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Mapping approximate inverse systems of compacta

by

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Abstract. Recently, T. Watanabe has extensively studied approximate mappings of inverse systems of spaces $f\colon X \dashrightarrow Y$. He showed if $p\colon X \to X$ and $q\colon Y \to Y$ are ANR-resolutions of topologically complete spaces X, Y, then $f\colon X \dashrightarrow Y$ induces a mapping $f\colon X \to Y$ and conversely, every mapping $f\colon X \to Y$ is obtainable in this way. In this paper it is shown that the basic results of Watanabe's theory are valid also for approximate mappings of approximate inverse systems $\boldsymbol{X}$, $\boldsymbol{Y}$ of compact ANR's and compact Hausdorff spaces $X = \lim \boldsymbol{X}$, $Y = \lim \boldsymbol{Y}$. Approximate systems, newly introduced by S. Mardešić and L. R. Rubin, have bonding maps $p_{aa'}$, $a \leqslant a'$, where in general $p_{a_1a_2}p_{a_2a_3}$ differs from $p_{a_1a_3}$, but in a controlled way.

1. Introduction. An inverse system of spaces $\boldsymbol{X} = (X_a, p_{aa'}, A)$ (in the usual sense) consists of a directed set $(A, \leqslant)$, spaces X_a, $a \in A$, and maps $p_{aa'}\colon X_{a'} \to X_a$, $a \leqslant a'$, such that $p_{aa} = \mathrm{id}$ and

(1) $$p_{a_1a_2}p_{a_2a_3} = p_{a_1a_3}, \quad a_1 \leqslant a_2 \leqslant a_3.$$

The (usual) inverse limit $X = \lim \boldsymbol{X}$ is the subspace $X \subseteq \Pi X_a$, which consists of all points $x = (x_a) \in \Pi X_a$ such that $p_{aa'}(x_{a'}) = x_a$, $a \leqslant a'$. Projections $p_a\colon X \to X_a$ are restrictions to X of the projections $\pi_a\colon \Pi X_a \to X_a$, $a \in A$.

A mapping of systems $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y} = (Y_b, q_{bb'}, B)$ consists of a function $f\colon B \to A$ and of mappings $f_b\colon X_{f(b)} \to Y_b$, $b \in B$, such that whenever $b_1 \leqslant b_2$, there exists an index $a \in A$, $a \geqslant f(b_1), f(b_2)$, such that

(2) $$f_{b_1}p_{f(b_1)a} = q_{b_1b_2}f_{b_2}p_{f(b_2)a}$$

(see, e.g., [8], I, § 1.1). It is well known that for any mapping of systems $\boldsymbol{f} = (f, f_b)\colon \boldsymbol{X} \to \boldsymbol{Y}$ there is a unique mapping $f\colon X \to Y$ of the limits $X = \lim \boldsymbol{X}$, $Y = \lim \boldsymbol{Y}$ such that

(3) $$f_b p_{f(b)} = q_b f, \quad b \in B.$$

This mapping is called the *limit of* $\boldsymbol{f}$ and is denoted by $f = \lim \boldsymbol{f}$ (see, e.g., [8], I, § 5.1).

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If $f\colon X\to Y$ is a mapping between compact Hausdorff spaces, then there exist inverse systems $\boldsymbol{X}$, $\boldsymbol{Y}$ of compact ANR's (compact polyhedra) and there exists a map of systems $\boldsymbol{f}\colon \boldsymbol{X}\to\boldsymbol{Y}$ such that $X=\lim\boldsymbol{X}$, $Y=\lim\boldsymbol{Y}$ and $f=\lim\boldsymbol{f}$ (see, e.g., [4] for the case when X and Y are metric and [3], [1] for the general case). However, if one chooses $\boldsymbol{Y}$ in advance, it may be impossible to find $\boldsymbol{X}$ and $\boldsymbol{f}$ such that (3) holds for a given f. Such examples exist even when X and Y are metric compacta and $\boldsymbol{Y}$ is a sequence of compact ANR's (polyhedra) (see, e.g., [2], [12] or [13]). This is the reason why various authors have studied more general mappings of systems $\boldsymbol{f}=(f,f_b)\colon \boldsymbol{X}\to\boldsymbol{Y}$, where the commutativity relation (2) holds only approximately (see, e.g., [11] for the case of inverse sequences).

An extensive and very general study of this phenomenon is due to T. Watanabe (see [12], [13]). He combined the idea of approximate mapping with the idea of resolution, introduced in [3]. This enabled him to also consider non-compact spaces. However, all inverse systems in Watanabe's work are usual (commutative) systems, which satisfy (1).

Recently, S. Mardešić and L. R. Rubin have introduced and studied approximate inverse systems of metric compacta, i.e., systems where (1) holds only approximately [4]. These systems have proven very effective in constructing various compact Hausdorff spaces using polyhedra as terms of the system (see [4], [9] and [5]).

The purpose of the present paper is to establish all the basic features of Watanabe's theory also for approximate mappings between *approximate* systems of compact ANR's (polyhedra). In particular, such mappings $\boldsymbol{f}\colon \boldsymbol{X}\to\boldsymbol{Y}$ induce a limit mapping $f\colon X\to Y$ between the limit spaces $X=\lim\boldsymbol{X}$, $Y=\lim\boldsymbol{Y}$ and conversely, every mapping $f\colon X\to Y$ between compact Hausdorff spaces is the limit of an approximate mapping $\boldsymbol{f}\colon\boldsymbol{X}\to\boldsymbol{Y}$ between two arbitrary approximate systems of compact ANR's (polyhedra) with $\lim\boldsymbol{X}=X$, $\lim\boldsymbol{Y}=Y$.

As a consequence of our general results, we can prove, e.g., the following. Every mapping $f\colon X\to Y$ between compact Hausdorff spaces with $\dim X\leqslant m$, $\dim Y\leqslant n$ is the limit of an approximate mapping $\boldsymbol{f}\colon\boldsymbol{X}\to\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are approximate systems of polyhedra X_a and Y_b respectively with $\dim X_a\leqslant m$ and $\dim Y_b\leqslant n$. This is so because, by [4], every compact Hausdorff space X with $\dim X\leqslant m$ is the limit of an approximate system of polyhedra of dimension $\leqslant m$.

If one insists on using commutative systems, such an expansion theorem in general does not hold, because there are compact Hausdorff spaces Y, $\dim Y=1$, which are not representable as limits of 1-dimensional (commutative) inverse systems (see [4]).

In the last section of this paper we show that approximate systems of compact ANR's can be used in the shape theory of compact Hausdorff spaces.

We believe that the basic results of this paper can be generalized to topologically complete spaces, approximate systems of non-compact ANR's and (non-commutative) approximate resolutions.

2. Approximate inverse systems of compacta. We quote from [4] the basic definitions.

DEFINITION 1. An *approximate inverse system* of metric compacta $\boldsymbol{X}=(X_a,u_a,p_{aa'},A)$ consists of the following: An unbounded directed set $(A,\leqslant)$; for each $a\in A$, a compact metric space X_a with metric $d_a=d$ and a real number $u_a>0$ (called a mesh); for each pair $a\leqslant a'$ from A, a mapping $p_{aa'}\colon X_{a'}\to X_a$, satisfying the following conditions:

(A1) $$d(p_{a_1a_2}p_{a_2a_3},p_{a_1a_3})\leqslant u_{a_1},\qquad a_1\leqslant a_2\leqslant a_3;\qquad p_{aa}=\mathrm{id}.$$

(A2) $$(\forall a_1\in A)(\forall\eta>0)(\exists a_1'\geqslant a_1)(\forall a_3\geqslant a_2\geqslant a_1')\quad d(p_{a_1a_2}p_{a_2a_3},p_{a_1a_3})\leqslant\eta.$$

(A3) $$(\forall a\in A)(\forall\eta>0)(\exists a'\geqslant a)(\forall a''\geqslant a')(\forall x,x'\in X_{a''})\quad d(x,x')\leqslant u_{a''}\Rightarrow d(p_{aa''}(x),p_{aa''}(x'))\leqslant\eta.$$

Here $d(f,g)=\sup d(f(x),g(x))$.

