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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

A fixed point index for bimaps

by

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Abstract. A bimap $\varphi\colon X\to X$ is a continuous multifunction on a topological space X for which the image of each point consists of either one or two points. Bimaps are the common special cases of $\{1, n\}$-valued multifunctions and of symmetric product maps. A fixed point index for bimaps on compact polyhedra is defined which has not only the properties of localization, additivity and homotopy invariance, but also the property that an isolated fixed point of index zero can be removed by an arbitrarily small deformation of the bimap. The definition and proofs use results about homotopy groups of symmetric products of spheres and about fix-finite approximations of bimaps and their homotopies.

1. Introduction. A bimap $\varphi\colon X\to Y$ from a topological space X to a topological space Y is a continuous (i.e. upper and lower semicontinuous) multifunction for which the image or each point consists ot either one or two points, and a bihomotopy is a bimap of the form $\Phi\colon X\times I\to[0, 1]$. The aim of this paper is the introduction of a fixed point index for a bimap $\varphi\colon X\to X$ which will be used to define a Nielsen number $N(\varphi)$ so that $N(\varphi)$ is for many spaces a sharp lower bound for the least number of fixed points in the bihomotopy class of φ [12].

Bimaps can be considered as the simplest multifunctions, and occur naturally e.g. as $\varphi\colon D\to D$, where $D=\{z\in C\mid |z|\leqslant 1\}$ is the disk in the complex plane and $\varphi(z)=\sqrt{z}$. Bimaps generalize single-valued maps, and belong to two classes of multifunctions for which a Nielsen number has been defined. The first class consists of the $\{1, n\}$-valued multifunctions first studied by B. O'Neil [7]. The second class consists of the multifunctions induced by symmetric product maps. We write X_n for the symmetric product space X^n/S_n which is defined as the orbit space of the n-fold Cartesian product X^n of a topological space X on which the symmetric group S_n acts by permuting its factors with the topology induced by the quotient map $q_n\colon X^n\to X_n$. A map (i.e. a single-valued continuous function) $f\colon X\to X_n$ is called a *symmetric product map*, and a point $x\in X$ is a *fixed point of the symmetric product*

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map f if $f(x) = q(z)$ implies that x is a coordinate of z. Fixed points of symmetric product maps were first studied by C.N. Maxwell [6]. If $n = 2$, then $f\colon X \to X_2$ defines a bimap $\varphi\colon X \to X$ by $\varphi = \pi \circ q^{-1} \circ f$, where $\pi\colon X^2 \to X$ is the multivalued projection $\pi((x_1, x_2)) = \{x_1, x_2\}$, and φ also defines f. We say that φ is *induced by* f, and that f is induced by φ. If the fixed point set of φ is defined, as usual, by $\mathrm{Fix}\,\varphi = \{x \in X \mid x \in \varphi(x)\}$, then $\mathrm{Fix}\,\varphi = \mathrm{Fix}\,f$, and hence the fixed point theories of symmetric product maps $f\colon X \to X_2$ and of bimaps $\varphi\colon X \to X$ are equivalent.

A fixed point index for $\{1, n\}$-valued multifunctions was introduced by Z. Dzedzej [2] and for symmetric product maps by S. Masih [5]. Hence two fixed point index theories for bimaps exist. But neither Dzedzej nor Masih prove one property which is crucial in Nielsen fixed point theory: they do not show that a fixed point of index zero can be removed. Both Dzedzej and Masih define their fixed point index with the help of homology theory, as is usually done in the classical case of single-valued maps. But the proof of the removability of an isolated fixed point of index zero for maps uses the Hurewicz isomorphism $H_n(S^n) \simeq \pi_n(S^n)$ to homotope the fixed point away, and it is not clear whether this proof can be adapted to either $\{1, n\}$-valued multifunctions or to symmetric product maps. Hence it is not clear whether the Nielsen number $N(f)$ for symmetric product maps defined by Masih is actually a sharp lower bound (and not only a lower bound) for the number of fixed points in the homotopy class of f.

As we plan to define a Nielsen number $N(\varphi)$ for a bimap φ and prove that it is a sharp lower bound for the number of fixed points in the bihomotopy class of φ, we introduce here a fixed point index for bimaps which permits the removal of fixed points of index zero. To do so, we use homotopy rather than homology groups, and define the index of an isolated fixed point in R^n with the help of the homomorphism $\pi_{n-1}(S^{n-1}) \to \pi_{n-1}(S_2^{n-1})$ induced by a bivector field near the fixed point. This definition is a natural extension of the geometric idea which underlies the definition of the fixed point index for single-valued maps. The fixed point index for bimaps is still an integer, and equals twice the usual index in the special case where the bimap is single-valued. The extension of the definition from bimaps of R^n to bimaps of compact polyhedra is modelled on [3], but we use methods from [8] and [10] to prove the properties of the index, and exploit both the algebraic properties of symmetric product maps and the geometric properties of bimaps in our arguments.

The paper is organized as follows. In § 2 we give some necessary background on $\pi_n(S_2^n)$. Next we study, in § 3, fix-finite bimaps and fix-finite bihomotopies, where a bimap $\varphi\colon X \to X$ is called *fix-finite* if its fixed point set is finite, and a bihomotopy $\Phi = \{\varphi_t\}\colon X \to X$ is called fix-finite if φ_t is a fix-finite bimap for all $0 \leqslant t \leqslant 1$. The results of § 3 are closely related to corresponding ones for fix-finite n-valued multifunctions and their fix-finite homotopies [9] (see also [11]), and some proofs are only sketched. These results are used in § 4, where the fixed point index for bimaps is defined. This is done first for isolated fixed points in R^n, then extended additively to finite fixed point sets in R^n, and finally to arbitrary fixed point sets on compact polyhedra with the help of retraction and fix-finite approximation. The fixed point index has the properties of localization, additivity and homotopy invariance (Theorems 4.5–4.7). Finally, in § 5, we prove the removability of an isolated fixed point of index zero on compact polyhedra (Theorem 5.3).

