alpha}$ and $(t_\beta)_{\beta<\alpha}$. E has won if $\big((x_\beta)_{\beta<\gamma}, (y_\beta)_{\beta<\gamma}\big) \in W_\gamma$ for all $\gamma \leqslant \alpha$ ($\big((x_\beta)_{\beta<\gamma}, (y_\beta)_{\beta<\gamma}\big) \in W_\gamma$ for some $\gamma \leqslant \alpha$).

1.6. Definition. If G is a sa-game then a *strategy* F of $A(E)$ for the game G^T is a set $F = \{f_\alpha\colon \alpha < \varkappa\}$ of functions $f_\alpha\colon \Pi_{\beta<\alpha} J_\beta \to T \times I_\alpha$ ($f_\alpha\colon \Pi_{\beta<\alpha} T \times I_\beta \to J_\alpha$).

In the case of se-games G the strategies for G^T are defined similarly.

Let T and T' be ordered sets. We say that a function $g\colon T \to T'$ is order-preserving if for any $x, y \in T$, $g(x) < g(y)$ if $x < y$.

1.7. Lemma 1. *Let G be an sa-game of length $\varkappa$.*

(i) *If E has a winning strategy for G, then for all $\infty, \varkappa$-trees T she has a winning strategy for G^T, too.*

(ii) *Let T and T' be $\infty, \varkappa$-trees. If there is an order-preserving function $g\colon T \to T'$ and E has a winning strategy for $G^{T'}$, then she has a winning strategy for G^T, too.*

2. *Let G be an se-game of length $\varkappa$.*

(i) *If for some $\infty, \varkappa$-tree T, E has a winning strategy for G^T, then she has a winning strategy for G, too.*

(ii) *Let T and T' be $\infty, \varkappa$-trees. If there is an order-preserving function $g\colon T \to T'$ and E has a winning strategy for G^T, then she has a winning strategy for $G^{T'}$, too.*

The lemma follows immediately from the definitions.

2. Approximation theorems for standard games. We begin this chapter by proving an Approximation Theorem for open se-games. Originally the idea of the proof of this theorem is due to M. Karttunen (see [Ka]).

2.1. Approximation Theorem for open se-games. *Let $G = E(I, J, Q)$ be an open $\lambda, \varkappa$-e-game for some cardinals λ and $\varkappa$. If E has a winning strategy for G then for some $\lambda, \varkappa$-tree T she has a winning strategy for G^T, too.*

Proof. Let $F = \{f_\alpha\colon \alpha < \varkappa\}$ be a winning strategy of E for G. Let T be the set of all sequences $\big((x_\beta)_{\beta<\alpha}, (y_\beta)_{\beta<\alpha}\big)$ such that

(i) always $x_\beta = f_\beta(\ldots, y_\gamma, \ldots)_{\gamma<\beta}$,

(ii) for all $\beta \leqslant \alpha$, $\big((x_\gamma)_{\gamma<\beta}, (y_\gamma)_{\gamma<\beta}\big) \notin W_\beta$,

where W_β, $\beta < \varkappa$, are as in the definition of openness. We order T by the initial segment relation, which makes T a $\lambda, \varkappa$-tree.

It is easy to see that E has a winning strategy for G^T: Let us assume that players are on move α. On earlier moves A has chosen elements $y_\beta \in J_\beta$, $\beta < \alpha$, and E elements $x_\beta \in I_\beta$ and $t_\beta \in T$, $\beta < \alpha$. Then on move α E chooses t_α to be $((x_\beta)_{\beta<\alpha}, (y_\beta)_{\beta<\alpha})$ and x_α to be $f_\alpha(\ldots, y_\beta, \ldots)_{\beta<\alpha}$. Clearly this is a winning strategy. ■

Next we give an approximation theorem for closed *sa*-games. The proof can be found in [Hy].

2.2. The Approximation Theorem for closed *sa*-games. *Let λ and $\varkappa$ be infinite cardinals, let $G = A(I, J, Q)$ be a closed $\lambda, \varkappa$-a-game and let μ satisfy the condition below. If for all $\mu, \varkappa$-trees U the player E has a winning strategy for G^U, then she has a winning strategy for G, too.*

The condition for μ is the following: If λ is a successor or $\mathrm{cf}(\lambda) \geqslant \varkappa$ *then*

$$\mu = \bigcup \{(2^{(\beta^\gamma)})^+ : \beta, \gamma \text{ cardinals and } \beta < \lambda, \gamma < \varkappa\}$$

and otherwise

$$\mu = \bigcup \{(2^{(\lambda^\gamma)})^+ : \gamma \text{ cardinal and } \gamma < \varkappa\}.$$

In Chapter 4 we will prove several consequences of these approximation theorems.

3. Infinite quantifier languages. In this chapter we prepare for the last chapter by defining the languages in which we are interested.

Let λ and $\varkappa$ be cardinals and let μ be a set of relation-, function- and constant symbols. The set μ is called the *signature*. We recall the definition of the language $L_{\lambda\varkappa}(\mu)$ ($L_{\lambda\varkappa}$ in short):

3.1. Definition. If all the relation and function symbols in μ are of arity $< \varkappa$ then the *language* $L_{\lambda\varkappa}(\mu)$ is defined and it is the least class X such that

(1) every atomic formula of the signature μ belongs to X;

(2) if $\varphi \in X$ then $\neg\varphi \in X$;

(3) if Φ is a subset of X of cardinality $< \lambda$ and the number of free variables in Φ is $< \varkappa$ then $\bigwedge \Phi$ and $\bigvee \Phi$ belong to X;

(4) if $\varphi \in X$ and $\bar{x}$ is a set of variables of cardinality $< \varkappa$ then $\forall \bar{x}\varphi$ and $\exists \bar{x}\varphi$ belong to X.

The semantics of $L_{\lambda\varkappa}$ is defined in the usual and obvious way.

The languages $M_{\lambda\varkappa}(\mu)$ ($M_{\lambda\varkappa}$ in short) were first introduced by M. Karttunen in [Ka]. Prior to that, J. Hintikka and V. Rantala had introduced in [HR] N-languages that were defined by a similar technique. The idea behind these languages is the following. Let T be a syntax tree of some formula $\varphi \in L_{\omega\omega}$. Then T has, among others, the following properties:

(1) Every node has $< \omega$ immediate successors.

(2) Every branch has length $< \omega$.

We can generalize $L_{\omega\omega}$ by increasing the number of immediate successors and length of branches. The resulting language is $M_{\lambda\varkappa}$.

3.2. Definition. If all the relation and function symbols in μ are of arity $< \varkappa$ then $M_{\lambda\varkappa}(\mu)$ is defined and a formula of $M_{\lambda\varkappa}(\mu)$ is a pair (T, l) where

1. T is a $\lambda, \varkappa$-tree;

2. l is a labeling function with the properties:

(a) if $t \in T$ does not have any successors then $l(t)$ is either an atomic or negated atomic formula of the signature μ;

(b) if $t \in T$ has exactly one immediate successor then $l(t)$ is of the form $\exists x$ or $\forall x$, x variable;

(c) if $t \in T$ has more than one immediate successor then $l(t)$ is either $\bigvee$ or $\bigwedge$.

To be able to define a semantics for $M_{\lambda\varkappa}$, we must define a certain semantic game. Let $\mathscr{A}$ be a model (of the signature μ) and let $\varphi = (T, l)$ be a sentence of $M_{\lambda\varkappa}$.

3.3. Definition. The *semantic game* $S(\mathscr{A}, \varphi)$ is a game of two players, A and E. When the game begins, the players are in the root of T and during the game the players go up the tree T. At each move the players are in some node $t \in T$ and it depends on $l(t)$ how they continue the game:

(1) If $l(t) = \bigvee$ ($\bigwedge$) then $E(A)$ chooses one immediate successor of t to be the node where the players go next.

(2) If $l(t) = \exists x$ ($\forall x$) then $E(A)$ chooses an element $x^{\mathscr{A}}$ from $\mathscr{A}$ to be an interpretation of x. The players go then to the immediate successor of t.

(3) If $l(t) = \varphi(\bar{x})$ then the game is over and E has won if

$$\mathscr{A} \models \varphi(\bar{x})[\overline{x^{\mathscr{A}}}].$$

The concepts a strategy of A for $S(\mathscr{A}, \varphi)$ and a strategy of E for $S(\mathscr{A}, \varphi)$ are defined essentially as in the case of *sa*-games and *se*-games.

