Separation in sequential spaces under PMEA

by

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Abstract. We show that the Product Measure Extension Axiom implies that normal (countably paracompact) sequential spaces are collectionwise normal (strongly collectionwise normal with respect to compact sets) and use the method of proof to show that, under PMEA, normal or countably paracompact, metalindelöf, locally compact (or, more generally, k-) spaces are paracompact.

I. Introduction. The Product Measure Extension Axiom (PMEA) has been used to show that normal spaces of character $< \mathfrak{c}$ are CWN [N], and that countably paracompact spaces of character $< \mathfrak{c}$ are strongly CWN w.r.t. compact sets [Bu]. Nyikos asked whether the "character $< \mathfrak{c}$" in his result could be replaced by "weak character $< \mathfrak{c}$". Junnila showed that the answer is "yes" [J]. We were able to generalize Junnila's result to show that PMEA implies normal (countably paracompact) sequential spaces are CWN (strongly CWN w.r.t. compact sets). We also discovered that certain quotient spaces of normal or countably paracompact, metalindelöf, locally compact (or, more generally, k-) spaces have enough sequential-like properties to enable us to show that they are also strongly CWN w.r.t. compact sets (and thus paracompact). These "applications" of our results on sequential spaces are less interesting now that Balogh has proved that if supercompact-many Cohen or random reals are added to a model of set theory (an assumption stronger than PMEA), then in the resulting model, normal, locally compact spaces are CWN, although our results use a weaker hypothesis [Ba_1].

In this paper we prove that PMEA implies normal (countably paracompact) sequential spaces are CWN (strongly CWN w.r.t. compact sets) and indicate how to apply these results to metalindelöf, locally compact spaces.

II. Definitions. The following concepts are used in the next section.

Product Measure Extension Axiom (PMEA): For each cardinal λ, the usual product measure on ${}^{\lambda}2$ can be extended to a $\mathfrak{c}$-additive measure defined on all subsets of ${}^{\lambda}2$.

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X is *collectionwise normal* (*with respect to compact sets*), abbreviated by CWN (w.r.t. compact sets), provided that every discrete collection of closed (compact) sets can be separated by disjoint open sets; if every such collection can be separated by a discrete collection of open sets, the space is said to be *strongly* CWN (*w.r.t. compact sets*).

X is *sequential* provided that a set U is open in X if, and only if, for each $x \in U$ and sequence $\langle x_n \rangle \to x$, there exists an m such that for each $n \geqslant m$, $x_n \in U$.

X is *countably paracompact* provided that every countable open cover has a locally finite open refinement.

III. On sequential spaces. Normal or countably paracompact spaces have the property that every countable discrete collection of closed sets has a locally finite open expansion. We will use this property to separate the desired sets. Our first theorem is analogous to Burke's result for spaces of character $< \mathfrak{c}$:

THEOREM 1 (PMEA). *Suppose X is sequential, $\mathscr{P} = \{P_\alpha: \alpha < \lambda\}$ is a discrete collection of closed sets, and every countable subcollection of $\mathscr{P}$ has a locally finite (point finite) open expansion. Then there is a sequence $\langle \mathscr{H}_n \rangle_{n \in Z^+}$ of open refinements of the canonical open cover such that for each $x \in X$ there is an $n \in \omega$ such that $\mathscr{H}_n$ is locally finite (point finite) at x.*

Proof. We omit the "point finite" case — its proof is analogous to the one presented. For each $f \in {}^{\omega \times \lambda}2$, let $A_{\omega,f} = \{\beta \in \lambda: f(n, \beta) = 0$ for each $n \in \omega\}$, and let $A_{nf} = \{\beta \in \lambda: f(n, \beta) = 1$ and $f(m, \beta) = 0$ for each $m < n\}$. For each $\alpha \leqslant \omega$, let $\mathscr{P}_{\alpha f} = \{P_\beta: \beta < \lambda$ and $\beta \in A_{\alpha,f}\}$. By hypothesis, we may let $\mathscr{U}_f = \{U_{\alpha f}: \alpha \leqslant \omega\}$ be a locally finite open expansion of $\{\bigcup \mathscr{P}_{\alpha f}: \alpha < \omega\}$, with

$$U_{\alpha f} \subset X \setminus \bigcup \{P_\beta: \beta < \lambda, \beta \in A_{\alpha,f}\}.$$

For each $x \in X$, let $U_{f,x}$ be an open set containing x that meets only finitely many elements of $\mathscr{U}_f$, with $U_{f,x} \subset \bigcap \{U \in \mathscr{U}_f: x \in U\}$.

