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CARLETON UNIVERSITY,
Ottawa, Canada K1S 5B6

Received 7 March 1988;
in revised form 15 June 1988

Imbeddings in R^{2m} of m-dimensional compacta with $\dim(X\times X)<2m$

by

Stanisław Spież (Warszawa)

Abstract. In this note we give a sufficient condition for two maps from compacta into balls to be transversely trivial. As a corollary we conclude that any m-dimensional compactum X with $\dim(X\times X)<2m$ admits a dense set of imbeddings into R^{2m} provided $m\geqslant 3$.

Introduction. Recall that a mapping f from a space X into the p-dimensional cube I^p, $I=[-1,1]$, is said to be *inessential* (in the sense of Alexandrov–Hopf) if there exists a mapping $g\colon X\to\partial I^p$ (∂I^p denotes the boundary of I^p) such that $g(x)=f(x)$ for each $x\in f^{-1}(\partial I^p)$. Two maps $f\colon X\to I^p$ and $g\colon Y\to I^q$ are said to be *transversely trivial* (see [Kr], compare also [K–L], Problem (2)) if there exist two mappings $F\colon X\to I^p\times I^q$ and $G\colon Y\to I^p\times I^q$ satisfying the following conditions:

(i) $F|f^{-1}(\partial I^p)=(f,0)|f^{-1}(\partial I^p)$,
(ii) $G|g^{-1}(\partial I^q)=(0,g)|g^{-1}(\partial I^q)$,
(iii) $F(X)\cap G(Y)=\emptyset$.

In this note we prove the following:

THEOREM. *Let X and Y be compacta with $\dim(X)=p$ and $\dim(Y)=q$. Suppose $f\colon X\to I^p$ and $g\colon Y\to I^q$ are mappings such that $f\times g\colon X\times Y\to I^p\times I^q$ is inessential. If $p\geqslant 3$, $q\geqslant 2$ then f, g are transversely trivial.*

This result gives a positive partial answer to Problem (2) in [K–L].

In [M–R] D. McCullough and L. R. Rubin proved the following interesting result: for each $m\geqslant 2$ there exists an m-dimensional continuum X such that the space $E(X,R^{2m})$ of imbeddings from X into R^{2m} is dense in the space $C(X,R^{2m})$ of continuous maps from X into R^{2m}. They asked whether this property is related to the phenomenon of m-dimensional compacta whose squares have dimension less than $2m$.

J. Krasinkiewicz (see [Kr]) proved that if $E(X,R^{2m})$ is dense in $C(X,R^{2m})$ then $\dim(X\times X)<2m$ for any m-dimensional compactum X, and he asked whether the converse is true.

A consequence of Theorem (2.2) in [Kr] and our theorem is the following:

COROLLARY. *If X is an m-dimensional compactum, $m\geqslant 3$, with $\dim(X\times X)<2m$ then the space $E(X,R^{2m})$ is dense in $C(X,R^{2m})$.*

Thus the well-known examples of m-dimensional continua X with $\dim(X\times X)<2m$ (which can be very easily constructed using the Boltyanski's Example, see [Bo]) satisfy the conclusion of the theorem in [M–R] provided $m\geqslant 3$. The property $E(X, \boldsymbol{R}^{2m})$ is dense in $C(X, \boldsymbol{R}^{2m})$ is equivalent to the property $\dim(X\times X)<2m$ for any m-dimensional compactum, $m\neq 2$. The case $m=2$ is open. (Added in proof: see note at the end of the paper.)

1. Intersection cocycle. We choose once and for all an orientation in $\boldsymbol{R}^m$. Let c and c' be p-dimensional and respectively q-dimensional singular chains in $\boldsymbol{R}^m$. The *intersection number* $c\wedge c'$ of c and c' is defined classically (see [S–T]) whenever $\mathrm{Carr}(c)\cap\mathrm{Carr}(\partial c')=\emptyset=\mathrm{Carr}(\partial c)\cap\mathrm{Carr}(c')$ and $p+q=m$. The following properties of the intersection number are well known:

$$c\wedge c'=(-1)^{pq}c'\wedge c\,,\qquad p+q=m\,,$$

$$(\partial c)\wedge c'=(-1)^{p}c\wedge(\partial c')\,,\qquad p+q=m+1\,.$$

An oriented p-cell σ is a p-cell $|\sigma|$ together with a homeomorphism h_σ of the standard p-simplex Δ_p onto $|\sigma|$. If f is a continuous map of $|\sigma|$ in $\boldsymbol{R}^m$, we shall denote by $f(\sigma)$ the singular p-simplex $f\circ h_\sigma\colon \Delta_p\to\boldsymbol{R}^m$. For a p-dimensional chain $c=\Sigma n_i\sigma_i$ and a continuous map f in $\boldsymbol{R}^m$ defined on the carrier of c by $f(c)$ we denote the singular p-chain $\Sigma n_i f(\sigma_i)$.