Let φ be a sentence of $M_{\lambda\varkappa}$ and $\mathscr{A}$ a model.

3.4. Definition. $\mathscr{A} \models \varphi$ if E has a winning strategy for $S(\mathscr{A}, \varphi)$.

We list below a couple of properties of these languages that are apparent but still worth noting.

(1) For all λ and $\varkappa$ $L_{\lambda\varkappa}$ is a sublanguage of $M_{\lambda\varkappa}$ ($L_{\lambda\varkappa} \leqslant M_{\lambda\varkappa}$), i.e. for any sentence φ of $L_{\lambda\varkappa}$ there is a sentence ψ of $M_{\lambda\varkappa}$ which is equivalent to φ.

(2) For all λ $L_{\lambda\omega} = M_{\lambda\omega}$, i.e. $L_{\lambda\omega} \leqslant M_{\lambda\omega}$ and $M_{\lambda\omega} \leqslant L_{\lambda\omega}$.

(3) If $\varkappa > \omega$ then there is no obvious reason why $M_{\lambda\varkappa}$ would be closed under negation. One might think that we could get the negation of $\varphi = (T, l)$ by putting $\neg\varphi = \sim\varphi = (T, l')$ where l' is such that $l'(t) = \bigwedge$ if and only if $l(t) = \bigvee$ and so on. But this is not the case because it may happen that for some model $\mathscr{A}$ the semantic game $S(\mathscr{A}, \varphi)$ is non-determined, i.e. neither E nor A has a winning strategy. In this case $\mathscr{A} \not\models \varphi$ and $\mathscr{A} \not\models \sim\varphi$ (see [Hy]). The question whether all the languages $M_{\lambda\varkappa}$ are closed under negation or not is open to the author.

In Chapter 4 we will need the next theorem from [Ka].

3.5. Theorem. *Assume $\lambda^{<\varkappa} = \lambda$. If $\varphi \in M_{\lambda^+\varkappa}(\mu)$, $|\mu| \leqslant \lambda$, has a model then it has a model of cardinality $\leqslant \lambda$.*

Proof. Simply take some model $\mathscr{A}$ of φ and some element $a \in \mathscr{A}$ and close $\{a\}$ under the winning strategy of E for $S(\mathscr{A}, \varphi)$ and under the functions of μ. ■

Theorem 3.5 is true also for the language $\bigwedge_{\lambda^+\varkappa}$, which will be defined below (see [Ka]).

Let λ and $\varkappa$ be infinite cardinals. The $\lambda\varkappa$-Vaught sentences are defined as follows. Originally Vaught sentences were defined by R. L. Vaught in [Va].

3.6. Definition. We assume that all the relation and function symbols in the signature μ are of arity $<\varkappa$. Then φ is a *conjunctive $\lambda\varkappa$-Vaught sentence* (of the signature μ) and ψ is a *disjunctive $\lambda\varkappa$-Vaught formula* it they are of the following form:

$$\varphi = (\forall x_\alpha \bigwedge_{i_\alpha \in I_\alpha} \exists y_\alpha \bigvee_{j_\alpha \in J_\alpha})_{\alpha<\varkappa} \bigwedge_{\alpha<\varkappa} \varphi^{i_0 j_0 \dots i_\alpha j_\alpha},$$

$$\psi = (\exists x_\alpha \bigvee_{i_0 \in I_\alpha} \forall y_\alpha \bigwedge_{j_\alpha \in J_\alpha})_{\alpha<\varkappa} \bigvee_{\alpha<\varkappa} \psi^{i_0 j_0 \dots i_\alpha j_\alpha}$$

where I_α and J_α, $\alpha<\varkappa$, are sets of cardinality $<\lambda$ and $\varphi^{i_0 j_0 \dots i_\alpha j_\alpha}$ and $\psi^{i_0 j_0 \dots i_\alpha j_\alpha}$, $\alpha<\varkappa$, are atomic or negated atomic formulas (of the signature μ) with the variables from the set $\{x_0, y_0, \dots, x_\alpha, y_\alpha\}$.

We will write $\bigwedge_{\lambda\varkappa}$ for the language of all conjunctive $\lambda\varkappa$-Vaught sentences and $\bigvee_{\lambda\varkappa}$ for the language of all disjunctive $\lambda\varkappa$-Vaught sentences. Again, in defining semantics for the languages $\bigwedge_{\lambda\varkappa}$ and $\bigvee_{\lambda\varkappa}$, we need a certain semantic game. Let $\mathscr{A}$ be a model, φ a conjunctive $\lambda\varkappa$-Vaught sentence and ψ a disjunctive $\lambda\varkappa$-Vaught sentence.

3.7. Definition. The semantic game $S(\mathscr{A}, \varphi)$ is a game of two players, A and E. For each move $\alpha<\varkappa$, first A chooses an element $x_\alpha^{\mathscr{A}}$ from $\mathscr{A}$ to be an interpretation for x_α and then he chooses some $i_\alpha \in I_\alpha$. When A has chosen $x_\alpha^{\mathscr{A}}$ and i_α E chooses some $y_\alpha^{\mathscr{A}}$ from $\mathscr{A}$ to be an interpretation for y_α and then she chooses some $j_\alpha \in J_\alpha$. After $\varkappa$ moves E *wins* if

$$\mathscr{A} \models \varphi^{i_0 j_0 \dots i_\alpha j_\alpha}(x_0^{\mathscr{A}}, y_0^{\mathscr{A}}, \dots, x_\alpha^{\mathscr{A}}, y_\alpha^{\mathscr{A}}),$$

for all $\alpha<\varkappa$.

The semantic game $S(\mathscr{A}, \psi)$ is again a game of two players, A and E. For each move $\alpha<\varkappa$, first E chooses an element $x_\alpha^{\mathscr{A}}$ from $\mathscr{A}$ to be an interpretation for x_α and then she chooses some $i_\alpha \in I_\alpha$. When E has chosen $x_\alpha^{\mathscr{A}}$ and i_α A chooses some $y_\alpha^{\mathscr{A}}$ from $\mathscr{A}$ to be an interpretation for y_α and then he chooses some $j_\alpha \in J_\alpha$. After $\varkappa$ moves E *wins* if

$$\mathscr{A} \models \psi^{i_0 j_0 \dots i_\alpha j_\alpha}(x_0^{\mathscr{A}}, y_0^{\mathscr{A}}, \dots, x_\alpha^{\mathscr{A}}, y_\alpha^{\mathscr{A}}),$$

for some $\alpha<\varkappa$.

There is also another way to define $S(\mathscr{A}, \varphi)$: we put

$$I = \{\mathscr{A} \times I_\alpha\colon \alpha<\varkappa\},$$

$$J = \{\mathscr{A} \times J_\alpha\colon \alpha<\varkappa\}$$

and define $W \subseteq (\Pi_{\alpha<\varkappa}\mathscr{A} \times I_\alpha) \times (\Pi_{\alpha<\varkappa}\mathscr{A} \times J_\alpha)$ to be such that

$$\big(((a_\alpha, i_\alpha))_{\alpha<\varkappa}, ((b_\alpha, j_\alpha))_{\alpha<\varkappa}\big) \in W$$

if and only if

$$\mathscr{A} \models \varphi^{i_0 j_0 \dots i_\alpha j_\alpha}(a_0, b_0, \dots, a_\alpha, b_\alpha)$$

for all $\alpha<\varkappa$. Then we can define $S(\mathscr{A}, \varphi)$ to be $A(I, J, W)$. So $S(\mathscr{A}, \varphi)$ is a closed sa-game of length $\varkappa$. Similarly we can see that $S(\mathscr{A}, \psi)$ is an open se-game of length $\varkappa$.

3.8. Definition. $\mathscr{A} \models \varphi$ ($\mathscr{A} \models \psi$) if the player E has a winning strategy for $S(\mathscr{A}, \varphi)$ ($S(\mathscr{A}, \psi)$).

We make the following remarks:

1. In Definition 3.6 we could have let $\varphi^{i_0 j_0 \dots i_\alpha j_\alpha}$ and $\psi^{i_0 j_0 \dots i_\alpha j_\alpha}$ be conjunction or disjunction of $<\lambda$ atomic or negated atomic formulas, e.g., and still get the same languages $\bigwedge_{\lambda\varkappa}$ and $\bigvee_{\lambda\varkappa}$.