For each $x, y \in X$, let $A(x, y) = \{f: y \in U_{f,x}\}$, and for each $n \in Z^+$, let $U_n(x) = \left\{y: \mu(A(x, y)) > 1 - \frac{1}{16n \cdot 2^{n^2}}\right\}$. We now show $U_n(x)$ is open in X, using the fact that X is sequential. Suppose $y \in U_n(x)$ and $y_n \to y$. For each $f \in A(x, y)$, $y \in U_{f,x}$, so let $m_f \in \omega$ be such that for each $k \geqslant m_f$, $y_k \in U_{f,x}$. For each $k \in \omega$, let $A(x, y, k) = \{f \in A(x, y): m_f \leqslant k\}$. Then $A(x, y, 0) \subset A(x, y, 1) \subset A(x, y, 2) \subset \ldots \subset \bigcup \{A(x, y, k): k \in \omega\} = A(x, y)$, so let $k \in \omega$ be such that $\mu(A(x, y, k)) > 1 - \frac{1}{16n \cdot 2^{n^2}}$. Suppose $m \geqslant k$. If $f \in A(x, y, k)$, then $m_f \leqslant k \leqslant m$, so $y_m \in U_{f,x}$, and so $f \in A(x, y_m)$. Thus $A(x, y, k) \subset A(x, y_m)$, which means that $\mu(A(x, y_m)) > 1 - \frac{1}{16n \cdot 2^{n^2}}$, and so $y_m \in U_n(x)$, for every $m \geqslant k$. Therefore $U_n(x)$ is open in X.

Now let $H_{n,\alpha} = \bigcup \{\bigcap_{m \leqslant n} U_m(x): x \in P_\alpha\}$, and let $\mathscr{H}_n = \{H_{n,\alpha}: \alpha < \lambda\}$, for each $n \in Z^+$. We now show $\langle \mathscr{H}_n \rangle_{n \in \omega}$ is the desired sequence.

Suppose $x \in X$ and for each n, $|\{H \in \mathscr{H}_n: H \cap \bigcap_{m \leqslant n} U_m(x) \neq \emptyset\}| \geqslant \omega$. Choose distinct β_n such that $\bigcap_{m \leqslant n} U_m(x) \cap H_{n,\beta_n} \neq \emptyset$, and choose $p_n \in P_{\beta_n}$, $y_n \in X$ such that $y_n \in \bigcap_{m \leqslant n} U_m(x) \cap \bigcap_{m \leqslant n} U_m(p_n)$. For each n, let $B_n = \{f: U_{f,x}$ meets no more than n elements of $\mathscr{U}_f\}$. Then $\langle B_n: n \in \omega \rangle$ is an increasing sequence with union $2^{\omega \times \lambda}$, so we may let $m \in \omega$ be such that $\mu(B_m) > 7/8$. The idea is to obtain a contradiction by finding an $f \in B_m$ that puts $m+1$ of the p's into different elements of $\mathscr{U}_f$ and cause $U_{f,x}$ to meet each of these. For each $j \geqslant 1$, let $I_j = \{\beta_{j(m+1)+i}: 0 \leqslant i \leqslant m\}$. (The I_j's partition the β's into sets of size $m+1$.) Let M_j be a basic neighborhood in $2^{\omega \times \lambda}$ defined as follows: let $\sigma_j: (m+1) \times I_j$ be the function such that $\sigma_j(r, \beta_{j(m+1)+i}) = 1$ iff $r = i$, and let $M_j = [\sigma_j]$. Burke showed that under these conditions, $\mu(\bigcup_{j=1}^{n} M_j) > 3/8$ for any $n \geqslant 2^{(m+1)^2-1}$[B]. Note that for $f \in M_j$ and $0 \leqslant i \leqslant m$, $\beta_{j(m+1)+i} \in A_{rf}$ iff $r = i$ (so M_j puts $m+1$ of the p's into different elements of $\mathscr{U}_f$).