I wish to thank Professor Satya Deo of the University of Jammu, India, for his help with the proofs in § 2.

2. Some facts about homotopy groups of symmetric products on spheres. We gather here some results concerning $\pi_n(S_2^n)$ which are needed for the definition of the fixed point index and the proofs of some of its properties. It follows from [4], p. 539, (11.2) that S_2^{n+1} is homeomorphic to a space obtained from the suspension $S(S_2^n)$ by attaching a $(2n+2)$-cell. Hence a homomorphism

$$s_n'\colon \pi_n(S_2^n) \to \pi_{n+1}(S_2^{n+1})$$

is defined by sending $\alpha = [f] \in \pi_n(S_2^n)$ to $[i_n \circ S(f)]$, where $S(f)\colon S(S^n) \to S(S_2^n)$ is the suspension and $i_n\colon S(S_2^n) \to S_2^{n+1}$ the inclusion.

LEMMA 2.1. *$s_n'\colon \pi_n(S_2^n) \to \pi_{n+1}(S_2^{n+1})$ is an isomorphism for all $n \geqslant 1$.*

Proof. This follows immediately from the fact that $i_{k*}\colon \pi_k(S(S_2^n)) \to \pi_k(S_2^{n+1})$ is an isomorphism for all $k \leqslant 2n-2$ ([13], p. 402, Lemma 15).

We write $id_n\colon S^n \to S^n$ for the identity map, $c_n\colon S^n \to S^n$ for a constant map and $q_n\colon S^n \times S^n \to S_2^n$ for the quotient map. Hence a homomorphism $\eta_n\colon \pi_n(S^n) \to \pi_n(S_2^n)$ can be defined by $\eta_n = q_{n*} \circ (id_n \times c_n)_*$.

LEMMA 2.2. *$\eta_n\colon \pi_n(S^n) \to \pi_n(S_2^n)$ is an isomorphism for all $n \geqslant 1$ and $\eta_n = q_{n*} \circ (c_n \times id_n)_*$.*

Proof. As $q_n \circ (id_n \times c_n)(S^n) = \Gamma_2^n$ is a regular imbedding ([4], § 3) of S^n we see from [4], p. 528, (5.1) that $H^s(S_2^n, \Gamma_2^n; Z) = 0$ for $s \leqslant n+1$. Hence $H^n(S_2^n; Z) \simeq H^n(\Gamma_2^n; Z) \simeq Z$, and so [4], p. 522, (1.4) and the Hurewicz isomorphism theorem show that η_n is an isomorphism. The fact that $\eta_n = q_{n*} \circ (c_n \times id_n)_*$ follows from the commutativity of the diagram

$$\begin{tikzcd}[column sep=small] & S^n \times S^n \arrow[dr, "q_n"] & \\ S^n \arrow[ur, "id_n \times c_n"] \arrow[dr, "c_n \times id_n"'] & & S_2^n . \\ & S^n \times S^n \arrow[ur, "q_n"'] & \end{tikzcd}$$

We now choose a generator e_1 of $\pi_1(S^1)$ and let e_n be the generator of $\pi_n(S^n)$ given by $s_{n-1}(e_{n-1}) = e_n$, where $s_{n-1}\colon \pi_{n-1}(S^{n-1}) \to \pi_n(S^n)$ is the suspension isomorphism. If e_n' denotes the generator of $\pi_n(S_2^n)$ given by $\eta_n(e_n) = e_n'$, then clearly $s_{n-1}'(e_{n-1}') = e_n'$.

LEMMA 2.3. *If $h\colon S^n \to S_2^n$ is a symmetric product map which factors as $h = q_n \circ (g_1 \times g_2)$ and if $g_{k*}(e_n) = d_k e_n$ for $k = 1, 2$, then $h_*(e_n) = (d_1 + d_2) e_n'$,* Proof. We have

$$\begin{aligned} h_*(e_n) &= q_{n*}\circ(g_1\times g_2)_*(e_n) = q_{n*}(d_1 e_n, d_2 e_n)\\ &= d_1 q_{n*}(e_n, 0)+d_2 q_{n*}(0, e_n)\\ &= d_1[q_{n*}\circ(id_n\times c_n)_*(e_n)]+d_2[q_{n*}\circ(c_n\times id_n)_*(e_n)] = (d_1+d_2)e'_n\,. \end{aligned}$$

LEMMA 2.4. *Let $h_n\colon S^n\to S_2^n$ and $h_{n+1}\colon S^{n+1}\to S_2^{n+1}$ be two symmetric product maps such that the diagram*

$$\begin{tikzcd} S^{n+1} \arrow[rr, "h_{n+1}"] \arrow[dr, "S(h_n)"'] & & S_2^{n+1} \\ & S(S_2^n) \arrow[ur, "i_n"'] & \end{tikzcd}$$

is homotopy commutative. Then $h_{n*}(e_n) = de'_n$ if and only if $h_{n+1*}(e_{n+1}) = de'_{n+1}$.

Proof. This follows easily from $s_n(e_n) = e_{n+1}$ and $s'_n(e'_n) = e'_{n+1}$.