2. In all cases $M_{\lambda\varkappa} \leqslant \bigwedge_{\lambda^+\varkappa}$ and $M_{\lambda\varkappa} \leqslant \bigvee_{\lambda^+\varkappa}$. If $\lambda>\varkappa$ then $\bigwedge_{\lambda\varkappa} \leqslant M_{\lambda\varkappa^+}$ and $\bigvee_{\lambda\varkappa} \leqslant M_{\lambda\varkappa^+}$. If λ is a successor cardinal or a regular limit cardinal with $\lambda^{<\varkappa}=\lambda$ then $M_{\lambda\varkappa} \leqslant \bigwedge_{\lambda\varkappa}$ and $M_{\lambda\varkappa} \leqslant \bigvee_{\lambda\varkappa}$.

3. As in the usual proof of the Gale–Stewart Theorem, we can see that if $\varphi \in \bigwedge_{\lambda\omega}$ then $S(\mathscr{A}, \varphi)$ is determined, i.e. either E or A has a winning strategy. If $\varphi \in \bigwedge_{\lambda\varkappa}$ for some $\varkappa>\omega$ then $S(\mathscr{A}, \varphi)$ does not have to be determined (see [Hy]). The same is also true for disjunctive Vaught sentences, because the semantic games of disjunctive Vaught sentences are duals of the semantic games of conjunctive Vaught sentences.

4. The languages $\bigwedge_{\lambda\varkappa}$ and $\bigvee_{\lambda\varkappa}$ are not always closed under negation. For example $\bigwedge_{\omega\omega}(\emptyset)$ is not closed under negation:

$$\varphi = \forall x_0 \forall x_1 \dots \bigwedge_{i<\omega} \wedge \{x_j \neq x_i\colon j<i\}$$

belongs to $\bigwedge_{\omega\omega}(\emptyset)$ but $\neg\varphi$ does not because $\bigwedge_{\omega\omega}$ is compact (see [Ka]). In Corollary 4.18 we have another example of a language $\bigwedge_{\lambda\varkappa}$ that is not closed under negation.

We say that φ belongs to $\bigwedge_{\infty\varkappa}(\bigvee_{\infty\varkappa}, M_{\infty\varkappa}, L_{\infty\lambda})$ if it belongs to $\bigwedge_{\lambda\varkappa}(\bigvee_{\lambda\varkappa}, M_{\lambda\varkappa}, L_{\lambda\varkappa})$ for some λ. Similarly we define the languages $\bigwedge_{\lambda\infty}$ $(\bigvee_{\lambda\infty}, M_{\lambda\infty}, L_{\lambda\infty})$ and $\bigwedge_{\infty\infty}$ $(\bigvee_{\infty\infty}, M_{\infty\infty}, L_{\infty\infty})$.

We conclude this chapter by giving a characterization for $\bigwedge_{\infty\varkappa}$-elementary equivalence.

Let $\mathscr{A}$ and $\mathscr{B}$ be models and α an ordinal.

3.9. Definition. The *Ehrenfeucht–Fraisse game of length α* $F_\alpha(\mathscr{A}, \mathscr{B})$ is a game of two players, A and E. For each move $\beta<\alpha$ first A chooses an element $a_\beta \in \mathscr{A}$ or $b_\beta \in \mathscr{B}$ and then E chooses an element $b_\beta \in \mathscr{B}$ if A has chosen $a_\beta \in \mathscr{A}$, otherwise she chooses $a_\beta \in \mathscr{A}$. After α moves E wins if the function that takes a_β to b_β for all $\beta<\alpha$ is a partial isomorphism.

As with the semantic game for the conjunctive $\lambda\varkappa$-Vaught sentences, we could find such I, J and W that $F_\alpha(\mathscr{A},\mathscr{B})$ is $A(I,J,W)$. So $F_\alpha(\mathscr{A},\mathscr{B})$ is an s-game of length α and it is closed if all the relations and functions are of arity $<\mathrm{cf}(\alpha)$, as one can immediately see.

Let $\varkappa$ be a cardinal and $\mathscr{A}$ a model of cardinality λ.

3.10. DEFINITION. (i) $\varphi^\varkappa_{\mathscr{A}}$ is the following sentence of $\bigwedge_{\lambda^+\varkappa}$:

$$\varphi^\varkappa_{\mathscr{A}} = (\forall x_\alpha \bigwedge_{i_\alpha\in\mathscr{A}} \exists y_\alpha \bigvee_{j_\alpha\in\mathscr{A}})_{\alpha<\varkappa} \bigwedge_{\alpha<\lambda} \varphi^{i_0j_0\ldots i_\alpha j_\alpha}(x_0,y_0,\ldots,x_\alpha,y_\alpha)$$

where $\varphi^{i_0j_0\ldots i_\alpha j_\alpha}(x_0,y_0,\ldots,x_\alpha,y_\alpha)$ is the conjunction of all atomic or negated atomic formulas $\psi(x_0,y_0,\ldots,x_\alpha,y_\alpha)$ that satisfy

$$\mathscr{A}\models\psi(j_0,i_0,\ldots,j_\alpha,i_\alpha).$$

(ii) $\psi^\varkappa_{\mathscr{A}}$ is the following sentence of $\bigvee_{\lambda^+\varkappa}$:

$$\psi^\varkappa_{\mathscr{A}} = (\exists x_\alpha \bigvee_{i_\alpha\in\mathscr{A}} \forall y_\alpha \bigwedge_{j_\alpha\in\mathscr{A}})_{\alpha<\varkappa} \bigvee_{\alpha<\varkappa} \psi^{i_0j_0\ldots i_\alpha j_\alpha}(x_0,y_0,\ldots,x_\alpha,y_\alpha)$$

where $\psi^{i_0j_0\ldots i_\alpha j_\alpha}(x_0,y_0,\ldots,x_\alpha,y_\alpha)$ is the disjunction of all atomic or negated atomic formulas $\psi(x_0,y_0,\ldots,x_\alpha,y_\alpha)$ that satisfy

$$\mathscr{A}\not\models\psi(j_0,i_0,\ldots,j_\alpha,i_\alpha).$$

The next lemma and its corollary are from [Ka].

3.11. LEMMA (M. Karttunen). *If all the relations and functions are of arity* $<\mathrm{cf}(\varkappa)$ *then for all models* $\mathscr{B}$:

(i) *E has a winning strategy for* $S(\mathscr{B},\varphi^\varkappa_{\mathscr{A}})$ *if and only if she has a winning strategy for* $F_\varkappa(\mathscr{A},\mathscr{B})$;

(ii) *E has a winning strategy for* $S(\mathscr{B},\psi^\varkappa_{\mathscr{A}})$ *if and only if A has a winning strategy for* $F_\varkappa(\mathscr{A},\mathscr{B})$.

3.12. COROLLARY (M. Karttunen). *We assume that* $\mathscr{A}$ *and* $\mathscr{B}$ *are models and all the relations and functions are of arity* $<\mathrm{cf}(\varkappa)$. *Then* $\mathscr{A}\equiv\mathscr{B}(\bigvee_{\infty\varkappa})$ *if and only if E has a winning strategy for* $F_\varkappa(\mathscr{A},\mathscr{B})$.

4. Model theory for infinite quantifier languages. Our first goal in this chapter is an approximation theorem for disjunctive $\lambda\varkappa$-Vaught sentences.

Let

$$\psi = (\exists x_\alpha \bigvee_{i_0\in I_\alpha} \forall y_\alpha \bigwedge_{j_\alpha\in J_\alpha})_{\alpha<\varkappa} \bigvee_{\alpha<\varkappa} \psi^{i_0j_0\ldots i_\alpha j_\alpha}$$

be a disjunctive $\lambda\varkappa$-Vaught sentence, T a $\nu,\varkappa$-tree and let $\xi=\max\{\lambda,\nu\}$.

4.1. LEMMA. *There is a T-approximation of* ψ, ψ^T, *in* $M_{\xi\varkappa}$, *such that for all models* $\mathscr{A}$:

(1) *E has a winning strategy for* $[S(\mathscr{A},\psi)]^T$ *if and only if she has a winning strategy for* $S(\mathscr{A},\psi^T)$ (*i.e.* $\mathscr{A}\models\psi^T$);

(2) *A has a winning strategy for* $[S(\mathscr{A},\psi)]^T$ *if and only if he has a winning strategy for* $S(\mathscr{A},\psi^T)$.