DEFINITION 2. A point $x=(x_a)\in\Pi X_a$ belongs to $X=\lim\boldsymbol{X}$,

$$\boldsymbol{X}=(X_a,u_a,p_{aa'},A)$$

provided, for every $a\in A$,

(AL) $$x_a=\lim_{a'}p_{aa'}(x_{a'}).$$

The projections $p_a\colon X\to X_a$ are given by $p_a=\pi_a|X$.

We now quote (as propositions) several results from [4], [5], [9] and [10], which we need later. We assume that $\boldsymbol{X}=(X_a,u_a,p_{aa'},A)$ is an approximate system of metric compacta with limit X and projections $p_a\colon X\to X_a$, $a\in A$.

PROPOSITION 1. *If $\boldsymbol{X}$ is a commutative system, then $X=\lim\boldsymbol{X}$ as defined in Definition 2, coincides with the usual inverse limit* (see [4], Proposition 1).

PROPOSITION 2. *If $\boldsymbol{X}=(X_a,p_{aa'},A)$ is a commutative system of metric compacta and A is unbounded (has no maximal element) and is cofinite (every element has only finitely many predecessors), then there exist numbers $u_a>0$, $a\in A$, such that $(X_a,u_a,p_{aa'},A)$ is an approximate system* ([4], Remark 2).

PROPOSITION 3. *If $X_a\neq\emptyset$ for all $a\in A$, then $X=\lim\boldsymbol{X}$ is a compact Hausdorff space and $X\neq\emptyset$* (see [4], Theorem 1 and 2).

PROPOSITION 4. *For every $a\in A$,*

$$\lim_{a'}d(p_{aa'}p_{a'},p_a)=0$$

(see [4], Lemma 4).

PROPOSITION 5. *Let* $X = (X_a, u_a, p_{aa'}, A)$ *be an approximate system with limit* X *and projections* p_a. *Let* $<'$ *be a binary relation on* A *such that*

$$a_1 <' a_2 \Rightarrow a_1 < a_2 ,$$

$$a_1 <' a_2 \text{ and } a_2 \leqslant a_3 \Rightarrow a_1 <' a_3 ,$$

$$(\forall a \in A)(\exists a' \in A) a <' a' .$$

Moreover, let $a_1 \leqslant' a_2$ *mean that* $a_1 <' a_2$ *or* $a_1 = a_2$. *Then* $A' = (A, \leqslant')$ *is a directed set and* $X' = (X_a, u_a, p_{aa'}, A')$ *is an approximate system with limit* $X' = X$ *and projections* $p'_a = p_a$ (see [5], Proposition 9).

PROPOSITION 6. *The following statements hold*:

(B1) *Let* $a \in A$ *and let* $U \subseteq X_a$ *be an open set which contains* $p_a(X)$. *Then there is an* $a' \geqslant a$ *such that* $p_{aa''}(X_{a''}) \subseteq U$ *for each* $a'' \geqslant a'$.

(B2) *For every open covering* $\mathcal{U}$ *of* X *there exists an* $a \in A$ *such that for any* $a_1 \geqslant a$ *there exists an open covering* $\mathcal{V}$ *of* X_{a_1} *for which* $(p_{a_1})^{-1}(\mathcal{V})$ *refines* $\mathcal{U}$ (see [4], Theorem 3 and [9], Theorem 1).

PROPOSITION 7. *The following statements hold*:

(R1) *For every compact* ANR P, $\eta > 0$ *and mapping* h: $X \to P$, *there is an* $a \in A$ *such that for any* $a' \geqslant a$ *there is a mapping* f: $X_{a'} \to P$ *for which* $d(fp_{a'}, h) \leqslant 2\eta$.

(R2) *Let* P *be a compact* ANR *and* $\eta > 0$. *Whenever* $a \in A$ *and* f, f': $X_a \to P$ *are mappings with the property* $d(fp_a, f'p_a) < \eta$, *then there is an* $a' \in A$ *such that for any* $a'' \geqslant a'$ *one has* $d(fp_{aa''}, f'p_{aa''}) < \eta$.

(R1) was proven in [5] as Proposition 7. (R2) is an immediate consequence of (B2).

DEFINITION 3 [10]. An *almost commutative system* $X = (X_a, p_{aa'}, A)$ consists of an unbounded directed set $(A, \leqslant)$, of metric compacta X_a and of maps $p_{aa'}$ such that it is possible to associate with every $a \in A$ a number $u_a > 0$ so as to obtain an approximate system as in Definition 1. Such numbers u_a are called *admissible meshes*.

Remark 1. If (u_a) and (u'_a) are admissible meshes for the same almost commutative system and $X = (X_a, u_a, p_{aa'}, A)$, $X' = (X_a, u'_a, p_{aa'}, A)$, then $X = \lim X$ coincides with $X' = \lim X'$, because the limit space does not depend on the meshes (see Definition 2). Therefore, the limit space of an almost commutative system is well defined.

Remark 2. If u_a and u'_a are admissible meshes for the same almost commutative system, then $\max(u_a, u'_a)$ and $\min(u_a, u'_a)$ are also admissible meshes for that system.

PROPOSITION 8. *For every approximate system* $X = (X_a, u_a, p_{aa'}, A)$ *over a cofinite set* A *there exist admissible meshes* $u'_a > u_a$, $a \in A$ (see [10], Lemma 1).

3. Mappings of approximate systems and their limits.

DEFINITION 4. Let $X = (X_a, u_a, p_{aa'}, A)$ *and* $Y = (Y_b, v_b, q_{bb'}, B)$ be approximate systems of metric compacta. A *mapping of approximate systems* f: $X \to Y$ (or approximate mapping) consists of a function f: $B \to A$ and of mappings f_b: $X_{f(b)} \to Y_b$, $b \in B$, such that whenever $b \leqslant b'$, then there exists an index $a \in A$, $a \geqslant f(b), f(b')$, such that for every $a' \geqslant a$ the following condition holds

(AM1) $$d(f_b p_{f(b)a'}, q_{bb'} f_{b'} p_{f(b')a'}) \leqslant v_b .$$

(see [12], § 2, condition (AM2)).

The main purpose of this section is to prove the following theorem.

THEOREM 1. *If* $f = (f, f_b)$: $X \to Y$ *is a mapping of approximate systems, then there exists a unique mapping* f: $X \to Y$, $X = \lim X$, $Y = \lim Y$, *such that*

(LM) $$d(f_b p_{f(b)}, q_b f) \leqslant v_b , \quad b \in B .$$

The mapping f: $X \to Y$ is called the *limit of* f and is denoted by $f = \lim f$. (It should not be confused with f: $B \to A$ from $f = (f, f_b)$: $X \to Y$.)

To prove the existence of f we need the following lemma.

LEMMA 1. *Let* X *be a compact Hausdorff space and let* h_b: $X \to Y_b$, $b \in B$, *be maps such that*

(1) $$d(h_b, q_{bb'} h_{b'}) \leqslant v_b , \quad b \leqslant b' .$$

Then there exists a mapping f: $X \to Y = \lim Y$ *such that*

(2) $$d(h_b, q_b f) \leqslant v_b , \quad b \in B .$$

Proof of Lemma 1. For a given $b \in B$ we consider all $b' \geqslant b$ and the maps $q_{bb'} h_{b'}$: $X \to Y_b$. We will show that these maps form a Cauchy net and therefore

(3) $$f^b = \lim_{b'} q_{bb'} h_{b'}$$

is a well-defined map f^b: $X \to Y_b$.