3. Fix-finite bimaps and fix-finite bihomotopies. It follows from [11], Theorem 1 that any bimap $\varphi\colon |K|\to|K|$ from a compact polyhedron to itself is homotopic to a bimap $\varphi'\colon |K|\to|K|$ which has a finite fixed point set. In order to define the fixed point index of φ and to establish its properties, in particular its homotopy invariance, we need a stronger form of this result which specifies the location of the fixed points (Theorem 3.4), and an extension to bihomotopies which say that two fix-finite and bihomotopic bimaps $\varphi_0, \varphi_1\colon |K|\to|K|$ with well located fixed points can be connected by a bihomotopy $\{\varphi_t\}\colon |K|\to|K|$ for which each φ_t is a fix-finite bimap with well located fixed points (Theorem 3.5). The main tool in the construction of fix-finite bimaps and bihomotopies is an extension of the Hopf construction for single-valued maps [1], pp. 117–118 to our setting (Lemma 3.3).

We use ϱ to denote the Hausdorff metric on the compact polyhedron $|K|$ with the barycentric metric, and $\bar{d}(\varphi_0, \varphi_1)$ for the distance between two bimaps $\varphi_0, \varphi_1\colon |K|\to|K|$ in the sup metric given by

$$\bar{d}(\varphi_0, \varphi_1) = \sup\{\varrho(\varphi_0(x), \varphi_1(x))|\ x\in|K|\}\,.$$

Two bimaps $\varphi_0, \varphi_1\colon A\to|K|$ are ε-homotopic if there exists a bihomotopy $\{\varphi_t\}\colon A\to|K|$ with $\bar{d}(\varphi_s, \varphi_t)<\varepsilon$ for all $s, t\in[0, 1]$. Other notation and terminology can be found in [9], § 3 and § 4 and [10], § 3 where corresponding results for n-valued multifunctions are developed.

LEMMA 3.1. *Let $|K|$ be a compact polyhedron and $A\subset|K|$ a closed subset of $|K|$. Given $\varepsilon>0$, there exists a $\delta>0$ such that any two bimaps $\varphi_0, \varphi_1\colon A\to|K|$ with $\bar{d}(\varphi_0, \varphi_1)<\delta$ are ε-homotopic.*

Proof. We write $f_0, f_1\colon A\to|K|$ for the symmetric product maps induced by φ_0 and φ_1, d_M for the metric on $|K|_2$ defined in [6], p. 808, and $\bar{d}'$ for the sup metric

$$\bar{d}'(f_0, f_1) = \sup\{d_M(f_0(x), f_1(x))|\ x\in A\}\,.$$

It follows from [1], p. 40 that given $\varepsilon>0$ there exists a $\delta'>0$ such that $\bar{d}'(f_0, f_1)<\delta'$ implies that $f_0, f_1\colon A\to|K|_2$ are ε-homotopic. Let $\{f_t\}\colon A\to|K|_2$ be such an ε-homotopy, and let $\delta = \delta'/\sqrt{2}$. It is an easy consequence of the definitions of ϱ and d_M that $\varrho(\varphi_0(x), \varphi_1(x))<\delta$ implies $d_M(f_0(x), f_1(x))<\delta'$ and that $d_M(f_s(x), f_t(x))<\varepsilon$ implies $\varrho(\varphi_s(x), \varphi_t(x))<\varepsilon$. Hence any two bimaps $\varphi_0, \varphi_1\colon A\to\to|K|$ with $d(\varphi_0, \varphi_1)<\delta$ induce two ε-homotopic symmetric product maps $f_0, f_1\colon A\to|K|_2$, and this ε-homotopy in return defines an ε-homotopy from φ_0 to φ_1.

As in [9] and [11] we say that a bimap $\varphi\colon |K'|\to|K''|$ is *simplicial* if, for every closed simplex $|\bar{\sigma}|$ of $|K'|$, the restriction $\varphi|\,|\bar{\sigma}|$ splits into two (not necessarily distinct) maps f_1, f_2 so that each f_k maps $|\bar{\sigma}|$ affinely onto a simplex $|\bar{\tau}_k|$ of $|K''|$.

LEMMA 3.2. *Let $|K|$ be a compact polyhedron and $\varphi\colon |K|\to|K|$ be a bimap. Given $\varepsilon>0$, there exist subdivisions $|K'|$, $|K''|$ of $|K|$ and a simplicial bimap $\varphi\colon |K'|\to|K''|$ with $\bar{d}(\varphi, \varphi')<\varepsilon$.*

Proof. The fact that $d_M(f_0(x), f_1(x))<\varepsilon$ implies $\varrho(\varphi_0, \varphi_1)<\varepsilon$ if φ_0, φ_1 are the bimaps induced by the symmetric product maps f_0, f_1 shows that the proof of [11], Lemma 1, case $n = 2$ can be used after $|K|$ is replaced by a suitably fine subdivision $|K''|$.

LEMMA 3.3 (Hopf construction). *Let K' be a refinement of a finite simplicial complex K, let $\varphi\colon |K'|\to|K|$ be a simplicial bimap, and let σ be a p-dimensional non-maximal simplex of K' with $|\dot{\sigma}|\cap\operatorname{Fix}\varphi = \emptyset$ and $|\sigma|\cap\operatorname{Fix}\varphi\neq\emptyset$. Then there exists a simplicial bimap $\varphi'\colon |K'_L|\to|K|$, where $L = K'-st_{K'}\sigma$, such that*

(i) *all fixed points of $\varphi'|\,|K_L'^p|$ lie in $|L|$,*
(ii) *$\varphi = \varphi'$ on $|L|$,*
(iii) *$d(\varphi, \varphi')\leqslant 2\mu(K)$.*

Proof. As φ is simplicial, $\varphi||\bar{\sigma}| = \{f_1, f_2\}$ splits into two (not necessarily distinct) maps $f_k\colon |\bar{\sigma}|\to|\bar{\tau}_k|$. To obtain φ' on the vertices of K'_L, we distinguish three cases. $\sigma<\tau$ means that the simplex σ is a face of the simplex τ, and $v(\sigma)$ is the barycenter of σ.