Let (K, K_0) and (L, L_0) be pairs of simplicial complexes and let $f\colon |K|\to\boldsymbol{R}^m$ and $g\colon |L|\to\boldsymbol{R}^m$ be continuous maps such that

(i) $f(|K|)\cap g(|L_0|)=\emptyset=f(|K_0|)\cap g(|L|)$,

(ii) $f(|\sigma|)\cap g(|\tau|)=\emptyset$ whenever σ is a p-dimensional simplex of K, τ is a q-dimensional simplex of L and $p+q<m$.

One can define an m-cochain $c(f,g)$ in $(K, K_0)\times(L, L_0)$ with integral coefficients by the formula

$$c(f,g)(\sigma\times\tau)=(-1)^{q}f(\sigma)\wedge g(\tau)$$

where σ is an (oriented) p-simplex of K and τ is an oriented q-simplex of L and $p+q=m$.

Here we consider the group of (relative) n-cochains in $(K, K_0)\times(L, L_0)$ as the subgroup of $C^n(K\times L)$ consisting of those cochains that vanish on every oriented n-cell $\sigma\times\tau$ of $(K\times L_0)\cup(K_0\times L)$. In the same way as in [Sh] or [Wu], we shall prove the following lemma:

(1.1) Lemma. *The cochain $c(f,g)$ is a cocycle in $(K, K_0)\times(L, L_0)$.*

Proof. Let σ be a p-simplex of K and let τ be a q-simplex of L. If $p+q=m+1$ then

$$\begin{aligned}\delta c(f,g)(\sigma\times\tau)&=c(f,g)(\partial(\sigma\times\tau))=c(f,g)(\partial\sigma\times\tau)+(-1)^{p}c(f,g)(\sigma\times\partial\tau)\\&=(-1)^{q}f(\partial\sigma)\wedge g(\tau)+(-1)^{p+q-1}f(\sigma)\wedge g(\partial\tau)\\&=(-1)^{q+p}f(\sigma)\wedge g(\partial\tau)+(-1)^{p+q+1}f(\sigma)\wedge g(\partial\tau)=0\,.\end{aligned}$$

The cocycle $c(f,g)$ we shall call the *intersection cocycle* of f and g with respect to (K, K_0) and (L, L_0). The cohomology class

$$[c(f,g)]\in H^m((K, K_0)\times(L, L_0))$$

we shall call the *intersection cohomology class* of f, g with respect to (K, K_0), (L, L_0).

Remark. One can see that the above definition agrees with the following one. Let (X, A) and (Y, B) be pairs of compacta. Suppose $f\colon X\to\boldsymbol{R}^m$ and $g\colon Y\to\boldsymbol{R}^m$ are continuous maps such that $f(X)\cap g(B)=\emptyset=f(A)\cap g(Y)$. Consider the map

$$d\colon (X, A)\times(Y, B)\to(\boldsymbol{R}^m, \boldsymbol{R}^m\setminus 0)$$

given by $d(x,y)=f(x)-g(y)$. Now we define the intersection cohomology class of f and g with respect to (X, A) and (Y, B) as the image $H^m(d)(e)$ of the generator e of $H^m(\boldsymbol{R}^m, \boldsymbol{R}^m\setminus 0)$ (induced by the orientation of $\boldsymbol{R}^m$) by the homomorphism

$$H^m(d)\colon H^m(\boldsymbol{R}^m, \boldsymbol{R}^m\setminus 0)\to H^m((X, A)\times(Y, B))\,.$$

2. Some lemmas. In this section we apply some techniques developed in [Sh] and [Wu] to prove the main Lemma (2.4). Let M be an m-dimensional oriented PL-manifold (with boundary or without boundary). Let P and Q be two connected oriented transversal PL-submanifolds of M with $\dim P+\dim Q=m$. If $a\in P\cap Q$ one can define the *index* $\varepsilon(a)=\pm 1$ *of intersection of submanifolds* P, Q in M at a (see [R–S]).

Suppose T is a closed subpolyhedron of M with dimension

$$\dim T\leqslant\dim Q\,.$$

Suppose that $a, b\in P\cap Q$, $\varepsilon(a)=-\varepsilon(b)$ and that there are two PL-arcs $\alpha\subseteq\mathrm{Int}P$, $\beta\subseteq\mathrm{Int}Q$ joining a and b, and such that $(\alpha\cup\beta)\subseteq\mathrm{Int}M\setminus T$ and $(\alpha\cap Q)=\beta\cap P=\{a, b\}$. Checking the proof of the Whitney Lemma in [R–S], one can obtain the following version of the Whitney Lemma which allows us to reduce the intersection points a, b.