Indeed, for any $\eta > 0$, by (A2), there exists an index $b' \geqslant b$ such that

(4) $$d(q_{bb_1} q_{b_1 b_3}, q_{bb_3}) \leqslant \eta , \quad b' \leqslant b_1 \leqslant b_3 .$$

Moreover, by (A3), one can assume that, for any $b'' \geqslant b'$,

(5) $$d(y, y') \leqslant v_{b''} \Rightarrow d(q_{bb''}(y), q_{bb''}(y')) \leqslant \eta .$$

(1) and (5) imply

(6) $$d(q_{bb_1} h_{b_1}, q_{bb_1} q_{b_1 b_3} h_{b_3}) \leqslant \eta , \quad b' \leqslant b_1 \leqslant b_3 ,$$

which together with (4) yields

(7) $$d(q_{bb_1} h_{b_1}, q_{bb_3} h_{b_3}) \leqslant 2\eta , \quad b' \leqslant b_1 \leqslant b_3 .$$

Analogously, we obtain

(8) $$d(q_{bb_2} h_{b_2}, q_{bb_3} h_{b_3}) \leqslant 2\eta , \quad b' \leqslant b_2 \leqslant b_3 .$$

Since B is directed, (7) and (8) prove

(9) $$d(q_{bb_1} h_{b_1}, q_{bb_2} h_{b_2}) \leqslant 4\eta , \quad b' \leqslant b_1, b_2 ,$$

which proves that $(q_{bb'}h_{b'}, b' \geqslant b)$ is, indeed, a Cauchy net of mappings $X \to Y_b$.

For every $x \in X$ and $b' \leqslant b_1 \leqslant b_3$, by (4), we have

$$d(q_{bb_1}q_{b_1b_3}h_{b_3}(x), q_{bb_3}h_{b_3}(x)) \leqslant \eta . \tag{10}$$

Passing to the limit with b_3, we obtain

$$d(q_{bb_1}f^{b_1}(x), f^b(x)) \leqslant \eta , \quad b_1 \geqslant b' , \tag{11}$$

which shows that

$$f^b(x) = \lim_{b_1} q_{bb_1}f^{b_1}(x) . \tag{12}$$

This means that the mappings f^b: $X \to Y_b$, $b \in B$, determine a mapping f: $X \to Y \subseteq \Pi Y_b$ such that

$$q_b f = f^b , \quad b \in B . \tag{13}$$

Passing to the limit with b' in (1), we obtain

$$d(h_b, f^b) \leqslant v_b , \quad b \in B , \tag{14}$$

which, by (13), coincides with the desired formula (2).

Proof of Theorem 1. *Existence*. Given a mapping of approximate systems $\boldsymbol{f} = (f, f_b)$: $\boldsymbol{X} \to \boldsymbol{Y}$, we put

$$h_b = f_b p_{f(b)}, \quad b \in B . \tag{15}$$

By Lemma 1, it suffices to show that the maps h_b: $X \to Y_b$, $b \in B$, satisfy (1).

By Definition 4, if $b \leqslant b'$, there is an $a \geqslant f(b), f(b')$ such that (AM1) holds for all $a' \geqslant a$. This implies

$$d(f_b p_{f(b)a'} p_{a'}, q_{bb'} f_{b'} p_{f(b')a'} p_{a'}) \leqslant v_b . \tag{16}$$

Passing to the limit with a', we obtain (1), because, by Proposition 4, for any $a \in A$, one has

$$\lim_{a'} p_{aa'} p_{a'} = p_a . \tag{17}$$

The uniqueness of f is a consequence of the following lemma.

LEMMA 2. *Let X be a compact Hausdorff space and let h_b, h'_b: $X \to Y_b$, $b \in B$, be two families of mappings satisfying*

$$d(h_b, h'_b) \leqslant v_b , \quad b \in B . \tag{18}$$

Let f, f': $X \to Y$ be mappings such that f satisfies (2) *and f' satisfies the analogous relations with h'_b. Then $f = f'$.*

Proof of Lemma 2. Given any $b \in B$ and $\eta > 0$, by Proposition 4, there is an index $b' \geqslant b$ such that

$$d(q_{bb''}q_{b''}, q_b) \leqslant \eta , \quad b'' \geqslant b' . \tag{19}$$

By (A3), one can assume that for $b'' \geqslant b'$ (5) also holds. Then, by (2), one has

$$d(q_{bb''}h_{b''}, q_{bb''}q_{b''}f) \leqslant v_b , \quad b'' \geqslant b' . \tag{20}$$

Analogously,

$$d(q_{bb''}h'_{b''}, q_{bb''}q_{b''}f') \leqslant \eta_b , \quad b'' \geqslant b' . \tag{21}$$

Furthermore, (5) and (18) imply

$$d(q_{bb''}h_{b''}, q_{bb''}h'_{b''}) \leqslant \eta , \quad b'' \geqslant b' . \tag{22}$$

Now, (19), (20), (22), (21) and again (19) yield

$$d(q_b f, q_b f') \leqslant 5\eta , \quad b \in B . \tag{23}$$

Since $\eta > 0$ was arbitrary, we conclude that $q_b f = q_b f'$, for all $b \in B$, and, therefore, $f = f'$.

Proof of Theorem 1. *Uniqueness*. It suffices to apply Lemma 2 to $h_b = h'_b = f_b p_{f(b)}$ and f, f'.

DEFINITION 5. An *almost commutative mapping* $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ between almost commutative systems consists of a function f: $B \to A$ and of mappings f_b: $X_{f(b)} \to Y_b$, $b \in B$, such that $\boldsymbol{f} = (f, f_b)$ is an approximate mapping for some choice of admissible meshes. The *limit mapping* f: $X \to Y$ is defined as the limit of such an approximate mapping.

Remark 3. Let v_b and v'_b, $b \in B$, be admissible meshes of an almost commutative system $(Y_b, q_{bb'}, B)$ and let $\boldsymbol{f} = (f, f_b)$ be an approximate mapping into both $\boldsymbol{Y} = (Y_b, v_b, q_{bb'}, B)$ and $\boldsymbol{Y}' = (Y_b, v'_b, q_{bb'}, B)$ with limits f and f' respectively. Clearly, $\boldsymbol{f}$ is also an approximate mapping into $(Y_b, \max(v_b, v'_b), q_{bb'}, B)$. Moreover,

$$d(f_b p_{f(b)}, q_b f) \leqslant v_b \leqslant \max(v_b, v'_b) , \tag{24}$$

$$d(f_b p_{f(b)}, q_b f') \leqslant v'_b \leqslant \max(v_b, v'_b) . \tag{25}$$

By the uniqueness in Theorem 1, one concludes that $f = f'$, so that $\lim \boldsymbol{f}$ does not depend on the choice of admissible meshes.

Remark 4. In some cases an approximate mapping of systems $\boldsymbol{f} = (f, f_b)$: $\boldsymbol{X} \to \boldsymbol{Y}$ also satisfies the additional condition

(AM2) $(\forall b \in B)(\forall \eta > 0)(\exists b' \geqslant b)(\forall b'' \geqslant b')(\exists a \geqslant f(b), f(b''))(\forall a' \geqslant a)$

$$d(f_b p_{f(b)a'}, q_{bb''} f_{b''} p_{f(b'')a'}) \leqslant \eta .$$

In this case the limit map $f\colon X\to Y$ satisfies the commutative relation

(26) $$f_b p_{f(b)} = q_b f\,, \quad b\in B\,.$$

Indeed, by (13), (3) and 15,

(27) $$q_b f = \lim_{b''} q_{bb''} f_{b''} p_{f(b'')}\,.$$

Therefore, Proposition 4 and (AM2) imply

(28) $$d(f_b p_{f(b)}, q_b f) \leqslant \eta\,, \quad b\in B\,.$$

Since $\eta>0$ was arbitrary, (28) proves (26).