Case 1. $\varphi||\sigma|$ is single-valued, i.e. $f_1 = f_2$ and hence $\tau_1 = \tau_2$. In this case we modify φ by making a Hopf construction analogous to the single-valued case [1], p. 117. More precisely, we define φ' on the vertices of K'_L as follows: if v is a vertex of L, we put $\varphi'(v) = \varphi(v)$. If $\sigma<\sigma_l$ but $\sigma\neq\sigma_l$, we let $\varphi'(v(\sigma_l))$ be any vertex of τ_1. Finally we let $\varphi'(v(\sigma))$ be any vertex of $\tau_1^*-\tau_1$, where τ_1^* is a maximal simplex of K_1 containing a maximal simplex σ^* of K' with $\sigma<\sigma^*$.

Case 2. $\varphi||\sigma|$ is $(1, 2)$-valued, i.e. neither single-valued nor 2-valued. As there exists a point $x\in|\sigma|$ with $f_1(x) = f_2(x)$, we have $\tau_1 = \tau_2$. We index the f_k so that f_1 has a fixed point on $|\sigma|$. Hence (as in [1], p. 117) we have $|\bar{\sigma}|\subset|\bar{\tau}_1|$, and as by assumption $f_2\colon |\bar{\sigma}|\to|\bar{\tau}_1|$ has no fixed point on $|\dot{\sigma}|$, it must have a fixed point on $|\sigma|$ also. We now determine φ' on the vertices of K'_L in the same way as in Case 1.

Case 3. $\varphi||\sigma|$ is 2-valued, i.e. $f_1(x)\neq f_2(x)$ for all $x\in|\sigma|$. Then $\tau_1\neq\tau_2$, as the distinct affine maps f_k map barycenters to barycenters. Again we index f_k so that f_1 has a fixed point on $|\sigma|$, and hence we can determine, as in [1], p. 117, maximal simplexes σ^* of K' and τ_1^* of K with $\sigma < \sigma^*$ and $\sigma^* \subset \tau_1^*$. Then we determine φ' on the vertices of K_L' as follows: if v is a vertex of L, we put $\varphi'(v) = \varphi(v)$. If $\sigma < \sigma_l$ but $\sigma \neq \sigma_l$ we let $\varphi'(v(\sigma_l))$ consist of the union of any vertex of τ_1 and any vertex of τ_2. Finally we let $\varphi'(v(\sigma))$ consist of the union of any vertex of $\tau_1^* - \tau_1$ and any vertex of τ_2.

We leave to the reader the laborious task of checking that φ' can be extended from the vertices of $|K_L'|$ to a simplicial bimap $\varphi'\colon |K_L'| \to |K|$ which satisfies the conditions (i), (ii) and (iii) of Lemma 3.3. The argument consists of a straightforward extension of the one used in the single-valued case.

As in the single-valued case, i.e. as in the proof of [1], Theorem 2, p. 118–119, Lemmas 3.1, 3.2 and 3.3 can be used to construct arbitrarily close fix-finite approximations with nicely located fixed points. More precisely we can obtain the following result, which sharpens [11], Theorem 1 (iii) for $n = 2$.

THEOREM 3.4 (Fix-finite approximation for bimaps). *Let X be a compact polyhedron and $\varphi\colon X \to X$ a bimap. Given $\varepsilon > 0$, there exists a bimap $\psi\colon X \to X$ so that*

(i) *ψ has finitely many fixed points,*

(ii) *there exists a triangulation of X so that each fixed point of ψ lies in a maximal simplex,*

(iii) *ψ is ε-homotopic to φ.*

A combination of the techniques used in the proofs of Theorem 3.4 and [8], Theorem 2 leads to the following extension of [10], Theorem 2.3 (case $n = 2$) which we need in the next section. The fairly long and technical proof is omitted. By a *hyperface* of a polyhedron $|K|$ we mean, as in [10], p. 209, an open simplex σ so that $\bar\sigma = \bar\sigma' \cap \bar\sigma''$, where σ' and σ'' are maximal simplexes of K.

THEOREM 3.5 (Fix-finite approximations for bihomotopies). *Let $|K|$ be a compact polyhedron, let $|K_1| \subseteq |K|$ be a subpolyhedron and let $\Phi\colon |K_1| \times I \to |K|$ be a bihomotopy so that φ_0 and φ_1 are fix-finite and have all their fixed points located in maximal simplexes of $|K_1|$. Given $\varepsilon > 0$, there exists a bihomotopy $\Phi'\colon |K_1| \times I \to |K|$ from $\varphi_0' = \varphi_0$ to $\varphi_1' = \varphi_1$ so that*

(i) *Φ' is fix-finite,*

(ii) *the fixed points of each φ_t' are located in maximal simplexes or in hyperfaces of $|K_1|$,*

(iii) Fix *Φ' is a 1-dimensional compact polyhydron in $|K_1| \times I$ so that no edge lies in a section $|K_1| \times \{t\}$ of $|K_1| \times I$,*

(iv) *$\bar d(\Phi, \Phi') < \varepsilon$.*

4. A fixed point index for bimaps. Our aim is to introduce a fixed point index for bimaps of compact polyhedra. We start by considering simple cases to which the general case will be reduced.