We skip the easy proof.

We immediately get the next corollary from Lemmas 4.1 and 1.7.

4.2. COROLLARY. *Let* φ *be a disjunctive* $\infty\varkappa$*-Vaught sentence and T and T′* $\infty,\varkappa$*-trees. Then*

1. $\models\varphi^T\to\varphi$;

2. *If there is an order-preserving function* $g\colon T\to T'$ *then* $\models\varphi^T\to\varphi^{T'}$.

The next theorem follows immediately from Lemma 4.1 and Theorem 2.1.

4.3. APPROXIMATION THEOREM FOR DISJUNCTIVE VAUGHT SENTENCES (M. Karttunen [Ka]). *If* ψ *is a disjunctive* $\lambda\varkappa$*-Vaught sentence and* $\mathscr{A}$ *is of cardinality* $<\lambda$ *then*

$$\mathscr{A}\models\bigvee_{T\lambda,\varkappa\text{-tree}}\varphi^T\leftrightarrow\varphi.$$

4.4. COROLLARY. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are models of cardinality* $\varkappa$ *and* $\mathscr{A}\equiv\mathscr{B}(M_{\varkappa^+\varkappa})$ *then* $\mathscr{A}\cong\mathscr{B}$.

Proof. This corollary follows immediately from Lemma 3.11 (ii) and Theorem 4.3. ■

In the case $\varkappa=\omega$ the following corollary is due to D. Scott. In [Ra] V. Rantala has proved a weak version of it.

4.5. COROLLARY. *Let* $\mathscr{A}$ *be a model of cardinality* $\varkappa$ *and* $X\subseteq\mathscr{A}$. *Then the following are equivalent*:

(i) *X is closed under all automorphisms of* $\mathscr{A}$;

(ii) *There is a formula* $\psi(x)$ *of* $M_{\varkappa^+\varkappa}$ *such that* $\mathscr{A}\models\psi(a)$ *if and only if* $a\in X$.

Proof. "(ii)→(i)" Trivial. "(i)→(ii)" For each $a\notin X$ let $\varphi(x)$ be a disjunctive $\varkappa^+\varkappa$-Vaught formula such that for all $b\in\mathscr{A}$ $\mathscr{A}\models\varphi(b)$ if and only if there is no automorphism of $\mathscr{A}$ which takes a to b. This φ exists by Theorem 3.11 (ii). By Theorem 4.3 for each $b\in X$ there is an approximation of φ which is true in c. Let ψ_b be conjunction of these approximations for each $a\notin X$. Then $\psi_b\in M_{\varkappa^+\varkappa}$ and $\mathscr{A}\models\varkappa_b(b)$ and $\mathscr{A}\not\models\psi_b(a)$ for all $a\notin X$. But now $\psi=\bigvee\{\psi_b\colon b\in X\}$ is the required defining formula. ■

Next we study approximations of conjunctive Vaught sentences. Let

$$\varphi = (\forall x_\alpha \bigwedge_{i_\alpha\in I_\alpha} \exists y_\alpha \bigvee_{j_\alpha\in J_\alpha})_{\alpha<\varkappa} \bigwedge_{\alpha<\varkappa} \varphi^{i_0j_0\ldots i_\alpha j_\alpha}$$

be a $\lambda\varkappa$-Vaught sentence, T a $\nu,\varkappa$-tree and let $\xi=\max\{\lambda,\nu\}$.

4.6. LEMMA. *There is a T-approximation of* φ, φ^T, *in* $M_{\xi\varkappa}$, *such that for all models* $\mathscr{A}$:

1. *E has a winning strategy for* $[S(\mathscr{A},\varphi)]^T$ *if and only if she has a winning strategy for* $S(\mathscr{A},\varphi^T)$ (*i.e.* $\mathscr{A}\models\varphi^T$);

2. *A has a winning strategy for* $[S(\mathscr{A},\varphi)]^T$ *if and only if he has a winning strategy for* $S(\mathscr{A},\varphi^T)$.

We skip the easy proof.

Vaught himself has defined approximations φ^α for all $\omega_1\omega$-Vaught sentences φ, see [Ma] (or [Va]). One might ask how different the approximations defined here are from those defined by Vaught. The next lemma answers this. We skip the easy proof.

4.7. LEMMA. *Let φ be an $\omega_1\omega$-Vaught sentence.*

1. *For every ordinal α there is an ∞, ω-tree T such that $\models \varphi^T \leftrightarrow \varphi^\alpha$;*
2. *For every ∞, ω-tree T there is an ordinal α such that $\models \varphi^\alpha \rightarrow \varphi^T$.*

We immediately get the next corollary from Lemmas 4.6 and 1.7.

4.8. COROLLARY. *Let φ be an $\infty\varkappa$-Vaught sentence and T and T' $\infty, \varkappa$-trees. Then*

1. $\models \varphi \rightarrow \varphi^T$;
2. *If there is an order-preserving function $g\colon T \rightarrow T'$ then $\models \varphi^{T'} \rightarrow \varphi^T$.*

The next theorem follows immediately from Lemma 4.6 and Theorem 2.2. In the case $\varkappa = \omega$ it is due to R. L. Vaught ([Va]).

4.9. APPROXIMATION THEOREM FOR CONJUNCTIVE. VAUGHT SENTENCES. *If φ is $\lambda\varkappa$-Vaught sentence, $\mathscr{A}$ is of cardinality $< \lambda$ and μ satisfies the condition below then*

$$\mathscr{A} \models \bigwedge_{T\,\mu,\varkappa\text{-tree}} \varphi^T \leftrightarrow \varphi .$$

The condition for μ is the following: if λ is a successor or $\mathrm{cf}(\lambda) \geqslant \varkappa$ then

$$\mu = \bigcup \{(2^{(\beta\alpha)})^+ : \beta, \gamma \text{ cardinals and } \beta < \lambda, \gamma < \varkappa\}$$

and otherwise

$$\mu = \bigcup \{(2^{(\lambda\gamma)})^+ : \gamma \text{ cardinal and } \gamma < \varkappa\} .$$

4.10. COROLLARY. *For all $\infty\varkappa$-Vaught sentences φ,*

$$\models \varphi \leftrightarrow \bigwedge_{T\,\infty,\varkappa\text{-tree}} \varphi^T .$$

4.11. COROLLARY. *For all models $\mathscr{A}$ and $\mathscr{B}$*

$$\mathscr{A} \equiv \mathscr{B}(M_{\infty\varkappa}) \quad \text{if and only if} \quad \mathscr{A} \equiv \mathscr{B}(\textstyle\bigvee_{\infty\varkappa}) .$$

We recall from Chapter 3 that $\mathscr{A} \equiv \mathscr{B}(\bigvee_{\infty\varkappa})$ if and only if E has a winning strategy for $F_\varkappa(\mathscr{A}, \mathscr{B})$. By Theorem 4.3 $\mathscr{A} \equiv \mathscr{B}(M_{\infty\varkappa})$ if and only if $\mathscr{A} \equiv \mathscr{B}(\bigvee_{\infty\varkappa})$.

Next we aim to prove the main theorem of this paper, Theorem 4.14. To do this, we have to be able to construct models for sets of sentences of $L_{\varkappa^+\varkappa}$. For this reason we now present a technique to construct a model out of constants.

Throughout the rest of this chapter, we assume that $\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa} = \varkappa$ (we do not assume that $\varkappa$ is a limit cardinal). We also assume that the signature μ is of cardinality $\leqslant \varkappa$ and that all the relation and function symbols in μ are of arity $< \varkappa$.

Let Σ be a set of sentences of $L_{\varkappa^+\varkappa}(\mu)$, $|\Sigma| \leqslant \varkappa$, and let $C = \{c_i\colon i < \varkappa\}$ be a set of new constant symbols. We assume that in every sentence $\varphi \in \Sigma$ all negations are pushed in front of atomic formulas.

Let $\Delta(C) \supseteq \Sigma$ be a fragment of $L_{\varkappa^+\varkappa}(\mu \cup C)$ of cardinality $\varkappa$, i.e. $\Delta(C)$ is closed under subformulas and under substituting free variables by terms. This $\Delta(C)$ exists under the assumptions on $\varkappa$ we have made. Let $\underline{\Delta}(C)$ be the family of all subsets S of $\Delta(C)$ of cardinality $< \varkappa$ that have the property that for all atomic formulas φ of $L_{\varkappa^+\varkappa}(\mu \cup C)$ either $\varphi \notin S$ or $\neg\varphi \notin S$.