(2.1) Lemma. *Suppose that $\dim P\geqslant 3$, $\dim Q\geqslant 2$ and $\pi_1(M\setminus P)=0$. Then there are an arbitrarily small neighborhood (a ball of dimension $\dim P$) B of the PL-arc α in P and a PL-imbedding $h\colon P\to M$ such that*

(i) *$h(x)=x$ for each $x\in P\setminus B$,*

(ii) *$h(B)\cap(T\cup Q\cup\mathrm{Bd}\,M)=\emptyset$.*

Now we consider two pairs (K, K_0), (L, L_0) of finite simplicial complexes with $\dim K=p$, $\dim L=q$, $p+q=m$, and two PL-mappings $f\colon |K|\to M$, $g\colon |L|\to M$, where M is an m-dimensional PL-manifold. We say that f, g are in *general position with respect to* (K, K_0) and (L, L_0) if the following conditions are satisfied:

(G1) $f(|K|)\cap g(|L_0|)=\emptyset=f(|K_0|\cap g(|L|)$,

(G2) $f^{-1}(\partial M)\subseteq|K_0|$ and $g^{-1}(\partial M)\subseteq|L_0|$,

(G3) $f(|K|)\cap g(S(g))=\emptyset=g(|L|)\cap f(S(f))$, (G4) $\dim(S(f)\setminus|K_0|)\leqslant\max(p-q,\,-1)$, $\dim(S(g)\setminus|L_0|)\leqslant\max(q-p,\,-1)$,

(G5) $f|\,|\sigma|$ and $g|\,|\tau|$ are PL-imbeddings and $f(|\partial\sigma|)\setminus\partial M\neq\emptyset$ for any simplex σ of $K\setminus K_0$ and for any simplex τ of $L\setminus L_0$,

(G6) $f(|\sigma|)$ and $g(|\tau|)$ are transversal for any simplex σ of $K\setminus K_0$ and any simplex τ of $L\setminus L_0$.

Here by $S(f)$ and $S(g)$ we denote the singular sets of f and g respectively, i.e. $S(f)=\mathrm{cl}\{x\in|K|\mid f^{-1}f(x)\neq x\}$ and $S(g)=\mathrm{cl}\{y\in|L|\mid g^{-1}g(y)\neq y\}$. Let us recall that (G6) implies that if $\dim\sigma+\dim\tau<m$ then $f(|\sigma|)\cap g(|\tau|)=\emptyset$. If $\dim\sigma=p$ and $\dim\tau=q$ then the intersection $f(|\sigma|)\cap g(|\tau|)$ consists of a finite number of points (contained in $\mathrm{Int}f(|\sigma|)\cap\mathrm{Int}g(|\tau|)\cap\mathrm{Int}M$) and for any point a of $f(|\sigma|)\cap g(|\tau|)$ there are neighborhoods U_1, U_2, U_3 of a in $f(|\sigma|)$, $g(|\tau|)$, M, respectively, such that the triple (U_1, U_2, U_3) is PL-homeomorphic to some neighborhood of the origin in the triple $(\boldsymbol{R}^p\times 0,\,0\times\boldsymbol{R}^q,\,\boldsymbol{R}^p\times\boldsymbol{R}^q)$.

Now let (K, K_0), (L, L_0) be pairs of finite simplicial complexes with $\dim K=p\geqslant 2$ and $\dim L=q\geqslant 2$. Suppose $f\colon |K|\to\boldsymbol{R}^m$ and $g\colon |L|\to\boldsymbol{R}^m$, $m=p+q$, are PL-mappings in general position with respect to (K, K_0), (L, L_0). There exist subdivisions (K', K_0') of (K, K_0) and (L', L_0') of (L, L_0) such that $f|\,|\sigma'|$ and $g|\,|\tau'|$ are linear for any simplexes σ' of K' and τ' of L'.

Now suppose σ is a p-simplex (respectively $(p-1)$-simplex) of $K\setminus K_0$ and τ is a $(q-1)$-simplex (respectively q-simplex) of $L\setminus L_0$. We take a simplex σ' of K' with $|\sigma'|\subseteq|\sigma|$ and $\dim\sigma'=\dim\sigma$, and a simplex τ' of L' with $|\tau'|\subseteq|\tau|$ and $\dim\tau'=\dim\tau$. There exist points $x\in\mathrm{Int}|\sigma'|\setminus S(f)$ and $y\in\mathrm{Int}|\tau'|\setminus S(g)$, and such that $f(x)\notin g(|L|)$ and $g(y)\notin f(|K|)$. By the General Position Theorem there is a PL-arc α in $\boldsymbol{R}^m$ arbitrarily close to the straight line segment $\overline{f(x), g(y)}$ such that $\alpha\cap(f(|K|)\cup g(|L|))=\{f(x), g(y)\}$. In a similar way as in [Sh] or in [Wu] (by the tube construction near α, see [R–S], p. 67) one can prove the following lemma (compare Lemma (7.3) in [Sh] or [Wu], p. 234).