(a) *The fixed point index of an isolated fixed point in R^n* ($n \geqslant 2$). Let U be an open subset of R^n, let $\varphi\colon U \to R^n$ be a bimap and let $a \in U$ be an isolated fixed point of φ. Let $B_a^n \subset U$ be a closed ball centered at a with $\operatorname{Fix}\varphi \cap B_a^n = \{a\}$, and let S_a^{n-1}

be its boundary. If $x \in S_a^{n-1}$, then $\varphi(x) = \{y_1, y_2\}$ with $x \notin \{y_1, y_2\}$ defines two (not necessarily distinct) vectors $\vec v_k(x) = \overrightarrow{y_k x}$ (for $k = 1, 2$), and hence a bimap χ from S_a^{n-1} to the unit sphere S^{n-1} in R^n by

$$\chi(x) = \left\{\frac{\vec v_1(x)}{|\vec v_1(x)|}, \frac{\vec v_2(x)}{|\vec v_2(x)|}\right\}.$$

Let $h\colon S_a^{n-1} \to S_2^{n-1}$ be the symmetric product map induced by χ. We use an orientation preserving homeomorphism to identify $\pi_{n-1}(S^{n-1})$ with $\pi_{n-1}(S_a^{n-1})$ and choose generators e_{n-1} of $\pi_{n-1}(S_a^{n-1})$ and e_{n-1}' of $\pi_{n-1}(S_2^{n-1})$ as in § 2. Then $h_*(e_{n-1}) = de_{n-1}'$ defines an integer d, and we define the *fixed point index of φ at a* by $\operatorname{Ind}(\varphi, a) = d$. It is clearly independent of the choice of S_a^{n-1}. We can think of d as the *degree of the symmetric product map h*, and as the *degree of the bivector field χ*. If we write $\operatorname{ind}(f, a)$ for the index of the isolated fixed point a of the (single-valued) map $f\colon U \to R^n$ as defined e.g. in [3], p. 11, then the following examples are an immediate consequence of Lemma 2.3 and the definition of $\operatorname{ind}(f, a)$.

EXAMPLE 4.1. Let $\varphi = \{f_1, f_2\}\colon U \to R^n$ split into two (not necessarily distinct) maps. Then

$$\operatorname{Ind}(\varphi, a) = \operatorname{ind}(f_1, a) + \operatorname{ind}(f_2, a).$$

EXAMPLE 4.2. Let $\varphi = \{f, f\}\colon U \to R^n$ be single-valued. Then $\operatorname{Ind}(\varphi, a) = 2\operatorname{ind}(f, a)$.

The next result will be used in the proof of Theorem 5.3.

THEOREM 4.3 (Restriction). *Let $U \subset R^n$ and $\varphi\colon U \to R^n$ be such that $\varphi(U) \subset R^m$ with $m < n$. If $a \in U \cap R^m$ is an isolated fixed point of φ, then*

$$\operatorname{Ind}(\varphi, a) = \operatorname{Ind}(\varphi | U \cap R^m, a).$$

Proof. It is sufficient to consider the case $m = n-1$. In this case, Theorem 4.3 follows immediately from Lemma 2.4.

(b) *The index of a finite set of fixed points in R^n.* Let again U be open in R^n and let the bimap $\varphi\colon U \to R^n$ have a finite set of fixed points $a_1, a_2, \ldots, a_k$. We define the *fixed point index of φ on U* additively by

$$\operatorname{Ind}(\varphi, U) = \sum (\operatorname{Ind}(\varphi, a_j) |\ j = 1, 2, \ldots, k).$$

If Fix φ is contained in the interior of a ball B_c^n centered at a point c, then a map $h\colon \operatorname{Bd} B_c^n \to S_2^{n-1}$ (where Bd denotes the boundary) can be defined as in the definition of the index of an isolated fixed point, and the homomorphism

$$h_*\colon \pi_{n-1}(\operatorname{Bd} B_c^n) \to \pi_{n-1}(S_2^{n-1})$$

determines again an integer d. It is an immediate consequence of the definition of the index that

$$\operatorname{Ind}(\varphi, U) = d. \tag{4.4}$$

We will use this fact in the proof of the homotopy invariance of the index (Theorem 4.7). If $\mathrm{Fix}\varphi \cap U = \emptyset$, then we define $\mathrm{Ind}(\varphi, U) = 0$.

(c) *The fixed point index for compact polyhedra, fix-finite case.* Now let $\varphi\colon X \to X$ be a bimap of a compact polyhedron $X = |K|$, let a be an isolated fixed point of φ and let U be a neighborhood of a in X. We imbed X into R^N $(N \geq 2)$ with inclusion $i\colon X \to R^N$, and let W be a neighborhood of $i(X)$ in R^N which has a retraction $r\colon W \to X$. Then $i \circ \varphi \circ r\colon r^{-1}(U) \to R^N$ is a bimap with a as an isolated fixed point, and we define the *fixed point index of* φ *at* a by

$$\mathrm{Ind}(\varphi, a) = \mathrm{Ind}(i \circ \varphi \circ r, a).$$

If φ is fix-finite on the open subset $U \subset X$, then the *fixed point index of* φ *on* U is defined additively by

$$\mathrm{Ind}(\varphi, U) = \sum \big(\mathrm{Ind}(\varphi, a) |\ a \in \mathrm{Fix}\varphi\big).$$

It can be shown that the definition is independent of the choice of N, W, i and r, but this fact is not needed here. If $\varphi = f\colon U \to X$ is single-valued, then it follows from Example 4.2 that $\mathrm{Ind}(\varphi, U) = 2\,\mathrm{ind}(f, U)$, where $\mathrm{ind}(f, U)$ is the usual fixed point index as defined e.g. in [3], p. 14.