4. Representing mappings as limits of approximate mappings. In this section we prove the following expansion theorem.

THEOREM 2. *Let X and Y be almost commutative systems with limits X and Y respectively and let $f\colon X\to Y$ be a mapping. If all Y_b, $b\in B$, are compact* ANR's *and B is cofinite, then there exists an almost commutative mapping $\boldsymbol{f}\colon \boldsymbol{X}\to \boldsymbol{Y}$ such that $f=\lim \boldsymbol{f}$.*

We first prove a lemma which often enables us to verify that a collection of maps is an approximate mapping.

LEMMA 3. *Let X and $Y=(Y_b, v_b, q_{bb'}, B)$ be approximate systems, let $f\colon B\to A$ be a function and let $f_b\colon X_{f(b)}\to Y_b$, $b\in B$, be maps which satisfy the following condition:*

(1) $$d(f_b p_{f(b)}, q_{bb'} f_{b'} p_{f(b')}) \leqslant v_b\,, \quad b\leqslant b'\,.$$

Then $\boldsymbol{f}=(f, f_b)$ is a mapping of approximate systems $\boldsymbol{f}\colon \boldsymbol{X}\to \boldsymbol{Y}'=(Y_b, v'_b, q_{bb'}, B)$, for any choice of admissible meshes $v'_b>v_b$, $b\in B$.

Proof. Given $b_1\leqslant b_2$ we must find an index $a'\in A$, $a'\geqslant f(b_1), f(b_2)$, such that for all $a''\geqslant a'$ one has

(2) $$d(f_{b_1} p_{f(b_1)a''}, q_{b_1 b_2} f_{b_2} p_{f(b_2)a''}) \leqslant v'_{b_1}\,.$$

Choose numbers $\eta_1>0$, $\eta_2>0$ such that

(3) $$d(x,x')\leqslant \eta_1 \Rightarrow d(f_{b_1}(x), f_{b_1}(x')) \leqslant \tfrac{1}{5}(v'_{b_1}-v_{b_1})\,.$$

(4) $$d(x,x')\leqslant \eta_2 \Rightarrow d(q_{b_1 b_2} f_{b_2}(x), q_{b_1 b_2} f_{b_2}(x')) \leqslant \tfrac{1}{5}(v'_{b_1}-v_{b_1})\,.$$

By (A2), there is an index $a\geqslant f(b_1), f(b_2)$ such that for any $a''\geqslant a$ one has

(5) $$d(p_{f(b_i)a} p_{aa''}, p_{f(b_i)a''}) \leqslant \eta_i\,, \quad i=1,2\,.$$

Now (3) and (4) imply

(6) $$d(f_{b_1} p_{f(b_1)a} p_{aa''}, f_{b_1} p_{f(b_1)a''}) \leqslant \tfrac{1}{5}(v'_{b_1}-v_{b_1})\,,$$

(7) $$d(q_{b_1 b_2} f_{b_2} p_{f(b_2)a} p_{aa''}, q_{b_1 b_2} f_{b_2} p_{f(b_2)a''}) \leqslant \tfrac{1}{5}(v'_{b_1}-v_{b_1})\,.$$

Since $\lim_{a''} p_{aa''} p_{a''} = p_a$ and $\lim_{a''} p_{f(b_i)a''} p_{a''} = p_{f(b_i)}$, $i=1,2$ (Proposition 4), (6) and (7) yield

(8) $$d(f_{b_1} p_{f(b_1)a} p_a, f_{b_1} p_{f(b_1)}) \leqslant \tfrac{1}{5}(v'_{b_1}-v_{b_1})\,,$$

(9) $$d(q_{b_1 b_2} f_{b_2} p_{f(b_2)a} p_a, q_{b_1 b_2} f_{b_2} p_{f(b_2)}) \leqslant \tfrac{1}{5}(v'_{b_1}-v_{b_1})\,.$$

Applying the assumption (1), we conclude that

(10) $$d(f_{b_1} p_{f(b_1)a} p_a, q_{b_1 b_2} f_{b_2} p_{f(b_2)a} p_a) \leqslant \tfrac{2}{5}(v'_{b_1}-v_{b_1})+v_{b_1}\,.$$

By continuity, there is a neighborhood U of $p_a(X)$ in X_a such that

(11) $$d(f_{b_1} p_{f(b_1)a}|U, q_{b_1 b_2} f_{b_2} p_{f(b_2)a}|U) \leqslant \tfrac{3}{5}(v'_{b_1}-v_{b_1})+v_{b_1}\,.$$

We now use property (B1) (Proposition 6) to conclude that there is an $a'\geqslant a$ such that for any $a''\geqslant a'$ one has

(12) $$p_{aa''}(X_{a''}) \subseteq U\,.$$

Therefore, (11) yields

(13) $$d(f_{b_1} p_{f(b_1)a} p_{aa''}, q_{b_1 b_2} f_{b_2} p_{f(b_2)a} p_{aa''}) \leqslant \tfrac{3}{5}(v'_{b_1}-v_{b_1})+v_{b_1}\,, \quad a''\geqslant a'\,.$$

By (6), (7) and (13), we conclude that (2) indeed holds for all $a''\geqslant a'$.

Proof of Theorem 2. By assumption there are numbers $v_b>0$, $b\in B$, such that $Y=(Y_b, v_b, q_{bb'}, B)$ is an approximate system. By Proposition 8, there exist admissible meshes $v'_b>v_b$, $b\in B$. By Lemma 3, it suffices to produce a function $f\colon B\to A$ and mappings $f_b\colon X_{f(b)}\to Y_b$, $b\in B$, such that

(14) $$d(f_b p_{f(b)}, q_{bb'} f_{b'} p_{f(b')}) \leqslant v'_b\,,$$

(15) $$d(f_b p_{f(b)}, q_b f) \leqslant v'_b\,, \quad b\in B\,.$$

First note that (A1) implies

(16) $$d(q_{bb'} q_{b'b''} q_{b''}, q_{bb''} q_{b''}) \leqslant v_b\,, \quad b\leqslant b'\leqslant b''\,.$$

Passing to the limit with b'', we obtain (by Proposition 4)

(17) $$d(q_{bb'} q_{b'}, q_b) \leqslant v_b\,, \quad b\leqslant b'\,.$$

By uniform continuity and cofiniteness of B, for any $b\in B$ there is an $\eta_b>0$ such that for all $b_0\leqslant b$ one has

(18) $$d(y,y')\leqslant \eta_b \Rightarrow d(q_{b_0 b}(y), q_{b_0 b}(y')) \leqslant \tfrac{1}{2}(v'_{b_0}-v_{b_0})\,.$$

Since Y_b, $b\in B$, is an ANR, property (R1) (see Proposition 7) yields an $a=f(b)\in A$ and a mapping $f_b\colon X_{f(b)}\to Y_b$ such that

(19) $$d(f_b p_{f(b)}, q_b f) \leqslant \min\{\eta_b, \tfrac{1}{2}(v'_b-v_b), v'_b\}\,.$$

If $b \leqslant b'$, then (19) and (18) yield

(20) $$d(q_{bb'}f_{b'}p_{f(b')}, q_{bb'}q_{b'}f) \leqslant \tfrac{1}{2}(v_b'-v_b).$$

Now, (19), (17) and (20) yield (14). Moreover (19) implies (15).

5. Contiguous and equivalent approximate mappings. Different approximate mappings can have the same limit mapping. In this section we study this phenomenon.

DEFINITION 6. Let X and $Y = (Y_b, v_b, q_{bb'}, B)$ be approximate systems. Two approximate mappings $\boldsymbol{f} = (f, f_b)$, $\boldsymbol{f}' = (f', f_b')$: $X \to Y$ are called *contiguous*, denoted by $\boldsymbol{f} \equiv \boldsymbol{f}'$, provided for every $b \in B$ there is an $a \in A$, $a \geqslant f(b), f'(b)$, such that for any $a' \geqslant a$ one has

(1) $$d(f_b p_{f(b)a'}, f_b' p_{f'(b)a'}) \leqslant v_b.$$

Two almost commutative maps $\boldsymbol{f}, \boldsymbol{f}'$: $X \to Y$ are called *contiguous* provided they are contiguous as approximate mappings for some choice of admissible meshes $v_b > 0$, $b \in B$.