(2.2) Lemma. *For any neighborhood U of $\overline{f(x), g(y)}$ in $\boldsymbol{R}^m$ and any $\lambda=\pm1$ there is a PL-mapping $f'\colon |K|\to\boldsymbol{R}^m$ (respectively $g'\colon |L|\to\boldsymbol{R}^m$) such that*

(a) *f' and g (resp. f and g') are in general position with respect to (K, K_0), (L, L_0),*

(b) *f' differs from f (resp. g' differs from g) only on an arbitrarily small ball B which is a closed neighborhood of x in $\mathrm{Int}|\sigma'|$ (resp. of y in $\mathrm{Int}|\tau'|$),*

(c) *$c(f', g)$ (resp. $c(f, g')$) is equal to $c(f, g)+\lambda\delta\chi_{\sigma,\tau}$.*

Here for any oriented simplexes σ of $K\setminus K_0$ and τ of $L\setminus L_0$ the cocycle $\chi_{\sigma,\tau}$ is given by the formulas:

$$\chi_{\sigma,\tau}(\sigma\times\tau)=1$$

and

$$\chi_{\sigma,\tau}(\hat{\sigma}\times\hat{\tau})=0\quad\text{if}\quad|\hat{\sigma}|\neq|\sigma|\ \text{or}\ |\hat{\tau}|\neq|\tau|\,.$$

(2.3) Remark. Suppose additionally $f(|K|)\subseteq\boldsymbol{I}^m$, $g(|L|)\subseteq\boldsymbol{I}^m$ and $f^{-1}(\partial\boldsymbol{I}^m)\subseteq|K_0|$, $g^{-1}(\partial\boldsymbol{I}^m)\subseteq|L_0|$. Then the segment $\overline{f(x), g(y)}$ is contained in $\mathrm{Int}\boldsymbol{I}^m$, so we may take U in $\mathrm{Int}\boldsymbol{I}^m$. Thus in Lemma (2.2) we may then additionally require that the image of f' (resp. g') is contained in $\boldsymbol{I}^m$, and $(f')^{-1}(\partial\boldsymbol{I}^m)\subseteq|K_0|$ (resp. $(g')^{-1}(\partial\boldsymbol{I}^m)\subseteq|L_0|$).

Now we are ready to prove the main lemma of this section.

(2.4) Lemma. *Let (K, K_0), (L, L_0) be pairs of finite simplicial complexes with $\dim K=p\geqslant 3$, $\dim L=q\geqslant 2$. Suppose $f\colon |K|\to\boldsymbol{I}^m$ and $g\colon |L|\to\boldsymbol{I}^m$ are PL-mappings in general position with respect to (K, K_0), (L, L_0), and $m=p+q$. If $[c(f, g)]=0$ then there are PL-mappings $f'\colon |K|\to\boldsymbol{I}^m$, $g'\colon |L|\to\boldsymbol{I}^m$ with disjoint images and such that*

(a) *$f'\,|\,|K_0|=f\,|\,|K_0|$ and $g'\,|\,|L_0|=g\,|\,|L_0|$,*

(b) *$(f')^{-1}(\partial\boldsymbol{I}^m)\subseteq|K_0|$ and $(g')^{-1}(\partial\boldsymbol{I}^m)\subseteq|L_0|$.*

Proof. Since $[c(f, g)]=0$, there is an $(m-1)$-dimensional cochain z in $(K, K_0)\times(L, L_0)$ with $\delta z=c(f, g)$. Then

$$z=\sum_1^r\lambda_i\chi_{\sigma_i,\tau_i}$$

where $\lambda_i=\pm1$ and σ_i, τ_i are simplexes of $K\setminus K_0$ and $L\setminus L_0$, respectively, with $\dim\sigma_i+\dim\tau_i=p+q-1$. Thus

$$c(f, g)=\sum_1^r\lambda_i\delta\chi_{\sigma_i,\tau_i}\,.$$

By applying Lemma (2.2) r times (see also Remark (2.3)) one can obtain two PL-mappings $f_r\colon |K|\to\boldsymbol{I}^m$, $g_r\colon |L|\to\boldsymbol{I}^m$ in general position with respect to (K, K_0), (L, L_0) such that

$$f_r\,|\,|K_0|=f\,|\,|K_0|\,,\quad g_r\,|\,|L_0|=g\,|\,|L_0|\quad\text{and}\quad c(f_r, g_r)=0\,.$$

We know that $f_r(|\sigma|)\cap g_r(|\tau|)=\emptyset$ for any two simplexes σ of K and τ of L with $\dim\sigma+\dim\tau<m$. Let us consider now a p-simplex σ of $K\setminus K_0$ and a q-simplex τ of $L\setminus L_0$. Then $P=f_r(|\sigma|)$ and $Q=g_r(|\tau|)$ are transversal oriented PL-submanifolds of $\boldsymbol{I}^m$ with dimensions p and q, respectively, and $\pi_1(\boldsymbol{I}^m\setminus P)=0$. The intersection $P\cap Q$ consists of a finite number of points contained in $\mathrm{Int}P\cap\mathrm{Int}Q\cap\boldsymbol{I}^m$. The equality $c(f_r, g_r)=0$ implies $f_r(\sigma)\wedge g_r(\tau)=0$. Thus $P\cap Q$ has an even number of points, say

$$P\cap Q=\{a_1, \ldots, a_s, b_1, \ldots, b_s\}$$

with $\varepsilon(a_i)=-\varepsilon(b_i)$ for each $i=1, \ldots, s$. By (G3) the set $P\cap Q$ is disjoint from the set $W=f_r(|K|\setminus\mathrm{Int}|\sigma|)\cup g_r(|L|\setminus\mathrm{Int}|\tau|)$. By (G4) we obtain

$$\dim\big(f_r(|K|\setminus\mathrm{Int}|\sigma|)\cap\mathrm{Int}P\big)\leqslant\dim\big(S(f_r)\setminus|K_0|\big)\leqslant\max(p-q,\,-1)\,.$$