(d) *The fixed point index for compact polyhedra, general case.* We proceed in a way analogous to that in [10], § 4, but we can omit the smallness conditions caused by the gap of n-valued multifunctions. We say that a triple (X, φ, U) is *admissible* if $\varphi\colon X \to X$ is a bimap on a compact polyhedron X and U an open subset of X with $\mathrm{Fix}\varphi \cap \mathrm{Bd}\,U = \emptyset$. If $U \neq \emptyset$, let $\varepsilon = \inf\{\bar{d}(x, \varphi(x)) |\ x \in \mathrm{Bd}\,U\} > 0$ and use Theorem 3.4 to find a bimap $\varphi'\colon X \to X$ which is fix-finite, has all its fixed points located in maximal simplexes of a triangulation of X and is $\varepsilon/2$-homotopic to φ. Then $\bar{d}(x, \varphi'(x)) > 0$ for all $x \in \mathrm{Bd}\,U$, and hence $\mathrm{Ind}(\varphi', U)$ is defined. Therefore we can define for any admissible triple (X, φ, U) the *fixed point index of* φ *on* U by

$$\mathrm{Ind}(\varphi, U) = \mathrm{Ind}(\varphi', U).$$

A proof that this definition is independent of the choice of φ' can be obtained as in [10], § 4. Note that such a proof needs an extension of [10], Lemma 3.2 to bihomotopies which can easily be obtained by using Theorem 3.5 instead of [10], Theorem 2.3 and generalizations of [8], Propositions 2 and 3 to bimaps. The generalization of [8], Proposition 2 to bimaps is immediate from the definition of $\mathrm{Ind}(\varphi, a)$. To obtain [8], Proposition 3 for bimaps we can clearly assume that $|t - t_k|$ is so small that x and all x_k lie in the interior of a ball B_c^N and then use (4.4). Again we define $\mathrm{Ind}(\varphi, \emptyset) = 0$.

As in the case of n-valued multifunctions we cannot prove commutativity of the index, for the composite of two bimaps need not be a bimap. But we can establish the other usual properties of the index with methods similar to those in [10], § 4.

THEOREM 4.5 (Localization). *Let* (X, φ, U) *and* (X, ψ, U) *be admissible and* $\varphi(x) = \psi(x)$ *for all* $x \in \mathrm{Cl}\,U$. *Then* $\mathrm{Ind}(\varphi, U) = \mathrm{Ind}(\psi, U)$.

Proof. As in [10], proof of Theorem 4.3, with $\varepsilon' = \varepsilon''$.

THEOREM 4.6 (Additivity). *Let* (X, φ, U) *be admissible and* $U_1, U_2, \ldots, U_s$ *be mutually disjoint open subsets so that* φ *has no fixed points on*

$$\mathrm{Cl}\,U - \bigcup (U_j |\ j = 1, 2, \ldots, s).$$

Then

$$\mathrm{Ind}(\varphi, U) = \sum \big(\mathrm{Ind}(\varphi, U_j) |\ j = 1, 2, \ldots, s\big).$$

Proof. As in [10], proof of Theorem 4.4, omitting $\eta \leq \gamma(\varphi)$.

THEOREM 4.7 (Homotopy invariance). *Let* $\varphi\colon X \times I \to X$ *be a bihomotopy so that* (X, φ_t, U) *is admissible for all* $t \in I$. *Then* $\mathrm{Ind}(\varphi_0, U) = \mathrm{Ind}(\varphi_1, U)$.

Proof. Similar to [10], proof of Theorem 4.5. As in the proof of the independence of $\mathrm{Ind}(\varphi, U)$ from φ' which was sketched above an extension of [10], Lemma 3.2 to bihomotopies has to be used.

From Theorems 4.6 and 4.7 we obtain two corollaries in the standard way. (See [10], Corollaries 4.6 and 4.7, or [1], p. 53, Corollaries 1 and 2.)

COROLLARY 4.8. *If* (X, φ, U) *is admissible and* $\mathrm{Ind}(\varphi, U) \neq 0$, *then* φ *has a fixed point on* U.

COROLLARY 4.9. *If* $\varphi\colon X \to X$ *is a bimap and* $\mathrm{Ind}(\varphi, X) \neq 0$, *then every bimap* $\psi\colon X \to X$ *which is bihomotopic to* φ *has a fixed point.*

We can also use Theorem 4.7 to obtain extensions or Examples 4.1 and 4.2.

EXAMPLE 4.10. If $\varphi = \{f_1, f_2\}\colon X \to X$ splits into two maps, then $\mathrm{Ind}(\varphi, X) = \mathrm{ind}(f_1, X) + \mathrm{ind}(f_2, X)$. In particular, if $\varphi = \{f, f\}$ is single-valued, then $\mathrm{Ind}(\varphi, X) = 2\,\mathrm{ind}(f, X)$.

Proof. We use the single-valued Hopf construction to homotope f_k, for $k = 1, 2$, to a fix-finite map $g_k\colon X \to X$. If $\psi = \{g_1, g_2\}$, then clearly ψ is bihomotopic to φ, and hence by Theorem 4.7 $\mathrm{Ind}(\varphi, X) = \mathrm{Ind}(\psi, X)$. But

$$i \circ \psi \circ r = \{i \circ g_1 \circ r, i \circ g_2 \circ r\}$$

in the definition of $\mathrm{Ind}(\varphi, X)$ part (c), and so Example 4.1 shows that

$$\begin{aligned}\mathrm{Ind}(\varphi, X) &= \sum \{\mathrm{Ind}(\psi, a) |\ a \in \mathrm{Fix}\psi\} = \sum \{\mathrm{Ind}(i \circ \psi \circ r, a) |\ a \in \mathrm{Fix}\psi\}\\ &= \sum \{\mathrm{ind}(i \circ g_1 \circ r, a) + \mathrm{ind}(i \circ g_2 \circ r, a) |\ a \in \mathrm{Fix}\psi\}\\ &= \mathrm{ind}(g_1, X) + \mathrm{ind}(g_2, X) = \mathrm{ind}(f_1, X) + \mathrm{ind}(f_2, X).\end{aligned}$$

5. Removability of fixed points of index zero. The removal of an isolated fixed point of index zero is crucial in Nielsen fixed point theory, and motivates our method for defining a fixed point index for bimaps. To show removability, we first prove a lemma which shows that it is only necessary to deal with fixed points at which the bimap is 2-valued, so that the Splitting Lemma [9], Lemma 1 and removal in the single-valued case can be used. The lemma is stated in a more general form than is needed here, as it will be used again in [12] to unite two fixed points in the same fixed point class.