The following is an easy consequence of Lemma 2.

THEOREM 3. *Let $\boldsymbol{f}, \boldsymbol{f}'$: $X \to Y$ be approximate maps of systems with limits f, f': $X \to Y$ respectively. If $\boldsymbol{f}$ and $\boldsymbol{f}'$ are contiguous, then $f = f'$.*

Proof. Let $h_b = f_b p_{f(b)}$, $h_b' = f_b' p_{f'(b)}$. By (LM), f satisfies § 3(2) and f' satisfies the analogous formula. By Lemma 2, it suffices to show that § 3(18) also holds.

By (1), we have

(2) $$d(f_b p_{f(b)a'} p_{a'}, f_b' p_{f'(b)a'} p_{a'}) \leqslant v_b,$$

for a' sufficiently large. Passing to the limit with a', by § 3(17), we indeed obtain § 3(18).

The following lemma is sometimes used to conclude that two approximate mappings are contiguous.

LEMMA 4. *Let X and $Y = (Y_b, v_b, q_{bb'}, B)$ be approximate systems and let $\boldsymbol{f} = (f, f_b)$, $\boldsymbol{f}' = (f', f_b')$: $X \to Y$ be approximate maps. If*

(3) $$d(f_b p_{f(b)}, f_b' p_{f'(b)}) \leqslant v_b, \quad b \in B,$$

then $\boldsymbol{f} \equiv \boldsymbol{f}'$: $X \to Y' = (Y_b, v_b', q_{bb'}, B)$, for any choice of admissible meshes $v_b' > v_b$, $b \in B$.

Proof. The proof is similar to the proof of Lemma 3. By (A2), there is an index $a \geqslant f(b), f'(b)$ such that for any $a'' \geqslant a$ one has

(4) $$d(f_b p_{f(b)a} p_{aa''}, f_b p_{f(b)a''}) \leqslant \tfrac{1}{5}(v_b'-v_b),$$

(5) $$d(f_b' p_{f'(b)a} p_{aa''}, f_b' p_{f'(b)a''}) \leqslant \tfrac{1}{5}(v_b'-v_b).$$

Since $\lim_{a''} p_{aa''} p_{a''} = p_a$ (Proposition 3), (4) and (5) imply

(6) $$d(f_b p_{f(b)a} p_a, f_b p_{f(b)}) \leqslant \tfrac{1}{5}(v_b'-v_b),$$

(7) $$d(f_b' p_{f'(b)a} p_a, f_b' p_{f'(b)}) \leqslant \tfrac{1}{5}(v_b'-v_b).$$

Using (3), one obtains

(8) $$d(f_b p_{f(b)a} p_a, f_b' p_{f'(b)a} p_a) \leqslant \tfrac{2}{5}(v_b'-v_b)+v_b.$$

By continuity, there is a neighborhood U of $p_a(X)$ in X_a such that

(9) $$d(f_b p_{f(b)a}|U, f_b' p_{f'(b)a}|U) \leqslant \tfrac{3}{5}(v_b'-v_b)+v_b.$$

By property (B2) (Proposition 4), there is an $a' \geqslant a$ such that for any $a'' \geqslant a'$ one has

(10) $$p_{aa''}(X_{a''}) \subseteq U,$$

and therefore,

(11) $$d(f_b p_{f(b)a} p_{aa''}, f_b' p_{f'(b)a} p_{aa''}) \leqslant \tfrac{3}{5}(v_b'-v_b)+v_b.$$

Now (4), (5) and (11) yield the desired relation

(12) $$d(f_b p_{f(b)a''}, f_b' p_{f'(b)a''}) \leqslant v_b'.$$

Remark 5. By Proposition 4, (1) implies (3).

The next two lemmas show that certain modification of approximate mappings, called *shifts*, produce contiguous approximate mappings.

LEMMA 5. *Let $\boldsymbol{f}$: $X \to Y = (Y_b, v_b, q_{bb'}, B)$ be an approximate mapping and let $v_b' > v_b$, $b \in B$, be admissible meshes. If B is cofinite, there exists a function $\varphi: B \to A$, $\varphi \geqslant f$, such that any function f': $B \to A$, $f' \geqslant \varphi$, together with the maps*

(13) $$f_b' = f_b p_{f(b)f'(b)}: X_{f'(b)} \to Y_b, \quad b \in B,$$

form an approximate mapping $\boldsymbol{f}' = (f', f_b')$: $X \to Y' = (Y_b, v_b', q_{bb'}, B)$, called an initial shift of $\boldsymbol{f}$. The approximate mappings $\boldsymbol{f}, \boldsymbol{f}'$: $X \to Y'$ are contiguous.

Proof. For each $b \in B$ choose a number $\eta_b > 0$ such that

(14) $$d(x, x') \leqslant \eta_b \Rightarrow d(q_{b_0 b} f_b(x), q_{b_0 b} f_b(x')) \leqslant \tfrac{1}{2}(v_{b_0}'-v_{b_0}),$$

for all $b_0 \leqslant b$.

By (A2), each $b \in B$ admits an index $\varphi(b) \geqslant f(b)$ such that for any $a' \geqslant a \geqslant \varphi(b)$,

(15) $$d(p_{f(b)a} p_{aa'}, p_{f(b)a'}) \leqslant \eta_b.$$

Let f': $B \to A$ satisfy $f' \geqslant \varphi$. Put

(16) $$f_b' = f_b p_{f(b)f'(b)}, \quad b \in B.$$

If $b_1 \leqslant b_2$, there is an $a \geqslant f(b_1), f(b_2)$ such that for any $a' \geqslant a$ one has

$$d(f_{b_1} p_{f(b_1)a'}, q_{b_1 b_2} f_{b_2} p_{f(b_2)a'}) \leqslant v_{b_1}. \tag{17}$$

We can choose $a \geqslant f'(b_1), f'(b_2)$. Then, by (15) and (14) for any $a' \geqslant a$ one has

$$d(f_{b_1} p_{f(b_1)f'(b_1)} p_{f'(b_1)a'}, f_{b_1} p_{f(b_1)a'}) \leqslant \tfrac{1}{2}(v'_{b_1} - v_{b_1}), \tag{18}$$

$$d(q_{b_1 b_2} f_{b_2} p_{f(b_2)f'(b_2)} p_{f'(b_2)a'}, q_{b_1 b_2} f_{b_2} p_{f(b_2)a'}) \leqslant \tfrac{1}{2}(v'_{b_1} - v_{b_1}). \tag{19}$$

(16), (18), (17) and (19) imply

$$d(f'_{b_1} p_{f'(b_1)a'}, q_{b_1 b_2} f'_{b_2} p_{f'(b_2)a'}) \leqslant v'_{b_1}, \tag{20}$$

which shows that $f'\colon X \to Y'$ is indeed a mapping of approximate systems. Obviously, $f \equiv f'$.

Lemma 6. *Let $f\colon X \to Y = (Y_b, v_b, q_{bb'}, B)$ be an approximate mapping and let $v'_b > v_b$, $b \in B$, be admissible meshes. If B is cofinite, there exists a function $\chi\colon B \to B$, $\chi(b) \geqslant b$, $b \in B$, such that for any function $\psi\colon B \to B$, $\psi \geqslant \chi$, the function $f' = f\psi$ and the maps*

$$f'_b = q_{b\psi(b)} f_{\psi(b)}\colon X_{f'(b)} \to Y_b, \quad b \in B, \tag{21}$$

form an approximate mapping $f' = (f' f'_b)\colon X \to Y' = (Y_b, v'_b, q_{bb'}, B)$, called a terminal shift of f. The approximate mappings $f, f'\colon X \to Y'$ are continuous.