By (G6) and (G1) it follows that the intersection of $g_r(|L|\setminus\mathrm{Int}|\tau|)$ and $\mathrm{Int}P$ is a finite set. Thus

$$\dim(W\cap\mathrm{Int}P)\leqslant\max(p-q,\,0)\leqslant p-2\,.$$

Similarly,

$$\dim(W \cap \operatorname{Int} Q) \leqslant \max(q-p, 0) \leqslant q-3 .$$

Thus by the General Position Theorem there exist a family $\alpha_1, \ldots, \alpha_s$ of pairwise disjoint arcs in $\operatorname{Int} P \setminus W$ and a family $\beta_1, \ldots, \beta_s$ of pairwise disjoint arcs in $\operatorname{Int} Q \setminus W$ such that α_i and β_i have the end points a_i, b_i and $\alpha_i \cap Q = \beta_i \cap P = \{a_i, b_i\}$ for each $i = 1, \ldots, s$. Since $\operatorname{Int} P \cap \operatorname{Int} Q \subseteq \operatorname{Int} I^m$ it follows that $(\alpha_i \cup \beta_i)$ is contained in $\operatorname{Int} I^m \setminus W$ for each i.

Now by Lemma (2.1) there exist a family $B_1, \ldots, B_s$ of pairwise disjoint p-dimensional balls in $\operatorname{Int} P \setminus W$ and a PL-imbedding $h\colon P \to I^m$ differing only from the inclusion of P into I^m on the set $B = B_1 \cup \ldots \cup B_s$ and such that $h(B)$ is disjoint from $g_r(|L|) \cup \partial I^m$ (take $T = g_r(|L| \setminus \operatorname{Int}|\tau|)$).

We define a map $f_r'\colon |K| \to I^m$ by the formulas

$$f_r'(x) = f_r(x) \quad \text{if} \quad x \notin |K| \setminus \operatorname{Int}|\sigma|$$

and

$$f_r'(x) = h \circ f_r(x) \quad \text{if} \quad x \in |\sigma| .$$

It follows that $f_r'(|\sigma|)$ and $g_r(|\tau|)$ are disjoint. Let us also observe that f_r' and g_r are in general position with respect to (K, K_0), (L, L_0) and $f_r' \mid |K_0| = f_r \mid |K_0|$ and $c(f_r', g_r) = 0$. By applying the same procedure to other pairs σ' and τ', where σ' is a p-simplex of $K \setminus K_0$ and τ' is a q-simplex of $L \setminus L_0$, we can get finally a map $f'\colon |K| \to I^m$ such that f' and g_r have disjoint images, f' and g_r are in general position with respect to (K, K_0), (L, L_0), and $f' \mid |K_0| = f_r \mid |K_0|$. Then f' and $g' = g_r$ are the desired maps.

3. Transversely trivial maps. The proof of the following lemma is standard.

(3.1) Lemma. *Let $f\colon (X, A) \to (I^p, \partial I^p)$, $g\colon (Y, B) \to (I^q, \partial I^q)$ be mappings of pairs of compacta such that the homomorphism $H^m(f \times g)$ of the relative cohomology groups is trivial, $m = p+q$. Then there are two pairs of simplicial complexes (K, K_0), (L, L_0) with $\dim K \leqslant \dim X$, $\dim L \leqslant \dim Y$ and there are factorizations*

(i) $(X, A) \xrightarrow{\varphi} (|K|, |K_0|) \xrightarrow{f'} (I^p, \partial I^p)$,

(ii) $(Y, B) \xrightarrow{\psi} (|L|, |L_0|) \xrightarrow{g'} (I^q, \partial I^q)$

such that $f' \circ \varphi$ and $g' \circ \psi$ are homotopic (as maps of pairs) to f and g, respectively, and the homomorphism $H^m(f' \times g')$ of the relative cohomology groups is trivial.

Now we are ready to prove the following theorem.

(3.2) Theorem. *Let X, Y be compacta with $\dim X = p$ and $\dim Y = q$. Let $f\colon X \to I^p$ and $g\colon Y \to I^q$ be maps such that $f \times g$ is inessential. If $p \geqslant 2$, $q \geqslant 2$ then f, g are transversely trivial.*

Proof. Let $A = f^{-1}(\partial I^p)$ and $B = g^{-1}(\partial I^q)$. Since $f \times g\colon X \times Y \to I^p \times I^q$ is inessential, the map

$$f \times g\colon (X, A) \times (Y, B) \to (I^p, \partial I^p) \times (I^q, \partial I^q)$$

induces the trivial homomorphism $H^m(f \times g)$ of the relative cohomology groups, where $m = p+q$. Now we consider factorizations (i) and (ii) satisfying the conditions of Lemma (3.1). Recall that $\dim K \leqslant p$ and $\dim L \leqslant q$. We may additionally assume that f', g' are linear on each simplex of K and L, respectively, and that $0 \notin f'(|\sigma|)$ for any $(p-1)$-simplex σ of K and that $0 \notin g'(|\tau|)$ for any $(q-1)$-simplex τ of L.