LEMMA 5.1. *Let σ be a maximal simplex of a compact polyhedron $|K|$ and $a \in \sigma$ an isolated fixed point of a bimap $\varphi\colon |K| \to |K|$ which is single-valued at a. Then there exist, for any integers d_1 and d_2 with $d_1 + d_2 = \operatorname{Ind}(\varphi, a)$, a neighborhood U of a with $\operatorname{Fix} \cap \operatorname{Cl} U = \{a\}$ and a bimap $\psi\colon |K| \to |K|$ which is homotopic to φ relative $|K| - U$ so that $\psi|\operatorname{Cl} U = \{f_1, f_2\}$ splits into two maps, $\operatorname{Fix} f_k = \{a_k\}$ is an arbitrary point of U, and $\operatorname{ind}(f_k, a_k) = d_k$ for $k = 1, 2$.*

Proof. Let σ be n-dimensional. We consider σ as a subset of R^n, write $B_r = \{x \in R^n \mid |x - a| \leqslant r\}$ and choose $0 < \delta \leqslant \varepsilon$ so that $\varphi(B_\delta) \subset B_\varepsilon \subset B_{3\varepsilon} \subset \sigma$ and $\operatorname{Fix} \varphi \cap B_\delta = \{a\}$. If $Y = \{x \in R^n \mid 0 < |x| \leqslant 2\varepsilon\}$ is a punctured n-ball, then a bimap $\chi_0\colon \operatorname{Bd} B_\delta \to Y$ can be defined by $\chi_0(x) = \{x - y_1, x - y_2\}$, where $\varphi(x) = \{y_1, y_2\}$. Let $g_k\colon \operatorname{Bd} B_\delta \to \operatorname{Bd} B_\varepsilon$ be two maps of degree d_k, let $i\colon \operatorname{Bd} B_\varepsilon \to Y$ be the inclusion, and let $\chi_1\colon \operatorname{Bd} B_\delta \to Y$ be the bimap given by $\chi_1 = \{i \circ g_1, i \circ g_2\}$. It follows from Lemmas 2.1 and 2.3, Theorem 4.3 and the fact that Y is of the same homotopy type as S^{n-1} and hence $\pi_{n-1}(Y_2) \simeq Z$ [4], (1.2) that χ_0 and χ_1 induce symmetric product maps which are homotopic, and thus χ_0 and χ_1 are bihomotopic. Let $\chi_t\colon \operatorname{Bd} B_\delta \to Y$ be this bihomotopy.

We now define a bimap $\varphi_1\colon B_\delta \to B_{3\varepsilon}$ as follows. If $x_s \in \operatorname{Cl}(B_\delta - B_{\delta/2})$ is the point on the segment from a to $x_1 \in \operatorname{Bd} B_\delta$ with $|x_s - a| = s\delta$ for $1/2 \leqslant s \leqslant 1$, we put $\varphi_1(x_s) = \{x_s - z_1, x_s - z_2\}$, where $\chi_{2-2s}(x) = \{z_1, z_2\}$. Hence $\varphi_1|\operatorname{Bd} B_{\delta/2}$ splits into two maps $\{f_1', f_2'\}$, say, and by coning $f_k'\colon \operatorname{Bd} B_{\delta/2} \to B_{3\varepsilon}$ from $a_k \in \operatorname{Int} B_{\delta/2}$ we obtain two maps $f_k\colon B_{\delta/2} \to B_{3\varepsilon}$ with fixed point a_k. We extend φ_1 over $B_{\delta/2}$ as $\varphi_1 = \{f_1, f_2\}$. Then the bimap $\psi\colon |K| \to |K|$ given by

$$\psi(x) = \begin{cases} \varphi_1(x) & \text{if } x \in B_\delta, \\ \varphi(x) & \text{if } x \in |K| - B_\delta \end{cases}$$

and the neighborhood $U = \operatorname{Int} B_{2/\delta}$ have the desired properties if we can show that φ is homotopic to ψ relative $|K| - B_{\delta/2}$.

To construct a suitable homotopy, let

$$\Psi'\colon (B_\delta \times 0) \cup (B_\delta \times 1) \cup (\operatorname{Bd} B_\delta \times I) \to B_{3\varepsilon}$$

be defined by

$$\Psi'(x, t) = \begin{cases} \varphi(x) & \text{if } (x, t) \in (B_\delta \times 0) \cup (\operatorname{Bd} B_\delta \times I), \\ \varphi_1(x) & \text{if } (x, t) \in B_\delta \times 1. \end{cases}$$

Then Ψ' induces a symmetric product map

$$H'\colon (B_\delta \times 0) \cup (B_\delta \times 1) \cup (\operatorname{Bd} B_\delta \times I) \to (B_{3\varepsilon})_2.$$

As $(B_{3\varepsilon})_2$ is contractible [4], (1.3), H' extends to a symmetric product map $H\colon B_\delta \times I \to (B_{3\varepsilon})_2$, and H induces a bihomotopy $\Psi\colon B_\delta \times I \to B_{3\varepsilon}$ from $\varphi|B_\delta$ to φ_1 relative to $\operatorname{Bd} B_\delta$. Thus a bihomotopy Φ from φ to ψ relative to $|K| - U$ can be obtained by

$$\Phi(x, t) = \begin{cases} \Psi(x, t) & \text{if } (x, t) \in B_\delta \times I, \\ \varphi(x) & \text{otherwise}. \end{cases}$$

We state a corollary which strengthens Theorem 3.4 and will be used in [12]. Its proof is immediate from Theorem 3.4 and Lemma 5.1 with $a_1 \neq a_2$.