4.12. DEFINITION. The *Hintikka game* $H_\varkappa(\Sigma, \underline{\Delta}(C))$ is a game of length $\varkappa$ played by A and E. During the game A interprets symbols of μ to the set C to make C a model of Σ. This is done so that at every move $\alpha < \varkappa$ first E asks a question and then A answers the question by choosing some $S_\alpha \in \underline{\Delta}(C)$. There are eight different ways to form the question:

(1) E chooses some $\varphi \in \Sigma$; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $\varphi \in S_\alpha$.

(2) E chooses a closed term t; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $t = t$, $t = c \in S_\alpha$ for some $c \in C$.

(3) E chooses $t = t' \in \bigcup_{\beta<\alpha} S_\beta$, where t and t' are closed terms; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $t' = t \in S_\alpha$.

(4) E chooses $\exists\bar{x}\varphi(\bar{x}) \in \bigcup_{\beta<\alpha} S_\beta$; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $\varphi(\bar{c}) \in S_\alpha$ for some $\bar{c} \in C$.

(5) E chooses $\forall\bar{x}\varphi(x) \in \bigcup_{\beta<\alpha} S_\beta$ and some sequence $\bar{t}$ of closed terms; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $\varphi(\bar{t}) \in S_\alpha$.

(6) E chooses $\bigvee\Phi \in \bigcup_{\beta<\alpha} S_\beta$; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $\varphi \in S_\alpha$ for some $\varphi \in \Phi$.

(7) E chooses $\bigwedge\Phi \in \bigcup_{\beta<\alpha} S_\beta$ and $\varphi \in \Phi$; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $\varphi \in S_\alpha$.

(8) E chooses $t = t'$ and $\varphi(t)$ from $\bigcup_{\beta<\alpha} S_\beta$, where t and t' are closed terms; then A must choose $S_\alpha \in \underline{\Delta}(C)$ so that $\varphi(t') \in S_\alpha$.

A must always choose S_α so that $\bigcup_{\beta<\alpha} S_\beta \subseteq S_\alpha$. E wins if for some $\alpha < \varkappa$ A cannot find S_α satisfying the rules. Otherwise A wins.

Again we notice that $H_\varkappa(\Sigma, \underline{\Delta}(C))$ is an open *se*-game of length $\varkappa$.

We call this game the Hintikka game because it is a generalization of the concept of Hintikka set: in the case $\varkappa = \omega$ E does not have a winning strategy for $H_\varkappa(\Sigma, \underline{\Delta}(C))$ if and only if Σ can be extended to a Hintikka set (see [Ma]). Prior to the author J. Oikkonen has considered the dual of this Hintikka game.

4.13. LEMMA. ($\varkappa$ *is a regular cardinal and* $\varkappa^{<\varkappa} = \varkappa$) *Let Σ and $\underline{\Delta}(C)$ be as above. If E does not have a winning strategy for $H_\varkappa(\Sigma, \underline{\Delta}(C))$ then Σ has a canonical model $\mathscr{A}$ (canonical means that for every $a \in \mathscr{A}$ there is $c \in C$ with $c^{\mathscr{A}} = a$).*

Proof. We assume that E does not have a winning strategy for $H_\varkappa(\Sigma, \underline{\Delta}(C))$. We let A and E play the game $H_\varkappa(\Sigma, \underline{\Delta}(C))$ so that E plays according to the strategy described below and A so that he wins. The idea here is to make E ask all the possible questions she can. For this let $g\colon \varkappa \rightarrow \varkappa \times \varkappa$ be one-one and onto with the property that if $g(x) = (y, z)$ then $y \leqslant x$. In each move $\alpha < \varkappa$ we let Q_α be the set of all possible questions E can ask in that move. This set depends on how the players have played earlier. In any case $|Q_\alpha| \leqslant \varkappa$ because $\varkappa^{<\varkappa} = \varkappa$ and we can enumerate it as $Q_\alpha = \{q^\alpha_\beta\colon \beta < \varkappa\}$. The question E chooses in move α is q^γ_β if $g(\alpha) = (\gamma, \beta)$.

Let $\{S_\alpha\colon \alpha<\varkappa\}$ be the set of answers of A in the game where A played so that he won and E according to the strategy described above. We let $S=\bigcup_{\alpha<\varkappa}S_\alpha$.

In S we have a complete description of $\mathscr{A}$. We get the universe of $\mathscr{A}$ from C as follows. In C we define an equivalence relation $\sim$ by $c\sim c'$ if $c=c'\in S$. Because of 2, 3 and 8 in the definition of the Hintikka game, $\sim$ is an equivalence relation. For all $c\in C$ we write $[c]=\{c'\in C\colon c'\sim c\}$ and define the universe of $\mathscr{A}$ to be $\{[c]\colon c\in C\}$.

All the symbols of $\mu\cup C$ are interpreted to $\mathscr{A}$ in the obvious way. For example if $c\in C$ then $c^{\mathscr{A}}=[c]$ and if $R(x,y)$ is a binary relation symbol then $R^{\mathscr{A}}([c],[c'])$ if $R(c,c')\in S$ and so on. It follows immediately from the definition of the Hintikka game that $\mathscr{A}$ is well-defined and a model of Σ. Trivially $\mathscr{A}$ is canonical. ■

We say that T is a *wide* $\lambda,\varkappa$*-tree* if it satisfies what we require from a $\lambda,\varkappa$-tree except that instead of neatness it is assumed to satisfy only the condition that for every $t\in T$ the set

$$\{t'\in T\colon \{u\in T\colon u<t'\}=\{u\in T\colon u<t\}\}$$

is of cardinality $<\lambda$. We use this concept in the next theorem instead of the concept of the $\lambda,\varkappa$-tree to make the construction of T^* easier. Notice that if for every $\lambda,\varkappa$-tree T there is an order-preserving function $g\colon T\to U$, then for every wide $\lambda,\varkappa$-tree T' there is an order-preserving function $g\colon T'\to U$. This is because every wide $\lambda,\varkappa$-tree can be extended to a $\lambda,\varkappa$-tree.

4.14. THEOREM. *($\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa}=\varkappa$) Let μ be any signature which includes a unary predicate symbol U and a binary predicate symbol $<$. We assume that φ is a sentence of $L_{\varkappa^+\varkappa}(\mu)$ and that for every wide $\varkappa^+,\varkappa$-tree T there is a model $\mathscr{A}$ of φ and an order-preserving function $g\colon T\to(U^{\mathscr{A}},<^{\mathscr{A}})$. Then there is a model $\mathscr{A}$ of φ such that $(U^{\mathscr{A}},<^{\mathscr{A}})$ contains an increasing sequence of length $\varkappa$.*

Proof. Let $D=\{d_\alpha\colon \alpha<\varkappa\}$ be a set of new constants. To prove the theorem it is enough to show that the set

$$\Sigma=\{\varphi\}\cup\{d_i<d_j\colon i<j<\varkappa\}\cup\{U(d_i)\colon i<\varkappa\}$$

has a model. Let $C=\{c_\alpha\colon \alpha<\varkappa\}$ be again a set of new constants and let $\Delta(C)$ and $\underline{\Delta}(C)$ be as in the definition of the Hintikka game (now the signature is $\mu\cup D$). By Lemma 4.13 it is enough to show that E does not have a winning strategy for $H_\varkappa(\Sigma,\underline{\Delta}(C))$. By the Approximation Theorem 2.1 it is enough to show that E does not have a winning strategy for $[H_\varkappa(\Sigma,\underline{\Delta}(C))]^T$ for every $\varkappa^+,\varkappa$-tree T.

We show this. Let T be an arbitrary $\varkappa^+,\varkappa$-tree. The idea here is the following. We define a model $\mathscr{A}$ of φ so that A can play the game $[H_\varkappa(\Sigma,\underline{\Delta}(C))]^T$ by putting to S_α only "what is true in $\mathscr{A}$". If he can do this, he must win. The only problem he must face in doing this is that E can make him interpret a lot of new constants d_α to $\mathscr{A}$ so that in all cases $d_\alpha^{\mathscr{A}}<^{\mathscr{A}}d_{\alpha'}^{\mathscr{A}}$ if $\alpha<\alpha'$. So we must make $(U^{\mathscr{A}},<^{\mathscr{A}})$ a very rich ordering.