By $i_1\colon I^p \to I^p \times I^q$ and $i_2\colon I^q \to I^p \times I^q$ we denote the inclusions defined by $i_1(x) = (x, 0)$ and $i_2(y) = (0, y)$. Let us observe that for any $\eta > 0$ there are maps

$$f''\colon |K| \to I^m \quad \text{and} \quad g''\colon |L| \to I^m$$

such that

(a) f'' and g'' are in general position with respect to $(|K|, |K_0|)$, $(|L|, |L_0|)$, additionally they are linear on each simplex of K and L, respectively, and $(f'')^{-1}(\partial I^m) = |K_0|$, $(g'')^{-1}(\partial I^m) = |L_0|$,

(b) f'' is η-close to $i_1 \circ f'$ and g'' is η-close to $i_2 \circ g'$.

Let $|\sigma_1|, \ldots, |\sigma_k|$ be all p-simplexes of K with the images by f' covering the point 0 of I^p and let $|\tau_1|, \ldots, |\tau_l|$ be all q-simplexes of L with the images by g' covering the point 0 of I^q. Let us assume that the orientation of each σ_i is coherent (by the map f') with the orientation on I^p, and that the orientation on each τ_j is coherent (by the map g') with the orientation on I^q. If η is sufficiently small then $f''(|\sigma_i|)$ and $g''(|\tau_j|)$ meet transversely and

$$f''(\sigma_i) \wedge g''(\tau_j) = i_1 \circ f'(\sigma_i) \wedge i_2 \circ g'(\tau_j) = 1$$

for each i and j, and also $f''(|\sigma|) \cap g''(|\tau|) = \emptyset$ for any two simplexes σ of K and τ of L such that $|\sigma| \notin \{|\sigma_1|, \ldots, |\sigma_k|\}$ or $|\tau| \notin \{|\tau_1|, \ldots, |\tau_l|\}$. Thus the cocycle of intersection $c(f'', g'')$ is given by

$$c(f'', g'')(\sigma \times \tau) = 1 \quad \text{if} \quad \sigma \in \{\sigma_1, \ldots, \sigma_k\} \text{ and } \tau \in \{\tau_1, \ldots, \tau_l\},$$
$$c(f'', g'')(\sigma \times \tau) = 0 \quad \text{if} \quad |\sigma| \notin \{|\sigma_1|, \ldots, |\sigma_k|\} \text{ or } |\tau| \notin \{|\tau_1|, \ldots, |\tau_l|\}$$

and $c(i_1 \circ f', i_2 \circ g') = c(f'', g'')$.

On the other hand (by the Künneth Formula), we have

$$H^m(f' \times g')(e) = H^p(f')(e_1) \otimes H^q(g')(e_2)$$

where e_1, e_2 and e are the generators of $H^p(I^p, \partial I^p)$, $H^q(I^q, \partial I^q)$ and $H^m(I^m, \partial I^m)$, respectively, induced by the orientations of I^p, I^q and $I^m = I^p \times I^q$ (here we identify the groups $H^p(|K|, |K_0|) \otimes H^q(|L|, |L_0|)$ and $H^m((|K|, |K_0|) \times (|L|, |L_0|))$ by the natural isomorphism). Observe that

$$H^p(f')(e_1) = [\sum_1^k \chi_i^p],$$

$$H^q(g')(e_2) = [\sum_1^l \chi_j^q],$$

where χ_i^p is the p-cochain of (K, K_0) given by

$$\chi_i^p(\sigma_i) = 1 \quad \text{and} \quad \chi_i^p(\sigma) = 0 \quad \text{if } |\sigma| \neq |\sigma_i|$$

and χ_j^q is the q-cochain of (L, L_0) given by

$$\chi_j^q(\tau_j) = 1 \quad \text{and} \quad \chi_j^q(\tau) = 0 \quad \text{if } |\tau| \neq |\tau_j| .$$

Thus

$$H^m(f' \times g')(e) = \sum_1^k \sum_1^l ([\chi_i^p] \otimes [\chi_j^q]) = [c(f'', g'')] .$$

Since $H^m(f' \times g')$ is trivial, it follows that $[c(f'', g'')] = 0$.