Let $S = B_m \cap \bigcup \{M_j: 1 \leqslant j \leqslant 2^{(m+1)^2-1}\}$. $\mu(S) > 1/8$, so let $1 \leqslant j \leqslant 2^{(m+1)^2-1}$ be such that $\mu(B_m \cap M_j) > \frac{1}{8 \cdot 2^{(m+1)^2-1}} = \frac{1}{4 \cdot 2^{(m+1)^2}}$. Finally, let

$$E = \bigcap_{i=0}^{m} A(p_{j(m+1)+i}, y_{j(m+1)+i}) \cap \bigcap_{i=0}^{m} A(x, y_{j(m+1)+i}).$$

Since

$$y_{j(m+1)+i} \in U_{j(m+1)+i}(x) \cap U_{j(m+1)+i}(p_{j(m+1)+i}),$$

the measure of each $A(p_{j(m+1)+i}, y_{j(m+1)+i})$ and each $A(x, y_{j(m+1)+i})$ is $> 1 - \frac{1}{16(j(m+1)+i) \cdot 2^{(j(m+1)+i)}}$, which is $\geqslant 1 - \frac{1}{16(m+1) \cdot 2^{(m+1)^2}}$. Thus $\mu(E^c) \leqslant$ $\leqslant \sum_{i=0}^{m} \mu(A(p_{j(m+1)+i}, y_{j(m+1)+i})^c) + \sum_{i=0}^{m} \mu(A(x, y_{j(m+1)+i})^c) \leqslant 2 \sum_{i=0}^{m} \frac{1}{16(m+1) \cdot 2^{(m+1)^2}} = \frac{1}{8 \cdot 2^{(m+1)^2}}$. So $\mu([(B_m \cap M_j) \cap E]^c) \leqslant \mu((B_m \cap M_j)^c) + \mu(E^c) \leqslant 1 - \frac{1}{4 \cdot 2^{(m+1)^2}} + \frac{1}{8 \cdot 2^{(m+1)^2}} < 1$. Thus we may choose an $f \in B_m \cap M_j \cap E$. Since $f \in B_m$, $U_{f,x}$ meets no more than m elements of $\mathscr{U}_f$. Since $f \in M_j$, $\beta_{j(m+1)+i} \in A_{i,f}$, and so $P_{\beta_{j(m+1)+i}} \subset U_{i,f}$; in particular, $U_{f,p_{j(m+1)+i}} \subset U_{i,f}$, for each i, $0 \leqslant i \leqslant m$. But since $f \in E$, $y_{j(m+1)+i} \in U_{f,p_{j(m+1)+i}} \cap U_{f,x}$ so $U_{f,x}$ meets each $U_{i,f}$, giving us a contradiction. So there must be an n with $\mathscr{H}_n$ locally finite at x.

The following results are analogous to those obtained by Burke, and may be proved by using our Theorem 1 in place of his Theorem 2.2 in his proofs:

COROLLARY 2 (PMEA). *In a sequential countably metacompact space X, any closed set which is the union of a discrete collection of G_δ-sets is itself a G_δ-set.*

COROLLARY 3 (PMEA). *In a sequential countably paracompact space X, any* *set which is the union of a discrete collection of regular G_δ-sets is itself a regular G_δ-set.*

THEOREM 4 (PMEA). *Suppose X is sequential, $\mathscr{P} = \{P_\alpha: \alpha < \lambda\}$ is a discrete collection of closed sets in X, and $\{U_{n,\alpha}: n \in Z^+, \alpha < \lambda\}$ is a collection of open sets in X such that $P_\alpha \subset \bigcap_n U_{n,\alpha} \subset \bigcap_n \overline{U_{n,\alpha}} = D_\alpha$ for each α.*