Proof. Since B is cofinite, by (A2), each $b \in B$ admits an index $\chi(b) \geqslant b$ such that, for $b'' \geqslant b' \geqslant \chi(b)$,

$$d(q_{bb'} q_{b'b''}, q_{bb''}) \leqslant \tfrac{1}{4}(v'_b - v_b) \tag{22}$$

and even

$$d(q_{b_0 b} q_{bb'} q_{b'b''}, q_{b_0 b} q_{bb''}) \leqslant \tfrac{1}{4}(v'_{b_0} - v_{b_0}), \tag{23}$$

for all $b_0 \leqslant b$.

By (A3), one can also assume that for $b'' \geqslant \chi(b)$

$$d(y, y') \leqslant v_{b''} \Rightarrow d(q_{b_0 b} q_{bb''}(y), q_{b_0 b} q_{bb''}(y')) \leqslant \tfrac{1}{4}(v'_{b_0} - v_{b_0}) \tag{24}$$

for all $b_0 \leqslant b$.

We will now show that χ has the desired property. Let $\psi\colon B \to B$ be a function with $\psi \geqslant \chi$. Let $b_1 \leqslant b_2$. Choose $b \geqslant \psi(b_1), \psi(b_2)$. By (AM1), there exists an index $a \geqslant f\psi(b_1), f\psi(b_2), f(b)$ such that for any $a' \geqslant a$ one has

$$d(f_{f\psi(b_i)} p_{f\psi(b_i)a'}, q_{\psi(b_i)b} f_b p_{f(b)a'}) \leqslant v_{\psi(b_i)}, \quad i = 1, 2. \tag{25}$$

Since $\psi(b_i) \geqslant \chi(b_i)$, (25) and (24) imply

$$d(q_{b_1\psi(b_1)} f_{\psi(b_1)} p_{f\psi(b_1)a'}, q_{b_1\psi(b_1)} q_{\psi(b_1)b} f_b p_{f(b)a'}) \leqslant \tfrac{1}{4}(v'_{b_1} - v_{b_1}), \tag{26}$$

$$d(q_{b_1 b_2} q_{b_2\psi(b_2)} f_{\psi(b_2)} p_{f\psi(b_2)a'}, q_{b_1 b_2} q_{b_2\psi(b_2)} q_{\psi(b_2)b} f_b p_{f(b)a'}) \leqslant \tfrac{1}{4}(v'_{b_1} - v_{b_1}). \tag{27}$$

By (22) and (23) we also have

$$d(q_{b_1\psi(b_1)} q_{\psi(b_1)b} f_b p_{f(b)a'}, q_{b_1 b} f_b p_{f(b)a'}) \leqslant \tfrac{1}{4}(v'_{b_1} - v_{b_1}), \tag{28}$$

$$d(q_{b_1 b_2} q_{b_2\psi(b_2)} q_{\psi(b_2)b} f_b p_{f(b)a'}, q_{b_1 b_2} q_{b_2 b} f_b p_{f(b)a'}) \leqslant \tfrac{1}{4}(v'_{b_1} - v_{b_1}). \tag{29}$$

Also, by (A1), we have

$$d(q_{b_1 b_2} q_{b_2 b} f_b p_{f(b)a'}, q_{b_1 b} f_b p_{f(b)a'}) \leqslant v_{b_1}. \tag{30}$$

Now (21), (26), (28), (30), (29) and (27) yield

$$d(f'_{b_1} p_{f'(b_1)a'}, q_{b_1 b_2} f'_{b_2} p_{f'(b_2)a'}) \leqslant v'_{b_1}, \tag{31}$$

which shows that $f' = (f', f'_b)$ is indeed an approximate mapping $f'\colon X \to Y'$.

By (AM1) applied to $b \leqslant \psi(b)$, we see that there is an $a \geqslant f(b), f'(b) = f\psi(b)$ such that for $a' \geqslant a$ one has

$$d(f_b p_{f(b)a'}, f'_b p_{f'(b)a'}) \leqslant v_b \leqslant v'_b. \tag{32}$$

In § 6 we will need the following lemma.

Lemma 7. *Let $f = (f, f_b)\colon X \to Y = (Y_b, v_b, q_{bb'}, B)$ be an approximate mapping and let $v'_b > v_b$, $b \in B$, be admissible meshes. If B is cofinite and $\delta_b > 0$ are arbitrary numbers, there exists an approximate mapping $f' = (f', f'_b)\colon X \to Y' = (Y_b, v'_b, q_{bb'}, B)$ such that*

$$d(x, x') \leqslant u_{f'(b)} \Rightarrow d(q_{b_0 b} f'_b(x), q_{b_0 b} f'_b(x')) \leqslant \delta_{b_0}, \tag{33}$$

for all $b_0 \leqslant b$. Moreover, $f' \equiv f\colon X \to Y'$ and f' is an initial shift of f.

Proof. By uniform continuity and cofiniteness of B, for each $b \in B$, there is a number $\eta_b > 0$ such that

$$d(x, x') \leqslant \eta_b \Rightarrow d(q_{b_0 b} f_b(x), q_{b_0 b} f_b(x')) \leqslant \delta_{b_0}, \quad b_0 \geqslant b. \tag{34}$$

By (A3), every $b \in B$ admits an index $\alpha(b) \geqslant f(b)$ such that, for any $a \geqslant \alpha(b)$,

$$d(x, x') \leqslant u_a \Rightarrow d(p_{f(b)a}(x), p_{f(b)a}(x')) \leqslant \eta_b. \tag{35}$$

We now choose a function $f'\colon B \to A$ such that $f' \geqslant \alpha$ and $f' \geqslant \varphi$, where φ is as in Lemma 5. Then f' and the maps $f'_b = f_b p_{f(b)f'(b)}$ form an approximate mapping $f'\colon X \to Y'$ and $f' \equiv f$. Moreover, (33) holds because of (34) and (35).

The relation of contiguity generates an equivalence relation in the set of all approximate mappings between two approximate systems.

Definition 7. Let X and Y be two approximate systems. Approximate maps $f, f'\colon X \to Y$ are called *equivalent*, written $f \sim f'$, provided there is a finite collection of approximate maps $f_i\colon X \to Y$, $i = 0, 1, \ldots, n$, such that $f_0 = f$, $f_n = f'$ and $f_i \equiv f_{i+1}$, $i = 0, \ldots, n-1$.

An analogous definition applies to almost commutative maps between almost commutative systems.

The equivalence class containing f will be denoted by $[f]$.

The main result of this section is the following theorem.

THEOREM 4. *Let $f, f'\colon X \to Y$ be almost commutative mappings between almost commutative systems with limits $f, f'\colon X \to Y$ respectively. If B is cofinite, then $f = f'$ if and only if $f \sim f'$.*

The sufficiency immediately follows from Definition 6 and Theorem 2. This part of the proof does not require the cofiniteness of B.

The necessity follows from the next lemma.