We put

$$T^*=\{(t,N,n)\colon t\in T,\ N\in{}^{\{u\in T\colon u\leqslant t\}}\varkappa,\ n\leqslant N(t)\}.$$

We define an ordering of T^* by putting $(t,N,n)<(t',N',n')$ if and only if

1. $t<t'$ and $N(x)=N'(x)$ for all $x\leqslant t$

or

2. $t=t'$ and $N=N'$ and $n<n'$.

Because $\varkappa$ is regular T^* is a wide $\varkappa^+,\varkappa$-tree.

Let $\mathscr{A}$ be a model of φ such that there is an order-preserving function $g\colon T^*\to(U^{\mathscr{A}},<^{\mathscr{A}})$. By using $\mathscr{A}$ we can now describe even a winning strategy of A for $[H_\varkappa(\Sigma,\underline{\Delta}(C))]^T$.

We assume that A and E have played α moves. E has chosen the elements $\{t_\beta\colon \beta<\alpha\}$ from T and asked the questions $\{q_\beta\colon \beta<\alpha\}$. A has answered with the sets $\{S_\beta\colon \beta<\alpha\}$. For some $i,j<\varkappa$, A has interpreted all c_β, $\beta<i$, and d_β, $\beta<j$, to $\mathscr{A}$ and nothing else. We write $c_\beta^{\mathscr{A}}$ and $d_\beta^{\mathscr{A}}$ for these interpretations. A has done this so that if some constant $c\in C$ or $d\in D$ exists in some sentence φ in $\bigcup_{\beta<\alpha}S_\beta$, then $c=c_\beta$ or $d=d_\gamma$ for some $\beta<i$ or $\gamma<j$. And everything that is in $\bigcup_{\beta<\alpha}S_\beta$ is true in $\mathscr{A}$ with given interpretations. A has also chosen numbers $N(t_\beta)$ for all $\beta<\alpha$.

On move α E chooses $t_\alpha\in T$ and asks a question q_α. The question q_α can be of one of the eight different types in the definition of the Hintikka game. We assume that q_α is of type 2 and describe how A answers. In all other cases A can answer similarly by keeping in mind that everything in $\bigcup_{\beta<\alpha}S_\beta$ is true in $\mathscr{A}$ with the given interpretations.

So E has chosen a closed term t. Let $i'<\varkappa$ and $j'<\varkappa$ be such that $i'>i$, $j'>j$ and if $c\in C$ or $d\in D$ exists in t then $c=c_\beta$ or $d=d_\gamma$ for some $\beta<i'$ or $\gamma<j'$. Then A puts $N(t_\alpha)=j'$ and for all β, $i\leqslant\beta<i'$, A interprets c_β arbitrarily and for all γ, $j\leqslant\gamma<j'$ A interprets d_γ to $g((t_\alpha,N',\gamma))$, where $N'\in{}^{\{t_\beta\colon\beta\leqslant\alpha\}}\varkappa$ and $N'(t_\beta)=N(t_\beta)$ for all $\beta\leqslant\alpha$ (recall that g is an order-preserving function $g\colon T^*\to(U^{\mathscr{A}},<^{\mathscr{A}})$). By interpreting new constants this way, A can be sure that if $\gamma<\beta$ then $d_\gamma^{\mathscr{A}}<d_\beta^{\mathscr{A}}$.

At this point the interpretation $t^{\mathscr{A}}$ of t is fixed and A interprets $c_{i'}$ to $t^{\mathscr{A}}$. Then he can answer by choosing S_α to be

$$\bigcup_{\beta<\alpha}S_\beta\cup\{t=t\}\cup\{t=c_{i'}\}.$$

By playing as explained above, A wins the game because he puts to S_α only what is true in $\mathscr{A}$. ■

4.15. COROLLARY. *($\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa}=\varkappa$) Let μ be any signature which includes a unary predicate symbol U and a binary predicate symbol $<$. Then in $L_{\varkappa^+\varkappa}(\mu)$ one cannot say that an ordering is a linear ordering which contains no descending sequence of length $\varkappa$.*

Proof. This corollary follows immediately from Theorem 4.14 and the fact that for each wide $\varkappa^+,\varkappa$-tree T there is a linear ordering S which contains no descending sequence of length $\varkappa$, and a function $f\colon T \to S$ such that if $x > y$ then $f(x) < f(y)$ (see [BP]). ■

As a Corollary of Theorem 4.14, we can also prove a generalization of Vaught's Covering Theorem. In the case $\varkappa = \omega$ it is due to R. L. Vaught ([Va]) and the proof we will give is analogous to that in [Va].

4.16. THEOREM. *($\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa} = \varkappa$) Let Φ be a conjunctive $\varkappa^+\varkappa$-Vaught sentence. If Φ does not have a model, then there is some $\varkappa^+$, $\varkappa$-tree T for which Φ^T does not have a model.*

Proof. For a contradiction we assume that

$$\Phi = (\forall x_\alpha \bigwedge_{i_\alpha \in I_\alpha} \exists y_\alpha \bigvee_{j_\alpha \in J_\alpha})_{\alpha<\varkappa} \bigwedge_{\alpha<\varkappa} \varphi^{i_0 j_0 \dots i_\alpha j_\alpha}$$

is a conjunctive $\varkappa^+\varkappa$-Vaught sentence of the signature μ and that for every $\varkappa^+$, $\varkappa$-tree T there is a model of Φ^T but there is no model of Φ. The idea in this proof is to construct a sentence Ψ of $L_{\varkappa^+\varkappa}$ that contradicts Theorem 4.14.

We take two new unary predicate symbols $M(x)$ and $U(x)$, one new binary predicate symbol $<$ and for every $\alpha < \varkappa$ and $(\dots, i_\beta, j_\beta, \dots)_{\beta<\alpha}$, $i_\beta \in I_\beta$, $j_\beta \in J_\beta$ for all $\beta < \alpha$, we take a new $1+\alpha$-ary predicate symbol $R^{(\dots, i_\beta, j_\beta, \dots)_{\beta<\alpha}}$.

Let Ψ be the conjunction of 1–6 below.

(1) "$(U, <)$ is a partial ordering and M is closed under the functions of μ".

(2) $\bigwedge \{\forall u \forall \{\dots, x_\beta, y_\beta, \dots\}_{\beta<\alpha} (R^{(\dots i_\beta j_\beta \dots)_{\beta<\alpha}}(u, x_0, y_0, \dots, x_\beta, y_\beta, \dots)_{\beta<\alpha}$
$\to \varphi^{(\dots i_\beta j_\beta \dots)_{\beta<\alpha}}(x_0, y_0, \dots, x_\beta, y_\beta, \dots)_{\beta<\alpha})\colon i_\beta \in I_\beta, j_\beta \in J_\beta \text{ for all } \beta < \alpha, \alpha < \varkappa\}$.

(3) $\forall u \in U(R^{\emptyset}(u))$.

(4) $\bigwedge \{\forall u, u' \in U \forall \{\dots, x_\beta, y_\beta, \dots\}_{\beta<\alpha} \big(u < u' \to$

$$(R^{(\dots i_\beta j_\beta \dots)_{\beta<\alpha}}(u, x_0, y_0, \dots, x_\beta, y_\beta, \dots)_{\beta<\alpha} \to$$
$$R^{(\dots i_\beta j_\beta \dots)_{\beta<\alpha}}(u', x_0, y_0, \dots, x_\beta, y_\beta, \dots)_{\beta<\alpha})\big)$$
$$\colon i_\beta \in I_\beta, j_\beta \in J_\beta \text{ for all } \beta < \alpha, \alpha < \varkappa\}$$

(5) $\bigwedge \Big\{\forall u \forall \{\dots, x_\beta, y_\beta, \dots\}_{\beta<\alpha} \Big(R^{(\dots i_\beta j_\beta \dots)_{\beta<\alpha}}(u, x_0, y_0, \dots, x_\beta, y_\beta, \dots)\beta<\alpha \to$

$$\forall u' \in U\big(u' > u \to$$
$$\forall x_\alpha \bigwedge_{i_\alpha \in I_\alpha} \exists y_\alpha \bigvee_{j_\alpha \in J_\alpha} \big(\neg M(x_\alpha) \vee (M(y_\alpha) \wedge R^{(i_0 j_0 \dots i_\alpha j_\alpha)}(u', x_0, y_0, \dots, x_\alpha, y_\alpha))\big)\big)\Big)$$
$$\colon i_\beta \in I_\beta, j_\beta \in J_\beta \text{ for all } \beta < \alpha, \alpha < \varkappa\Big\}.$$

(6) $\bigwedge \{\forall u \forall \{\dots, x_\beta, y_\beta, \dots\}_{\beta<\alpha} (\bigwedge_{\beta<\alpha} R^{(\dots i_\gamma j_\gamma \dots)_{\gamma<\beta}}(u, x_0, y_0, \dots, x_\gamma, y_\gamma, \dots)_{\gamma<\beta} \to$
$R^{(\dots i_\beta j_\beta \dots)_{\beta<\alpha}}(u, x_0, y_0, \dots, x_\beta, y_\beta, \dots)_{\beta<\alpha})\colon i_\beta \in I_\beta, j_\beta \in J_\beta$ for all $\beta < \alpha$,
α is a limit ordinal and $< \varkappa\}$.