(a) *If X is countably paracompact, and W is open such that $\bigcup_{\alpha<\lambda} D_\alpha \subset W$, then there is an open set U such that $\bigcup_{\alpha<\lambda} P_\alpha \subset U \subset \overline{U} \subset W$.*

(b) *If every countable subcollection of $\mathscr{P}$ has a locally finite open expansion, and if $D_\alpha \cap D_\beta = \emptyset$ for all $\alpha \neq \beta$, then $\mathscr{P}$ can be separated by disjoint open sets.*

COROLLARY 5 (PMEA). *A sequential countably paracompact space is strongly* CWN *w.r.t. compact sets and also strongly* CWN *w.r.t. regular G_δ-sets.*

THEOREM 6 (PMEA). *A sequential, countably paracompact, subparacompact (submetacompact) space is paracompact.*

It also follows from Theorem 4(b) that:

COROLLARY 7 (PMEA). *Every sequential normal space is (strongly)* CWN.

After hearing of our results, D. Fremlin proved that PMEA implies that normal, countably tight, k-spaces are CWN, which implies Corollary 7 [F]. It is difficult to find countably tight compact spaces that are not sequential – in fact, it is consistent that they are sequential.

IV. Applications.

THEOREM 8 (PMEA). *If X is normal or regular countably paracompact, metalindelöf, and every compact subset of X (point in X) is contained in a compact set of character less than $\mathfrak{c}$, then X is strongly* CWN *w.r.t. compact sets* (SCWH).

Proof. We do the case where every compact subset of X is contained in a compact set of character less than $\mathfrak{c}$. Suppose $\{C_\alpha: \alpha < \lambda\}$ is a discrete collection of compact sets in X. Let Y be the absolute of X [see E, p. 464], and let g be the natural (perfect) mapping from Y onto X. For each $x \notin \bigcup_\alpha C_\alpha$, let U_x be an open set containing x such that $\overline{U}_x \cap \bigcup_\alpha C_\alpha = \emptyset$ and for each $x \in C_\alpha$, let U_x be an open set containing x such that $\overline{\bigcup \{U_x: x \in C_\alpha\}} \cap \bigcup_{\beta\neq\alpha} C_\beta = \emptyset$ (use the compactness of C_α). For each $\alpha < \lambda$, let K_α be a compact set containing C_α having character less than $\mathfrak{c}$, and let $\{V_{\alpha,\beta}: \beta < \lambda_\alpha\}$ be a basis for K_α, where $\gamma_\alpha < \mathfrak{c}$. Let $\mathscr{W}$ be a point-countable open refinement of $\{U_x: x \in X\}$; for each $\alpha < \lambda$, let

$$V_\alpha = \bigcup \{V \in \mathscr{W}: V \cap C_\alpha \neq \emptyset\} \subset \bigcup \{U_x: x \in C_\alpha\}.$$

Let $\mathscr{V} = \{V_\alpha: \alpha < \tau\}$ list $\{V_\alpha: \alpha < \lambda\} \cup \{V \in \mathscr{W}: V \cap \bigcup_\alpha C_\alpha = \emptyset\}$. For each $\alpha < \lambda$, let $L_\alpha = K_\alpha \setminus \bigcup_{\beta\neq\alpha} V_\beta$. Since Y is regular and extremally disconnected and $g^{-1}(L_\alpha)$ is compact, let W_α be a clopen set such that $g^{-1}(L_\alpha) \subset W_\alpha \subset g^{-1}(V_\alpha)$ for each $\alpha < \lambda$.