Because f'' and g'' are in general position with respect to (K, K_0), (L, L_0) in I^m, thus by Lemma (2.4), there exist PL-mappings $\tilde{f}\colon |K| \to I^m$ and $\tilde{g}\colon |L| \to I^m$ with disjoint images and such that

$$\tilde{f} \mid |K_0| = f'' \mid |K_0| , \qquad \tilde{g} \mid |L_0| = g'' \mid |L_0|$$

and

$$(\tilde{f})^{-1}(\partial I^m) = |K_0| , \qquad (\tilde{g})^{-1}(\partial I^m) = |L_0| .$$

Now we are going to modify (in a standard way) the maps $\tilde{f} \circ \varphi$ and $\tilde{g} \circ \psi$ in order to get maps f^* and g^* with disjoint images and such that f^* coincides with $i_1 \circ f$ on A and g^* coincides with $i_2 \circ g$ on B. We may assume (it is enough to take $\eta < 1/2$) that

$$f''(|K_0|) \subseteq \partial I^p \times D^q \quad \text{and} \quad g''(|L_0|) \subseteq D^p \times \partial I^q ,$$

where $D^p = (-\frac{1}{2}, \frac{1}{2})^p$ and $D^q = (-\frac{1}{2}, \frac{1}{2})^q$.

Since

$$\tilde{f}(|K|) \cap (I^p \times \partial I^q) = \emptyset \quad \text{and} \quad \tilde{g}(|L|) \cap (\partial I^p \times I^q) = \emptyset ,$$

it follows that there are open neighborhoods V of ∂I^p in I^p and W of ∂I^q in I^q such that

$$\tilde{f}(|K|) \cap (I^p \times W) = \emptyset \quad \text{and} \quad \tilde{g}(|L|) \cap (V \times I^q) = \emptyset .$$

Observe that

$$\tilde{f} \circ \varphi(A) \subseteq f''(|K_0|) \subseteq V \times D^q ,$$

$$\tilde{g} \circ \psi(B) \subseteq g''(|L_0|) \subseteq D^p \times W .$$

There are closed neighborhoods A' of A in X and B' of B in Y such that

$$\tilde{f} \circ \varphi(A') \subseteq V \times D^q \quad \text{and} \quad \tilde{g} \circ \psi(B') \subseteq D^p \times W .$$

We may assume (we take η sufficiently small) that $\tilde{f} \circ \varphi|A$ is homotopic to $i_1 \circ f|A$ in $\partial I^p \times D^q$. By the Borsuk Homotopy Extension Theorem there exists a map $\hat{f}\colon A' \to V \times D^q$ such that

$$\hat{f}|A = i_1 \circ f|A \quad \text{and} \quad \hat{f}|\mathrm{Bdry}_X A' = \tilde{f} \circ \varphi|\mathrm{Bdry}_X A' .$$

Similarly, there exists a map $\hat{g}\colon B' \to D^p \times W$ such that

$$\hat{g}|B = i_2 \circ g|B \quad \text{and} \quad \hat{g}|\mathrm{Bdry}_Y B' = \tilde{g} \circ \psi|\mathrm{Bdry}_Y B' .$$

Now we define maps $f^*\colon X \to I^m$ and $g^*\colon Y \to I^m$ by the formulas:

$$f^*|A' = \hat{f}|A' \quad \text{and} \quad f^*|(X \setminus A') = \tilde{f} \circ \varphi|(X \setminus A') ,$$

$$g^*|B' = \hat{g}|B' \quad \text{and} \quad g^*|(Y \setminus B') = \tilde{g} \circ \psi|(Y \setminus B') .$$

Then f^* and g^* are well defined continuous maps such that $f^*|A = i_1 \circ f|A$ and $g^*|B = i_2 \circ g|B$. If we take V and W sufficiently small (i.e. such that the intersection of the sets $V \times D^q$ and $D^p \times W$ is empty) then f^* and g^* have disjoint images. Thus f and g are transversely trivial.

Let us note that it was proved in [Kr] (compare also [K–L]) that if $f\colon X \to I^p$, $g\colon Y \to I^q$ are transversely trivial then $f \times g\colon X \times Y \to I^p \times I^q$ is inessential.

4. Imbeddings into R^{2m}. Suppose X is an m-dimensional compactum with $\dim(X \times X) < 2m$, where $m \geqslant 3$. If $f\colon X \to I^m$, $g\colon X \to I^m$ are any continuous maps, then the map $f \times g\colon X \times X \to I^m \times I^m$ is inessential, thus by Theorem (3.2) f and g are transversely trivial. Now it follows by Theorem (2.2) [Kr] that $E(X, R^{2m})$ is dense in $C(X, R^{2m})$. Thus we have:

(4.1) Theorem. *If X is an m-dimensional compactum with $\dim(X \times X) < 2m$ and $m \geqslant 3$, then $E(X, R^{2m})$ is dense in $C(X, R^{2m})$.*

For the sake of completeness let us give a sketch of the proof of Theorem (4.1). In fact, we sketch the proof of a particular case of Lemma (2.1) in [Kr] and applying Theorem (3.2) we get Theorem (4.1).