In short Ψ says the following thing. Let $\mathscr{A}$ be a model for the signature $\mu \cup \{U, M, <\}$. Then

$$\mathscr{A} \models \exists \{R^{(\dots i_\beta j_\beta \dots)_{\beta<\alpha}}\colon i_\beta \in I_\beta, j_\beta \in J_\beta, \alpha < \varkappa\} \Psi$$

if and only if

$$\text{"}M^{\mathscr{A}} \models \Phi(U^{\mathscr{A}}, <^{\mathscr{A}})\text{"}.$$

Because for every $\varkappa^+$, $\varkappa$-tree T, Φ^T has a model, we see that for every wide $\varkappa^+$, $\varkappa$-tree T' there are a model $\mathscr{A}$ of Ψ and an order-preserving $g\colon T' \to (U^{\mathscr{A}}, <^{\mathscr{A}})$. Because Φ does not have a model, there is no model $\mathscr{A}$ where $(U^{\mathscr{A}}, <^{\mathscr{A}})$ contains an increasing $\varkappa$-sequence. Because $\Psi \in L_{\varkappa^+\varkappa}$, the existence of Ψ contradicts Theorem 4.14. ■

As a corollary of Theorem 4.16, we will prove a generalization of Theorem 4.14. We could also get this corollary from Theorem 4.14 itself by Skolemization. This Skolemization would go somewhat like the proof of Theorem 4.16.

4.17. COROLLARY. *($\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa} = \varkappa$) Let μ be any signature that includes a unary predicate symbol U and a binary predicate symbol $<$. We assume that φ is a sentence of $\bigwedge_{\varkappa^+\varkappa}(\mu)$ and that for every $\varkappa^+$, $\varkappa$-tree T there is an order-preserving function $g\colon T \to (U^{\mathscr{A}}, <^{\mathscr{A}})$. Then there is a model $\mathscr{A}$ of φ such that $(U^{\mathscr{A}}, <^{\mathscr{A}})$ consists of an increasing sequence of length $\varkappa$.*

Proof. To obtain a contradiction let $\varphi \in \bigwedge_{\varkappa^+\varkappa}(\mu)$ be such that for every $\varkappa^+$, $\varkappa$-tree T there is a model $\mathscr{A}$ of φ and an order-preserving $g\colon T \to (U^{\mathscr{A}}, <^{\mathscr{A}})$ but there is no model $\mathscr{A}$ of φ such that $(U^{\mathscr{A}}, <^{\mathscr{A}})$ consists of an increasing sequence of length $\varkappa$.

Let $\psi \in \bigwedge_{\varkappa^+\varkappa}$ be

$$\psi = (\exists x_\alpha)_{\alpha<\varkappa} \bigwedge_{\beta<\alpha<\varkappa} U(x_\beta) \wedge U(x_\alpha) \wedge (x_\beta < x_\alpha).$$

Then $\varphi \wedge \psi$ is a conjunctive $\varkappa^+\varkappa$-Vaught sentence and it does not have a model. On the other hand, for every $\varkappa^+\varkappa$-tree T $(\varphi \wedge \psi)^T$ has a model, because $\mathscr{A} \models \psi^T$ if and only if there is an order-preserving function $g\colon T \to (U^{\mathscr{A}}, <^{\mathscr{A}})$. This contradicts Theorem 4.16.

4.18. COROLLARY. *($\varkappa$ is regular cardinal and $\varkappa^{<\varkappa} = \varkappa$) Let μ be any signature which includes a unary predicate symbol U and a binary predicate symbol $<$. Then $\bigvee_{\varkappa^+\varkappa}(\mu)$ is not closed under negation.*

Proof. Let $\psi \in \bigvee_{\varkappa^+\varkappa}(\mu)$ be as in the proof of Corollary 4.17. Then $\neg\psi \notin \bigvee_{\varkappa^+\varkappa}(\mu)$ by Corollary 4.17. ■

We can also get a separation theorem for $M_{\varkappa^+\varkappa}$ as a corollary of Theorem 4.16. (The method in the proof is analogous to that used in [Va].) For this we need the following theorem. In its most general form it is due to J. Oikkonen [Oi1]. In the case $\varkappa = \omega$ it is due to L. Svenonius and R. L. Vaught ([Sv] and [Va]).

We recall that Ψ is Σ_1^1 over $L_{\lambda\varkappa}(\mu)$ (over $M_{\lambda\varkappa}(\mu)$) if it is of the form $\exists \bar{S}\psi$, where $\bar{S}$ is a set of relation (and function) symbols and $\psi \in L_{\lambda\varkappa}(\mu \cup \bar{S})$ $(\psi \in M_{\lambda\varkappa}(\mu \cup \bar{S}))$.

4.19. THEOREM. *($\varkappa$ is regular cardinal and $\varkappa^{<\varkappa}=\varkappa$)*

1. *For all Σ_1^1 over $L_{\varkappa^+\varkappa}$ sentences $\exists\bar{S}\psi$ there is a sentence φ of $\bigvee_{\varkappa^+\varkappa}$ such that*

(i) $\models\exists\bar{S}\psi\to\varphi$,

(ii) *for all models $\mathscr{A}$ of cardinality $\leqslant\varkappa$ $\mathscr{A}\models\varphi\to\exists\bar{S}\psi$.*

2. *For all sentences φ of $\bigvee_{\varkappa^+\varkappa}$ there is a Σ_1^1 over $L_{\varkappa^+\varkappa}$ sentence $\exists\bar{S}\psi$ such that $\models\varphi\leftrightarrow\exists\bar{S}\psi$.*

We omit the proof of this theorem. The proof of part 1 goes as in the case $\varkappa=\omega$ (see [Ma]). Part 2 can be proved by Skolemization.

Notice that if in part 1 in Theorem 4.19 the negation of $\exists\bar{S}\psi$ is also Σ_1^1 over $L_{\varkappa^+\varkappa}$, then $\models\varphi\leftrightarrow\exists\bar{S}\psi$ (use Downward–Löwenheim–Skolem Theorem).

4.20. SEPARATION THEOREM FOR $M_{\varkappa^+\varkappa}$. *($\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa}=\varkappa$) Let $\exists\bar{R}\varphi$ and $\exists\bar{S}\psi$ be Σ_1^1 over $M_{\varkappa^+\varkappa}$. If $\exists\bar{R}\varphi\wedge\exists\bar{S}\psi$ does not have a model then there is a sentence $\theta\in M_{\varkappa^+\varkappa}$ such that*

$$\models\exists\bar{R}\varphi\to\theta$$

and

$$\models\theta\to\neg\exists\bar{S}\psi\,.$$

Proof. We assume that $\exists\bar{R}\varphi\wedge\exists\bar{S}\psi$ does not have a model. By Skolemization we can assume that $\exists\bar{R}\varphi$ and $\exists\bar{S}\psi$ are Σ_1^1 over $L_{\varkappa^+\varkappa}$. By Theorem 4.19 there are sentences Φ and Ψ of $\bigwedge_{\varkappa^+\varkappa}$ such that

1. $\models\exists\bar{R}\varphi\to\Phi$ and $\models\exists\bar{S}\psi\to\Psi$

and

2. for all models $\mathscr{A}$ of cardinality $\leqslant\varkappa$ $\mathscr{A}\models\Phi\to\exists\bar{R}\varphi$ and $\mathscr{A}\models\Psi\to\exists\bar{S}\psi$.