Since g is closed let R_α be an open set such that $L_\alpha \subset R_\alpha \subset V_\alpha$ and $g^{-1}(R_\alpha) \subset W_\alpha$. $\{R_\alpha: \alpha < \lambda\} \cup \{U_x: x \notin \bigcup_\alpha R_\alpha\}$ covers X, so let $\mathscr{V}'$ be a point-countable open refinement. We claim that $\mathscr{R} = \{\overline{g^{-1}(R_\alpha)}: \alpha < \lambda\} \cup \{g^{-1}(V): V \in \mathscr{V}', V \not\subset \bigcup_\alpha R_\alpha\}$ is a point-countable open cover of Y: if p is in $\overline{g^{-1}(R_\alpha)}$ for uncountably many α, then p is in W_α for uncountably many α, so p is in $g^{-1}(V_\alpha)$ for uncountably many α, contradicting the point-countability of $\mathscr{V}$. Y is extremally disconnected, so each $\overline{g^{-1}(R_\alpha)}$ is clopen. Rename $\mathscr{R}$ as $\{S_\alpha: \alpha < \varkappa\}$, where $S_\alpha = \overline{g^{-1}(R_\alpha)}$ for each $\alpha < \lambda$. For each $\alpha < \lambda$, let $M_\alpha = g^{-1}(K_\alpha) \setminus \bigcup_{\beta\neq\alpha} S_\beta$. $\{M_\alpha: \alpha < \lambda\}$ is a discrete collection, and $g^{-1}(C_\alpha) \subset M_\alpha$.

For each $f \in {}^{\omega\times\lambda}2$, let $A_{\omega,f} = \{\beta \in \lambda: f(m, \beta) = 0$ for each $m \in \omega\}$, and let $A_{n,f} = \{\beta \in \lambda: f(m, \beta) = 0$ for $m < n$ and $f(n, \beta) = 1\}$ for $n \in \omega$. Let

$$\mathscr{M}_{\alpha,f} = \{M_\beta: \beta \in A_{\alpha,f}\}$$

for each $\alpha \leqslant \omega$. We now show $\{\bigcup \mathscr{M}_{\alpha,f}: \alpha \leqslant \omega\}$ has a locally finite open expansion. First we show $\mathscr{F} = \{g(M_\alpha): \alpha < \lambda\}$ is discrete. Suppose $p \in X$. Note that $M_\alpha \cap g^{-1}(R_\beta) = \emptyset$ for $\alpha \neq \beta$ in λ, so each R_β witnesses the discreteness of $\mathscr{F}$. If $p \notin \bigcup R_\beta$, let $p \in V \in \mathscr{V}'$ for some $V \not\subset \bigcup_\beta R_\beta$. $g^{-1}(V) = S_\beta$ for some β, and by definition of the M's, S_β misses each M_α, so V misses each $g(M_\alpha)$. For each $\alpha \leqslant \omega$, let $P_\alpha = \bigcup \{f(M_\beta): \beta \in A_{\alpha,f}\}$. Let $\{T_\alpha: \alpha \leqslant \omega\}$ be a locally finite open expansion of $\{P_\alpha: \alpha \leqslant \omega\}$. $\bigcup \mathscr{M}_{\alpha,f} \subset g^{-1}(T_\alpha)$ for $\alpha \leqslant \omega$, and it is easy to check that $\mathscr{U}_f = \{U_{\alpha,f}: \alpha \leqslant \omega\}$, where $U_{\alpha,f} = g^{-1}(T_\alpha)$ is a locally finite open expansion of $\{\bigcup \mathscr{M}_{\alpha,f}: \alpha \leqslant \omega\}$; without loss of generality, $T_\alpha \subset X \setminus \bigcup_{\beta\neq\alpha} P_\beta$, and so

$$g^{-1}(T_\alpha) \subset Y \setminus \bigcup \{M_\beta: \beta < \lambda, \beta \notin A_{\alpha,f}\}.$$

For each $x \notin \bigcup_\alpha C_\alpha$, let K_x be a compact set of character $< \mathfrak{c}$ containing x; let $\{V_{x,\beta}: \beta < \gamma_x\}$ be a basis for K_x, where $\gamma_x < \mathfrak{c}$; let $M_x = g^{-1}(K_x) \setminus \bigcup \{S_\beta: \beta \in \lambda$ and $g^{-1}(x) \cap S_\beta = \emptyset\}$. For each $x \in X$, let $F_x = \{\beta: g^{-1}(x) \cap S_\beta \neq \emptyset\}$; since $\mathscr{V}$ and $\mathscr{V}'$ are point-countable, F_x is countable, and so let $F_x = \{\alpha_{x,m}: m \in \omega\}$ and for each $n \in \omega$, let $F_x|n = \{\alpha_{x,m}: m < n\}$.