It is enough to prove that the set of ε-mappings from X into R^{2m} is dense in $C(X, R^{2m})$. Let $f\colon X \to R^{2m}$ be any continuous map. It is well known that for any $\delta > 0$ there is an m-dimensional finite complex K, an ε-map $\varphi\colon X \to |K|$ and a map $g\colon |K| \to R^{2m}$ such that $g \circ \varphi$ is δ-close to f. We may also assume that g is linear on each simplex of K, the images by g of vertices of K are in general position in R^{2m} (so the simplexes $g(|\sigma|)$, $g(|\tau|)$ are transversal in R^{2m} for any two simplexes σ of K, τ of L with $|\sigma| \cap |\tau| = \emptyset$) and g has only double critical values (i.e. the set $g^{-1}g(x)$ contains at most two points for each $x \in |K|$).

The set $C = \{g(x) |\ x \in |K| \text{ and } g^{-1}g(x) \text{ is not a point}\}$ of critical values of g is finite, say $C = \{y_1, \ldots, y_s\}$. For each $i = 1, \ldots, s$, there are exactly two simplexes σ_i, τ_i of K such that the simplexes $s_i = g(|\sigma_i|)$, $t_i = g(|\tau_i|)$ intersect transversely at the point y_i. For each i, there exists an arbitrarily small closed neighborhood U_i of y_i in R^{2m} such that the triple $(U_i, U_i \cap s_i, U_i \cap t_i)$ is PL-homeomorphic to $(I^m \times I^m, I^m \times 0, 0 \times I^m)$ and such that $U_i \cap g(|K|) = U_i \cap (s_i \cup t_i)$. Let us consider $A_i = (g \circ \varphi)^{-1}(U_i \cap s_i)$ and $B_i = (g \circ \varphi)^{-1}(U_i \cap t_i)$. If U_i's are sufficiently small then $\mathrm{diam}(A_i) < \varepsilon$ and $\mathrm{diam}(B_i) < \varepsilon$ and U_i's are pairwise disjoint. Since $\dim(A_i \times B_i) < 2m$, thus by Theorem (3.2) the maps

$$g \circ \varphi|A_i\colon A_i \to U_i \cap s_i \quad \text{and} \quad g \circ \varphi|B_i\colon B_i \to U_i \cap t_i$$

are transversely trivial. So we can find a continuous map $f'\colon X\to R^{2m}$ differing from $g\circ\varphi$ on pairwise disjoint sets $A_1, \ldots, A_s, B_1, \ldots, B_s$ and such that

$$f'(A_i)\cup f'(B_i)\subseteq U_i \quad \text{and} \quad f'(A_i)\cap f'(B_i)=\emptyset .$$

Thus f' is an ε-mapping. Since U_i can be chosen arbitrarily small, f' can be chosen arbitrarily close to $g\circ\varphi$ and so to f.

Comments. Suppose M is an m-dimensional oriented PL-manifold (with a chosen orientation). Let (X, A), (Y, B) be pairs of compacta and let $f\colon X\to M$, $g\colon Y\to M$ be continuous maps with

$$f(X)\cap g(B)=\emptyset=f(A)\cap g(Y).$$

Then one can define a *cohomology class of intersection* $[c(f,g)]$ (an element of $H^m((X, A)\times(Y, B))$) of the maps f, g with respect to (X, A), (Y, B). We say that f, g are *transversely trivial with respect to* (X, A), (Y, B) if there are homotopies $F\colon X\times[0,1]\to M$ and $G\colon Y\times[0,1]\to M$ such that

$$F(X\times[0,1])\cap G(B\times[0,1])=\emptyset=F(A\times[0,1])\cap(Y\times[0,1]),$$
$$F(x,0)=f(x) \text{ for } x\notin X \quad \text{and} \quad G(y,0)=g(y) \text{ for } y\notin Y,$$
$$F\{X\times 1\}\cap G\{Y\times 1\}=\emptyset$$

(equivalently we may additionally assume that F and G are fixed on A and B, respectively).

Now suppose that $\pi_1(M)=0$, $\dim X\geqslant 3$, $\dim Y\geqslant 2$, $\dim X+\dim Y=m$, and $f(X)\subset P$, $g(Y)\subset Q$, where P, Q are polyhedra with $\dim P<\dim X$ and $\dim Q<\dim Y$. Then one can prove that $[c(f,g)]=0$ if and only if f, g are transversely trivial with respect to (X, A), (Y, B). This gives a generalization of Theorem (3.2).

I would like to thank J. Krasinkiewicz for calling my attention to the problem mentioned in the introduction.

Added in proof. Since this paper was accepted there has been essential progress of investigations initiated in [M–R1], [K–L], [M–R], [Kr] and in the present paper. The Corollary of the Introduction has been proved using a different argument by A. N. Dranishnikov and E. V. Shchepin in [D–S]. Theorem (3.2) is true without any assumption on p and q: the case $p=1$ is treated in [M–R1] and [K–L] and the remaining case $p=q=2$ was established in [Sp2] by applying methods of the present paper and a 4-dimensional version of Whitney's Lemma. Consequently the Corollary is true for any m. (Slightly earlier the case $m=2$ of the Corollary was proved in [Sp1] and announced in [D–S]; see also [D–R–S]).

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ul. Śniadeckich 8
00-950 Warsaw, Poland

Received 10 March 1988