COROLLARY 5.2. *Let X be a compact polyhedron and $\varphi\colon X \to X$ a bimap. Then φ is homotopic to a bimap $\psi\colon X \to X$ so that*

(i) *ψ has finitely many fixed points,*

(ii) *there exists a triangulation of X so that each fixed point of ψ lies in a maximal simplex,*

(iii) *ψ is 2-valued at each of its fixed points.*

Finally we prove removability.

THEOREM 5.3 (Removability). *Let $|\sigma|$ be a maximal simplex of a compact polyhedron $|K|$ and a an isolated fixed point of index zero of the bimap $\varphi\colon |K| \to |K|$. Then there exist a neighborhood U of a in $|\sigma|$ with $\operatorname{Fix}\varphi \cap \operatorname{Cl} U = \{a\}$ and a bimap $\psi\colon |K| \to |K|$ homotopic to φ relative $|K| - U$ which is fixed point free on $\operatorname{Cl} U$.*

Proof. We can assume that φ is 2-valued at a, for we can otherwise use Lemma 5.1 with $d_1 = d_2 = 0$ to homotope φ relative $|K| - V$ for some $V \subset |\sigma|$ to a bimap which has two isolated fixed points of index zero on V. If φ is 2-valued at a, then it follows from the fact that the set of points at which φ is 2-valued is open that we can select a Euclidean neighborhood U of a with $\operatorname{Cl} U \subset |\sigma|$ so that φ is 2-valued on $\operatorname{Cl} U$ and $\operatorname{Fix}\varphi \cap \operatorname{Cl} U = \{a\}$. Thus $\varphi|\operatorname{Cl} U = \{g_1, g_2\}$ splits into two maps $g_k\colon \operatorname{Cl} U \to |K|$ ([8], Lemma 1) and we index the g_k so that $g_1(a) = a$ and g_2 is fixed point free. By definition of the fixed point index and Example 4.1 we have

$$\begin{aligned} \operatorname{Ind}(\varphi, a) &= \operatorname{Ind}(i \circ \varphi \circ r, r^{-1}(U)) \\ &= \operatorname{ind}(i \circ g_1 \circ r, r^{-1}(U)) + \operatorname{ind}(i \circ g_2 \circ r, r^{-1}(U)), \end{aligned}$$

and so $\operatorname{Ind}(\varphi, a) = 0$ implies $\operatorname{ind}(i \circ g_1 \circ r, r^{-1}(U)) = \operatorname{ind}(g_1, a) = 0$.

We can now use standard methods to homotope g_1 relative $\operatorname{Bd} U$ to a fixed point free map (see e.g. [1], p. 123 Theorem 4, or [3], p. 12), and thus obtain a bimap which satisfies Theorem 5.3.

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Imbeddings in R^{2m} of m-dimensional compacta with $\dim(X\times X)<2m$

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Abstract. In this note we give a sufficient condition for two maps from compacta into balls to be transversely trivial. As a corollary we conclude that any m-dimensional compactum X with $\dim(X\times X)<2m$ admits a dense set of imbeddings into R^{2m} provided $m\geq 3$.

Introduction. Recall that a mapping f from a space X into the p-dimensional cube I^p, $I=[-1,1]$, is said to be *inessential* (in the sense of Alexandrov–Hopf) if there exists a mapping $g\colon X\to\partial I^p$ (∂I^p denotes the boundary of I^p) such that $g(x)=f(x)$ for each $x\in f^{-1}(\partial I^p)$. Two maps $f\colon X\to I^p$ and $g\colon Y\to I^q$ are said to be *transversely trivial* (see [Kr], compare also [K–L], Problem (2)) if there exist two mappings $F\colon X\to I^p\times I^q$ and $G\colon Y\to I^p\times I^q$ satisfying the following conditions:

(i) $F|f^{-1}(\partial I^p)=(f,0)|f^{-1}(\partial I^p)$,

(ii) $G|g^{-1}(\partial I^q)=(0,g)|g^{-1}(\partial I^q)$,

(iii) $F(X)\cap G(Y)=\emptyset$.

In this note we prove the following:

THEOREM. *Let X and Y be compacta with $\dim(X)=p$ and $\dim(Y)=q$. Suppose $f\colon X\to I^p$ and $g\colon Y\to I^q$ are mappings such that $f\times g\colon X\times Y\to I^p\times I^q$ is inessential. If $p\geq 3$, $q\geq 2$ then f, g are transversely trivial.*

This result gives a positive partial answer to Problem (2) in [K–L].

In [M–R] D. McCullough and L. R. Rubin proved the following interesting result: for each $m\geq 2$ there exists an m-dimensional continuum X such that the space $E(X,R^{2m})$ of imbeddings from X into R^{2m} is dense in the space $C(X,R^{2m})$ of continuous maps from X into R^{2m}. They asked whether this property is related to the phenomenon of m-dimensional compacta whose squares have dimension less than $2m$.

J. Krasinkiewicz (see [Kr]) proved that if $E(X,R^{2m})$ is dense in $C(X,R^{2m})$ then $\dim(X\times X)<2m$ for any m-dimensional compactum X, and he asked whether the converse is true.

A consequence of Theorem (2.2) in [Kr] and our theorem is the following:

COROLLARY. *If X is an m-dimensional compactum, $m\geq 3$, with $\dim(X\times X)<2m$ then the space $E(X,R^{2m})$ is dense in $C(X,R^{2m})$.*