By the fact that Theorem 3.5 is true also for $\bigwedge_{\lambda^+\varkappa}$, as we noticed after the theorem, we see that $\Phi\wedge\Psi$ does not have a model. By Theorem 4.16 there is a $\varkappa^+$, $\varkappa$-tree T such that $(\Phi\wedge\Psi)^T$ does not have a model. But then $\Phi^T\wedge\Psi^T$ does not have a model. Now

$$\models\Phi^T\to\neg\Psi^T$$

and

$$\models\neg\Psi^T\to\neg\Psi$$

and

$$\models\neg\Psi\to\neg\exists\bar{S}\psi$$

and so

$$\models\Phi^T\to\neg\exists\bar{S}\psi\,.$$

On the other hand,

$$\models\exists\bar{R}\varphi\to\Phi$$

and

$$\models\Phi\to\Phi^T$$

and so

$$\models\exists\bar{R}\varphi\to\Phi^T\,.\ ■$$

4.21. INTERPOLATION THEOREM FOR $L_{\varkappa^+\varkappa}$. *($\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa}=\varkappa$) Let $\varphi\in L_{\varkappa^+\varkappa}(\mu)$ and $\psi\in L_{\varkappa^+\varkappa}(\mu')$. If $\varphi\models\psi$ then there is $\theta\in M_{\varkappa^+\varkappa}(\mu\cap\mu')$ such that $\varphi\models\theta$ and $\theta\models\psi$.* ■

Let

$$M_{\lambda\varkappa}^n=\{\varphi\in M_{\lambda\varkappa}\colon \text{the negation of } \varphi \text{ belongs to } M_{\lambda\varkappa}\}\,.$$

We recall that $\Delta(L_{\lambda\varkappa})(\Delta(M_{\lambda\varkappa}))$ is the set of those Σ_1^1 over $L_{\lambda\varkappa}$ (over $M_{\lambda\varkappa}$) formulas Ψ for which the negation of Ψ is also Σ_1^1 over $L_{\lambda\varkappa}$ (over $M_{\lambda\varkappa}$).

4.22. COROLLARY. *($\varkappa$ is a regular cardinal and $\varkappa^{<\varkappa}=\varkappa$)*

$$\Delta(L_{\varkappa^+\varkappa})\ (=\Delta(M_{\varkappa^+\varkappa}))=M_{\varkappa^+\varkappa}^n\,.\ ■$$

4.23. BETH'S THEOREM FOR $M_{\varkappa^+\varkappa}$. *($\varkappa$ is regular cardinal and $\varkappa^{<\varkappa}=\varkappa$) Let $\varphi(P)\in M_{\varkappa^+\varkappa}(\mu\cup\{P\})$ and let $\varphi(P')$ be the sentence formed by replacing P everywhere by P'. We assume that*

$$\varphi(P)\wedge\varphi(P')\models\forall\bar{x}(P(\bar{x})\leftrightarrow P'(\bar{x}))\,.$$

Then there is $\theta\in M_{\varkappa^+\varkappa}(\mu)$ such that

$$\varphi(P)\models\forall\bar{x}(P(\bar{x})\leftrightarrow\theta(\bar{x}))\,.$$

Proof. Let $\bar{c}$ be new constants. Then

$$(\varphi(P)\wedge P(\bar{c}))\wedge(\varphi(P')\wedge\neg P'(\bar{c}))$$

does not have a model and $\theta(\bar{x})$ will satisfy what we required if $\theta(\bar{c})$ is the separating sentence of these sentences. ■

References

[BP] R. Bonnet, M. Pouzet, *Linear extensions of ordered sets*, in *Ordered Sets*, edited by I. Rival. D. Reidel Publishing Company, Dordrecht, Boston, London, 1982, 125–170.

[HR] J. Hintikka, V. Rantala, *A new approach to infinitary languages*, Ann. Math. Logic, vol. 10, 1976, 95–115.

[Hy] T. Hyttinen, *Games and infinitary languages*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, vol. 64, 1987.

[Ka] M. Karttunen, *Model theory for infinitely deep languages*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, vol. 50, 1984.

[Lo] E. Lopez-Escobar, *On defining well-orderings*, Fund. Math., vol. 59, 1966, 13–21, 299–300.

[Ma] M. Makkai, *Admissible sets and infinitary logic*, in *Handbook of mathematical logic*, edited by K. J. Barwise. North-Holland Publishing Company, Amsterdam, London, New York, 1977, 233–281.

[Mo] M. Morley, *Omitting classes of elements*, in *The theory of models*, edited by J. Addison, L. Henkin and A. Tarski. North-Holland Publishing Company, Amsterdam, London, New York, 1965, 265–273.

[Oi1] J. Oikkonen, Σ_1^1 *over* $L_{\varkappa^+\varkappa}$, in *Vexing Questions, An urnful of essays in honour of Veikko Rantala on the occasion of his fiftieth birthday*, edited by I. Patoluoto, E. Saarinen and P. Stenman. Reports from the Department of Philosophy, University of Helsinki, no. 3/1983.

[Oi2] J. Oikkonen, *How to obtain interpolation for $L_{\varkappa^+\varkappa}$*, to appear.
[Ra] V. Rantala, *Aspects of definability*, Acta Philos. Fenn., vol. 29, Nos. 2–3, 1977.
[Sv] L. Svenonius, *On the denumerable models of theories with extra predicates*, in *The theory of models*, edited by J. W. Addison, L. A. Henkin and A. Tarski, North-Holland Publishing Company, Amsterdam, London, New York, 1965, 376–389.
[Va] R. L. Vaught, *Descriptive set theory in $L_{\omega_1\omega}$*, in *Cambridge summer school in mathematical logic*, edited by A. R. D. Mathias and H. Rogers, Springer-Verlag Lecture Notes in Mathematics, vol. 337, 1973, 574–598.

UNIVERSITY OF HELSINKI
DEPARTMENT OF MATHEMATICS
Hallituskatu 15
00100 Helsinki 10
Finland

Received 25 April 1988;
in revised form 27 June 1988

A proof of Saffe's conjecture

by

Ludomir Newelski (Wrocław)

Abstract. We prove that if T is weakly minimal, $p_0 \in S(\emptyset)$ is non-isolated and has infinite multiplicity, then T has $2^{\aleph_0}$ countable models, thus proving Saffe's conjecture. Together with [B2] this completes the proof of Vaught's conjecture for weakly minimal theories.

§ 0. Introduction. This paper may be regarded as a continuation of the proof of Vaught's conjecture for weakly minimal theories, which was initiated in [B2], and carried on in [B3]. We use a standard set- and model-theoretic terminology. First we shall review shortly what was proved in [B2], [B3], and sketch some proofs to make the paper more self-contained. The reader should know the basic ideas from [B1] and [B2] however, as well as be familiar with stable groups (see [Po]). Vaught's conjecture states that every 1^{st}-order theory has either countably or $2^{\aleph_0}$ many countable models. Up to now there has been made only a relatively small progress towards proving this conjecture (see [Ls]). Shelah proved Vaught's conjecture for ω-stable theories [SHM]. Thus the natural aim of attack became the case of weakly minimal T. In [B2], Buechler proved that if T is weakly minimal and satisfies

(S) For every finite A, if $p \in S(A)$ is non-isolated then it has finite multiplicity,

then Vaught's conjecture holds for T. Earlier this was also known to Jürgen Saffe. Saffe conjectured that if T is weakly minimal and does not satisfy (S) then T has $2^{\aleph_0}$ countable models. Buechler [B2, Lemma 2.4 Proposition 3.1] reduced proving Saffe's conjecture to proving it for T weakly minimal and unidimensional. This paper is devoted to the proof of Saffe's conjecture for weakly minimal 1-dimensional T. So throughout we assume that T is weakly minimal, 1-dimensional, not ω-stable, does not satisfy (S) and (wlog) is small (i.e. $S_n(\emptyset)$ is countable).

CB denotes Cantor–Bendixson rank defined on $S(A)$ (cf. [B2]), $\mathrm{CB}(a/A)$ abbreviates $\mathrm{CB}(\mathrm{tp}(a/A))$. Recall that by [B1] every non-algebraic weakly minimal strong 1-type over $\emptyset$ is locally modular. For the advantages that local modularity gives, see [B1], [B2], [H]. Also, every such type is non-trivial. This is essentially by [B2, 2.4 and 3.1]. Notice also that if T is weakly minimal, unidimensional, and a non-algebraic $p \in S(\emptyset)$ is trivial, then T is ω-stable.