For each $x \notin \bigcup_\alpha C_\alpha$ and each $y \in M_x$, let $U_{f,x,y}$ be an open set in Y containing y that meets only finitely many elements of $\mathscr{U}_f$, with $U_{f,x,y} \subset \bigcap \{U \in \mathscr{U}_f: y \in U\}$; let $\mathscr{U}_{f,x}$ be a finite subset of $\{U_{f,x,y}: y \in M_x\}$ which covers M_x, and let $U_{f,x} = \bigcup \mathscr{U}_{f,x}$. For each $x \in C_\alpha$, and each $y \in M_\alpha$, let $U_{f,x,y}$ be defined similarly; let $\mathscr{U}_{f,x}$ be a finite subset of $\{U_{f,x,y}: y \in M_\alpha\}$ which covers M_α, and let $U_{f,x} = \bigcup \mathscr{U}_{f,x}$.

For each $x \in X$ and $y \in Y$, let $A(x, y) = \{f: y \in U_{f,x}\}$, and for each $n \in \omega$ let $U_n(x) = \left\{y: \mu(A(x, y)) > 1 - \dfrac{1}{16n \cdot 2^{n^2}}\right\}$.

We now show $U_n(x)$ is a neighborhood of $g^{-1}(x)$ for $x \notin \bigcup_\alpha C_\alpha$. First we need a:

LEMMA. *If* $M_x \subset U$, *U an open set, then there is an* $\alpha < \gamma_x$ *and a* $C \in [\lambda \setminus F_x]^{<\omega}$ *such that* $g^{-1}(V_{x,\alpha}) \setminus \bigcup \{S_\beta\colon \beta \in C\} \subset U$.

Proof of Lemma. Assume the hypothesis. Suppose for each $C \in [\lambda \setminus F_x]^{<\omega}$, there is an $h_C \in g^{-1}(K_x) \setminus (\bigcup \{S_\beta\colon \beta \in C\} \cup U)$. Let $A = \overline{\{h_C\colon C \in [\lambda \setminus F_x]^{<\omega}\}}$. $A \subset g^{-1}(K_x) \setminus U$, so is compact. If $y \in A$, then $y \in M_x$, so there is a $\beta \in \lambda \setminus F_x$ such that $y \in S_\beta$. Let $C \in [\lambda \setminus F_x]^{<\omega}$ be such that $\{S_\beta\colon \beta \in C\}$ covers A. Since h_C cannot be so covered, we have a contradiction. Thus there is a $C \in [\lambda \setminus F_x]^{<\omega}$ with $g^{-1}(K_x) \setminus \{S_\beta\colon \beta \in C\} \subset U$. So $g^{-1}(K_x) \subset U \cup \bigcup \{S_\beta\colon \beta \in C\}$. Let R be an open set containing K_x such that $g^{-1}(K_x) \subset g^{-1}(R) \subset U \cup \bigcup \{S_\beta\colon \beta \in C\}$. Let $\alpha < \gamma_x$ be such that $K_x \subset V_{x,\alpha} \subset R$; then $M_x \subset g^{-1}(V_{x,\alpha}) \setminus \bigcup \{S_\beta\colon \beta \in C\} \subset U$.