LEMMA 8. *Let $f, f'\colon X \to Y = (Y_b, v_b, q_{bb'}, B)$ be approximate mappings with limits $f, f'\colon X \to Y$ respectively. If B is cofinite and $v'_b > v_b$, $b \in B$, are admissible meshes, then $f = f'$ implies $f \sim f'\colon X \to Y' = (Y_b, v'_b, q_{bb'}, B)$.*

Proof. Choose functions $\chi, \chi'\colon B \to B$ by applying Lemma 6 to f and f' respectively. By (A3), there exists a function $\psi\colon B \to B$ such that $\psi \geqslant \chi, \chi'$ and

(35) $$d(y, y') \leqslant v_{\psi(b)} \Rightarrow d\big(q_{b\psi(b)}(y), q_{b\psi(b)}(y')\big) \leqslant \tfrac{1}{2} v_b, \quad b \in B.$$

Put $g = f\psi$, $g' = f'\psi$ and

(36) $$g_b = q_{b\psi(b)} f_{\psi(b)},$$

(37) $$g'_b = q_{b\psi(b)} f'_{\psi(b)}.$$

By Lemma 6, $g = (g, g_b)$, $g' = (g', g'_b)$ are approximate mappings $g, g'\colon X \to Y'$ and $f \equiv g$, $f' \equiv g'$. Moreover, by (LM),

(38) $$d(f_{\psi(b)} p_{f\psi(b)}, q_{\psi(b)} f) \leqslant v_{\psi(b)},$$

and therefore, by (35) and (36),

(39) $$d(g_b p_{g(b)}, q_{b\psi(b)} q_{\psi(b)} f) \leqslant \tfrac{1}{2} v_b, \quad b \in B.$$

Analogously, (35) and (37) imply

(40) $$d(g'_b p_{g'(b)}, q_{b\psi(b)} q_{\psi(b)} f') \leqslant \tfrac{1}{2} v_b, \quad b \in B.$$

Therefore, one has

(41) $$d(g_b p_{g(b)}, g'_b p_{g'(b)}) \leqslant v_b \leqslant v'_b, \quad b \in B.$$

By Lemma 4, g and g' are contiguous approximate mappings $Y \to Y'$. Since $f \equiv g \equiv g' \equiv f'$, we obtain the desired relation $f \sim f'$.

6. Composition of approximate mappings

DEFINITION 8. Let X, Y and $Z = (Z_c, w_c, r_{cc'}, C)$ be approximate systems and let $f = (f, f_b)\colon X \to Y$, $g = (g, g_c)\colon Y \to Z$ be approximate maps. Let $h = fg\colon C \to A$ and let

(1) $$h_c = g_c f_{g(c)}\colon X_{h(c)} \to Z_c, \quad c \in C.$$

If $h = (h, h_c)\colon X \to Z' = (Z_c, w'_c, r_{cc'}, C)$ is an approximate mapping for some admissible mesh $w'_c \geqslant w_c$ and if $\lim h = gf$, then h is called the *composition of f and g* (for the meshes w'_c) and is denoted by $h = gf$.

Remark 6. In general, h is not an approximate mapping $X \to Y$. Even if $w'_c > w_c$, h may not be an approximate mapping $X \to Z'$. Nevertheless, we will see that, for cofinite C, the composition gf is often defined. This will enable us to define the composition of equivalence classes $[g][f]$, whenever C is cofinite.

LEMMA 9. *Let $f = (f, f_b)\colon X \to Y$ and $g = (g, g_c)\colon Y \to Z = (Z_c, w_c, r_{cc'}, C)$ be approximate maps and let $w'_c > w_c$ be arbitrary admissible meshes. If C is cofinite, there exists an approximate mapping $g' = (g', g'_c)\colon Y \to Z' = (Z_c, w'_c, r_{cc'}, C)$ (initial shift of g) such that $g \equiv g'$ and the composition $g'f\colon X \to Z'$ is a well-defined approximate mapping with*

(2) $$\lim(g'f) = gf,$$

where $g = \lim g = \lim g'$ and $f = \lim f$.

Proof. Choose numbers w''_c, $c \in C$, such that $w_c < w''_c < w'_c$. Clearly, $Z'' = (Z_c, w''_c, r_{cc'}, C)$ is an approximate system. By Lemma 7, applied to $g\colon Y \to Z$, $w''_c > w_c$ and $\delta_c = \frac{1}{2}(w'_c - w''_c)$, there exists an approximate mapping $g' = (g', g'_c)\colon Y \to Z''$ (initial shift of g) such that

(3) $$d(y, y') \leqslant v_{g'(c)} \Rightarrow d(r_{c_0c} g'_c(y), r_{c_0c} g'_c(y')) \leqslant \tfrac{1}{2}(w'_{c_0} - w''_{c_0}),$$

for all $c_0 \leqslant c$.

We will first show that $h' = fg'$ and the maps

(4) $$h'_c = g'_c f_{g'(c)}, \quad c \in C,$$

form an approximate mapping $h'\colon X \to Z'$. Indeed, if $c_1 \leqslant c_2$, then there is a $b \geqslant g'(c_1), g'(c_2)$ such that

(5) $$d(g'_{c_1} q_{g'(c_1)b}, r_{c_1c_2} g'_{c_2} q_{g'(c_2)b}) \leqslant w''_{c_1}.$$

Moreover, there is an $a \geqslant fg(c_1), fg(c_2), f(b)$ such that for any $a' \geqslant a$ one has

(6) $$d(f_{g'(c_i)} p_{fg'(c_i)a'} q_{g'(c_i)b} f_b p_{f(b)a'}) \leqslant v_{g'(c_i)}, \quad i = 1, 2.$$

Note that (6) and (3) imply

(7) $$d(g'_{c_1} f_{g'(c_1)} p_{fg'(c_1)a'}, g'_{c_1} q_{g'(c_1)b} f_b p_{f(b)a'}) \leqslant \tfrac{1}{2}(w'_{c_1} - w''_{c_1}),$$

(8) $$d(r_{c_1c_2} g'_{c_2} f_{g'(c_2)} p_{fg'(c_2)a'}, r_{c_1c_2} g'_{c_2} q_{g'(c_2)b} f_b p_{f(b)a'}) \leqslant \tfrac{1}{2}(w'_{c_1} - w''_{c_1}).$$

Now (4), (7), (5) and (8) yield

$$d(h'_{c_1}p_{h'(c_1)a'}, r_{c_1c_2}h'_{c_2}p_{h'(c_2)a'}) \leqslant w'_{c_1}, \tag{9}$$

which shows that h': $X \to Y'$ is an approximate mapping.

If $f = \lim f$, by (LM),

$$d(f_{g'(c)}p_{h'(c)}, q_{g'(c)}f) \leqslant v_{g'(c)}. \tag{10}$$

Therefore, by (3) and (4),

$$d(h'_c p_{h'(c)}, g'_c q_{g'(c)} f) \leqslant \tfrac{1}{2}(w'_c - w''_c). \tag{11}$$

If $g = \lim g$, then $g' \equiv g$ implies $g = \lim g'$ (Theorem 3). Therefore, by (LM),

$$d(g'_c q_{g'(c)}, r_c g) \leqslant w''_c. \tag{12}$$

Now, (11) and (12) yield

$$d(h'_c p_{h'(c)}, r_c gf) \leqslant \tfrac{1}{2}(w'_c + w''_c) \leqslant w'_c, \tag{13}$$

which shows that indeed

$$\lim h' = gf. \tag{14}$$

DEFINITION 9. Let X, Y and $Z = (Z_c, w_c, r_{cc'}, C)$ be approximate systems, where C is cofinite. Let $w'_c > w_c$ be arbitrary admissible meshes and let $Z' = (Z_c, w'_c, r_{cc'}, C)$. The composition $[g][f]$ of equivalence classes of approximate mappings f: $X \to Y$, g: $Y \to Z$ is the equivalence class $[g'f]$ of the approximate mapping $g'f$: $X \to Z'$, where $g' \sim g$ is chosen so that $g'f$ is defined.

Remark 7. By Lemma 9 such g' exists and $\lim(g'f) = gf$, because $\lim g' = \lim g$. If $g'' \sim g$ is another representative of the class g such that $g''f$ is defined, then by Definition 8, $\lim g''f = gf$. We now conclude, by Theorem 4, that $g''f \sim gf$ and therefore, $[g''f] = [g'f]$. This shows that $[g][f]$ is well defined and $\lim([g][f]) = gf$.