Returning to the proof of the theorem, given $f \in {}^{\omega \times \lambda}2$, let $\alpha_f < \gamma_x$ and $C_f \in [\lambda \setminus F_x]^{<\omega}$ be such that $M_x \subset g^{-1}(V_{x,\alpha_f}) \setminus \bigcup \{S_\beta\colon \beta \in C_f\} \subset U_{f,x}$. For each $\alpha < \gamma_x$, let $D_\alpha = \{f\colon \alpha_f = \alpha\}$. Since $\sum_{\alpha < \gamma_x} \mu(D_\alpha) = 1$, let $\{\alpha_i\colon i \leqslant l\} \subset \gamma_x$ be such that $\mu(\bigcup_{i \leqslant l} D_{\alpha_i}) > 1 - \frac{1}{16n \cdot 2^{n^2}}$, where $l \in \omega$. Let $\alpha < \gamma_x$ be such that $V_{x,\alpha} \subset \bigcap_{i \leqslant l} V_{x,\alpha_i}$. Suppose for each $F \in [\lambda \setminus F_x]^{<\omega}$, $g^{-1}(V_{x,\alpha}) \setminus \bigcup \{S_\beta\colon \beta \in F\} \not\subset U_n(x)$; let

$$y_l \in g^{-1}(V_{x,\alpha}) \setminus (\bigcup \{S_\beta\colon \beta \in \bigcup_{k<l} F_{\alpha(y_k)} | l \setminus F_x\} \cup U_n(x)).$$

By choice of the y's, for each $f \in {}^{\omega \times \lambda}2$ there is a $k_f \in \omega$ such that for each $k \geqslant k_f$, $y_k \notin \bigcup \{S_\beta\colon \beta \in C_f\}$, since otherwise infinitely many of the y's would be in the same S_β for some β. Therefore, for each $k \geqslant k_f$, $y_k \in g^{-1}(V_{x,\alpha}) \setminus \bigcup \{S_\beta\colon \beta \in C_f\}$.

For each $k \in \omega$, let $A(k) = \{f \in \bigcup_{i \leqslant l} D_{\alpha_i}\colon k_f \leqslant k\}$. $\langle A_k\colon k \in \omega \rangle$ is increasing, with union $\bigcup_{i \leqslant l} D_{\alpha_i}$, so let $k \in \omega$ be such that $\mu(A(k)) > 1 - \frac{1}{16n \cdot 2^{n^2}}$. We show $y_k \in U_n(x)$, a contradiction. For each $f \in A(k)$, $k_f \leqslant k$, so

$$y_k \in g^{-1}(V_{x,\alpha}) \setminus \bigcup \{S_\beta\colon \beta \in C_f\} \subset g^{-1}(V_{x,\alpha_f}) \setminus \bigcup \{S_\beta\colon \beta \in C_f\} \subset U_{f,x}$$

and so $f \in A(x, y_k)$. Thus $\mu(A(x, y_k)) > 1 - \frac{1}{16n \cdot 2^{n^2}}$, and so $y_k \in U_n(x)$.

Thus we may conclude that there is an $\alpha < \gamma_x$ and $F \in [\lambda \setminus F_x]^{<\omega}$ with $g^{-1}(x) \subset g^{-1}(V_{x,\alpha}) \setminus \bigcup \{S_\beta\colon \beta \in F\} \subset U_n(x)$; since $F \subset \lambda$ each S_β is clopen for $\beta \in F$, and so $U_n(x)$ is a neighborhood of $g^{-1}(x)$.

Now if $M_\alpha \subset U$ for some $\alpha < \lambda$ and some open set U, there is a $\gamma < \gamma_\alpha$ and $C \in [\varkappa \setminus \{\alpha\}]^{<\omega}$ with $g^{-1}(V_{\alpha,\gamma}) \setminus \bigcup \{S_\beta\colon \beta \in C\} \subset U$, so by similar reasoning there is a $\gamma < \gamma_\alpha$ and $F \in [\varkappa \setminus \{\alpha\}]^{<\omega}$ with $g^{-1}(V_{\alpha,\gamma}) \setminus \bigcup \{S_\beta\colon \beta \in F\} \subset U_n\{x\}$, for $x \in C_\alpha$. Now for $\beta \in F$, either $S_\beta = \overline{g^{-1}(R_\beta)}$, and so is closed, or $S_\beta = g^{-1}(V)$ for $V \in \mathscr{V}'$ with $V \not\subset \bigcup_\alpha R_\alpha$. In the latter case $V \subset U_z$ for some $z \notin \bigcup_\alpha R_\alpha$. If $y \in g^{-1}(x) \cap \bar{S}_\beta$, then $x \in g(\bar{S}_\beta) = g(\overline{g^{-1}(V)}) \subset \overline{g(g^{-1}(V))} = \bar{V} \subset \bar{U}_z$, but $\bar{U}_z$ misses C_α. Thus $g^{-1}(x) \subset g^{-1}(V_{\alpha,n}) \setminus \overline{\bigcup \{S_\beta\colon \beta \in F\}} \subset U_n(x)$, and so $U_n(x)$ is a neighborhood of $g^{-1}(x)$.