For any approximate system $Y = (Y_b, v_b, q_{bb'}, B)$ we define the identity mapping 1_Y: $Y \to Y$. It is given by the identity 1_B: $B \to B$ and by the identity maps 1_b: $Y_b \to Y_b$. Condition (AM1) is an immediate consequence of (A1).

THEOREM 5. *Almost commutative systems over cofinite index sets and equivalence classes of almost commutative maps between such systems form a category denoted by* Ap-Inv.

Proof. The associativity law

$$[h]([g][f]) = ([h][g])[f] \tag{15}$$

follows from $h(gf) = (hg)f$ and Theorem 4. Similarly $[1_Y][g] = [g]$ and $[g][1_X] = [g]$ follows from $1_Y g = g$ and $g 1_X = g$ respectively.

It is a consequence of Definition 8 that lim is a functor from Ap-Inv to the category $\mathscr{C}$ of compact Hausdorff spaces and continuous mappings. By Theorem 4, $\lim[f] = \lim[f']$ implies $[f] = [f']$, which shows that lim: (Ap-Inv)$\to \mathscr{C}$ is a faithful functor.

Let (Ap-Inv)$_{\mathrm{ANR}}$ denote the full subcategory of Ap-Inv whose objects are almost commutative systems of compact ANR's over cofinite index sets. By Theorem 2, lim: (Ap-Inv)$_{\mathrm{ANR}} \to \mathscr{C}$ is a full functor.

Furthermore, every compact Hausdorff space X is homeomorphic to the limit of an inverse system X of compact polyhedra over an unbounded cofinite index set (see e.g., [8], I, § 5.2, Theorem 8 and I, § 1.2, Theorem 2). By Propositions 1 and 2, X can be viewed as an almost commutative system of compact ANR's. This shows that lim on Ap-Inv and (Ap-Inv)$_{\mathrm{ANR}}$ is a representable functor.

Summarizing, we obtain the following theorem.

THEOREM 6. *The functor* lim *is an equivalence of categories between* (Ap-Inv)$_{\mathrm{ANR}}$ *and* $\mathscr{C}$.

7. Approximate systems and shape of compact spaces. The purpose of this section is to show that approximate systems of ANR's over cofinite sets can be used to study shape of compact Hausdorff spaces in the same way in which commutative ANR-systems were used in [7] (see also [8]).

In this section we are interested in approximate systems $X = (X_a, u_a, p_{aa'}, A)$ of compact ANR's which satisfy the following homotopy conditions.

$$p_{a_1a_2}p_{a_2a_3} \simeq p_{a_1a_3}, \quad a_1 \leqslant a_2 \leqslant a_3, \tag{H}$$

$$p_{a_1a_2}p_{a_2} \simeq p_{a_1}, \quad a_1 \leqslant a_2. \tag{LH}$$

The next lemma shows how to convert an arbitrary approximate system of ANR's into one having properties (H) and (LH).

LEMMA 10. *Let* $X = (X_a, v_a, p_{aa'}, A)$ *be an approximate system of compact* ANR's. *Then there exists an ordering* $\leqslant'$ *on* A *such that* $a <' a'$ *implies* $a < a'$, $A' = (A, \leqslant')$ *is directed and* $X' = (X_a, v_a, p_{aa'}, A')$ *is an approximate system satisfying* (H) *and* (LH). *Moreover,* $X' = \lim X'$ and $X = \lim X$ *coincide and so do the natural projections* p'_a: $X' \to X_a$ *and* p_a: $X \to X_a$, $a \in A$.

Proof. For every $a \in A$ there is an $\eta_a > 0$ such that η_a-near maps into X_a are homotopic (X_a is an ANR). By property (A2), there is a function φ: $A \to A$, $\varphi(a) \geqslant a$, such that

$$d(p_{a_1a_2}p_{a_2a_3}, p_{a_1a_3}) \leqslant \eta_{a_1}, \tag{1}$$

whenever $a_3 \geqslant a_2 \geqslant \varphi(a_1)$.

We put $a_1 <' a_2$ provided $a_2 > \varphi(a_1)$ and we put $a_1 \leqslant' a_2$ provided $a_1 <' a_2$ or $a_1 = a_2$. Clearly, $a_1 <' a_2$ and $a_2 \leqslant a_3$ imply $a_1 <' a_3$. Moreover, since A is unbounded, every $a \in A$ admits an $a' \in A$ such that $a <' a'$. Then, by Proposition 5, $X' = (X_a, v_a, p_{aa'}, A')$ is an approximate system and $X' = X$, $p'_a = p_a$, $a \in A$.

Note that $a_1 <' a_2 \leqslant' a_3$ implies $a_1 \leqslant \varphi(a_1) < a_2 \leqslant a_3$ so that (1) yields (H). Moreover, (1) and Proposition 4 imply

(2) $$d(p_{a_1 a_2} p_{a_2}, p_{a_1}) \leqslant \eta_{a_1}, \quad a_1 <' a_2,$$

which yields (LH).

Let $X = (X_a, p_{aa'}, A)$ be an almost commutative system of compact ANR's with limit X and projections p_a and let (H) and (LH) be satisfied. Then the homotopy functor converts X to a system $HX = (X_a, [p_{aa'}], A)$ in the homotopy category. Moreover, the homotopy classes $[p_a]$: $X \to X_a$, $a \in A$, form a morphism Hp: $X \to HX$ of pro-HTop (for terminology and notation see [8]).

Approximate ANR-systems (over cofinite index sets) with properties (H) and (LH) can be used in shape theory just as commutative ANR-systems because of the following theorem, which generalizes [8], I, § 5.3, Theorem 9.

THEOREM 7. *Let $X = (X_a, p_{aa'}, A)$ be an almost commutative system of compact* ANR's, *which has properties* (H) *and* (LH). *Let $X = \lim X$ and let p_a: $X \to X_a$, $a \in A$, be natural projections. Then $Hp = ([p_a])$: $X \to HX = (X_a, [p_{aa'}], A)$ is an* HTop-*expansion of X.*

Proof. We must verify the following two conditions (see [8], I, § 4.1 and § 5.3, Lemma 3).

(E1) For every mapping h: $X \to P$ into a compact ANR P there exist an $a \in A$ and a mapping f: $X_a \to P$ such that $h \simeq fp_a$.

(E2) If P is a compact ANR, $a \in A$ and f, f': $X_a \to P$ are mappings such that $fp_a \simeq f'p_a$, then there exists an $a' \geqslant a$ such that $fp_{aa'} \simeq f'p_{aa'}$.

Proof of (E1). Choose $\eta > 0$ so small that η-near maps into P are homotopic By Property (R1) (Proposition 7), there exists an $a \in A$ and a mapping f: $X_a \to P$ such that $d(fp_a, h) \leqslant \eta$. Consequently, $fp_a \simeq h$.

Proof of (E2). Let H: $X \times I \to P$ be a homotopy with $H_0 = fp_a$ and $H_1 = f'p_a$. By compactness, there are numbers $0 = t_0 < t_1 < \ldots < t_k = 1$ such that

(3) $$d(H_{t_i}, H_{t_{i+1}}) \leqslant \eta/4, \quad i = 0, \ldots, k-1.$$

By property (R1) (Proposition 7), there is an $a \in A$ and there are maps f_i: $X_a \to P$ such that

(4) $$d(H_{t_i}, f_i p_a) \leqslant \eta/4, \quad i = 0, \ldots, k-1.$$

(3) and (4) imply

(5) $$d(f_i p_a, f_{i+1} p_a) < \eta, \quad i = 0, \ldots, k-1.$$

By Property (R2) (Proposition 7), there is an $a' \geqslant a$ such that

(6) $$d(f_i p_{aa'}, f_{i+1} p_{aa'}) < \eta, \quad i = 0, \ldots, k-1,$$

and therefore,

(7) $$fp_{aa'} = f_0 p_{aa'} \simeq f_1 p_{aa'} \simeq \ldots \simeq f_k p_{aa'} = f' p_{aa'}.$$

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