Define $H_{n,\alpha} = \bigcup \{\bigcap_{m \leqslant n} U_m(x)\colon x \in C_\alpha\}$ for each $\alpha < \lambda$ and $\mathscr{H}_n = \{H_{n,\alpha}\colon \alpha < \lambda\}$ for each $n \in Z^+$. By an argument similar to the one presented in the proof of Theorem 1, for each $x \in X$ there is an $n \in Z^+$ such that $\mathscr{H}_n$ is locally finite at $g^{-1}(x)$.

Since each $U_m(x)$ is a neighborhood of $g^{-1}(x)$, $H_{n,\alpha}$ is a neighborhood of $g^{-1}(C_\alpha)$; for each $n \in Z^+$, let $J_{n,\alpha}$ be an open set containing C_α such that

$$g^{-1}(J_{n,\alpha}) \subset H_{n,\alpha}, \quad \overline{J_{n+1,\alpha}} \subset J_{n,\alpha},$$

and $\overline{J_{n,\alpha}} \subset \bigcup \{U_x\colon x \in C_\alpha\}$; let $\mathscr{J}_n = \{J_{n,\alpha}\colon \alpha < \lambda\}$. Then for each $x \in X$, there is an $n \in Z^+$ such that $\mathscr{J}_n$ is locally finite at x. Similar to Burke's proof of his Theorem 3.1 (b), let $K_{n,\alpha} = J_{n,\alpha} \setminus \overline{\bigcup \{J_{n,\beta}\colon \beta \in \lambda \setminus \{\alpha\}\}}$ for each $\alpha < \lambda$, $n \in Z^+$, and let $K_\alpha = \bigcup_n K_{\alpha,n}$. Then $C_\alpha \subset K_\alpha$ and $K_\alpha \cap K_\beta = \emptyset$ if $\alpha \neq \beta$, so $\{K_\alpha\colon \alpha < \lambda\}$ separates $\{C_\alpha\colon \alpha < \lambda\}$. Obviously if X is normal we can now get a strong separation of $\{C_\alpha\colon \alpha < \lambda\}$. If X is countably paracompact, proceed as in Burke's proof of his Corollary 3.2 (a): let A_α be open such that $C_\alpha \subset A_\alpha \subset \bar{A}_\alpha \subset K_\alpha$. Let $B_{n,\alpha} = J_{n,\alpha} \cap A_\alpha$. Let $V_n = X \setminus \overline{\bigcup_\alpha B_{n,\alpha}}$. $\{\bigcup_\alpha K_\alpha\} \cup \{V_n\colon n \in Z^+\}$ covers X, so let $\{\bigcup_\alpha K_\alpha\} \cup \{G_n\colon n \in Z^+\}$ be a precise locally finite open refinement. Then $\bigcup_\alpha C_\alpha \subset X \setminus \overline{\bigcup_n G_n} \subset X \setminus \overline{\bigcup_n G_n} \subset \bigcup_\alpha K_\alpha$, so we may get a strong separation of $\{C_\alpha\colon \alpha < \lambda\}$.

COROLLARY 9. *Normal or countably paracompact, locally compact, metalindelöf spaces are paracompact.*

Proof. Use Theorem 8 along with Balogh's result that locally Lindelöf, submetalindelöf spaces that are SCWH are paracompact [Ba2].

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