Finite-dimensional complement theorems in shape theory and their relation to S-duality

by

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Abstract. The finite-dimensional category isomorphism in shape theory determines a functor ϑ assigning to each isotopy class of homeomorphisms $D^n - X \to D^n - Y$, $X, Y \subset S^{n-1}$ compact, a shape isomorphism $X \to Y$. The finite-dimensional complement theorem is used to prove that the stabilization ϑ^* of ϑ is a full functor; orientation and boundary obstructions are applied to show that ϑ^* is not faithful. We specify a functor R, defined for shape isomorphisms induced by simple homotopy equivalences between compact subpolyhedra of S^{n-1}, which is a right inverse for ϑ^* and effectively controls these obstructions. R is then used to exhibit the relation between S-duality and complement theorems.

Introduction. Let us recall

1. THE FINITE-DIMENSIONAL COMPLEMENT THEOREM IN SHAPE THEORY ([4], [16], [18]). *Two compacta X, Y in the sphere S^n, satisfying suitable embedding conditions, have the same shape iff $S^n - X$, $S^n - Y$ are homeomorphic*;

2. THE S-DUALITY COMPLEMENT THEOREM ([17]). *Two compact subpolyhedra X, Y of S^n have the same stable homotopy type iff $S^n - X$, $S^n - Y$ have the same stable homotopy type.*

The striking similarity of these two results suggests that they might not be independent but rather be two partial aspects of a more complex geometric phenomenon.

Let us consider the two obvious functors ϑ and ∂ assigning to each isotopy class μ of homeomorphisms $D^{n+1} - X \to D^{n+1} - Y$, where X, Y are compacta in the boundary of the ball D^{n+1}, isomorphisms $\vartheta(\mu)\colon X \to Y$ in the shape category and $\partial(\mu)\colon S^n - X \to S^n - Y$ in the ordinary homotopy category (ϑ involves the well-known category isomorphism theorems in shape theory [3], [13]; ∂ is the "boundary" functor). All what seems to be known about ϑ, ∂ and their interrelations concerns the image of ϑ: The shape-theoretic complement theorems supply information about which shape isomorphisms occur as images of isotopy classes; cf. Proposition 4.1. The first step towards something new is to recognize ϑ and ∂ as *operator* *functors* between *operator categories* (this means that ϑ resp. ∂ are "compatible" with certain endofunctors operating on domain and range of ϑ resp. ∂; for details see Sections 1–3). These additional structures enable us to transform ϑ and ∂ into operator functors ϑ^* and ∂^* assigning to each "*stabilized*" *isotopy class* μ^*: $D^{n+1}-X\to D^{n+1}-Y$ isomorphisms $\vartheta^*(\mu^*)$: $X\to Y$ in the shape category and $\partial^*(\mu^*)$: $S^n-X\to S^n-Y$ in the stable homotopy category; cf. Section 4. In this stabilized setting we get rid of all technical conditions appearing in the finite-dimensional complement theorems: It turns out that *each* shape isomorphism $X\to Y$ between compacta in S^n is in the image of ϑ^*. However, the *boundary obstruction* given by ∂^* shows that ϑ^* is not faithful, and thus we would considerably improve our stable version of the complement theorem if we were able to specify a *section* R of ϑ^* controlling the boundary obstruction, i.e. an operator functor assigning to each shape isomorphism Φ: $X\to Y$ a stabilized isotopy class $R(\Phi)$: $D^{n+1}-X\to D^{n+1}-Y$ such that $\vartheta^*R(\Phi)=\Phi$ holds and $\partial^*R(\Phi)$ is made explicit. We do not know a complete solution of this problem but for our objective it will suffice to construct the desired $R(\Phi)$ for each Φ which is induced by a *simple homotopy equivalence* $X\to Y$ between compact subpolyhedra X, Y of S^n: This is accomplished in Theorem 4.10.

So far we have exclusively been dealing with functorial versions of complement theorems. However, once we have our partial section R of ϑ^*, it is a straightforward exercise (see Section 5) to derive Spanier–Whitehead duality in its original form [17], thereby exhibiting the intimate relationship between ϑ, ∂ and the "geometric" Spanier–Whitehead duality functor D. The point is that D factors through the stabilizations $R^{\#}$ of R and $\partial^{\#}$ of ∂^* (Theorem 5.4): For each stable homotopy equivalence α, $D(\alpha)=\partial^{\#}R^{\#}(\alpha^{-1})$.

Finally, we sketch in Section 5 how our results are related to the following theorem (which is essentially contained, though not explicitly stated, in Lima [11]; cf. also Nowak [14]): Two compacta X, $Y\subset S^n$ have the same stable shape iff S^n-X, S^n-Y have the same stable homotopy type.

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1. Preliminaries. We begin with some notation. If A is a category, we let $\operatorname{Ob}A$ denote the class of objects of A, and for any two objects X, Y of A we let $A(X,Y)$ denote the set of morphisms from X to Y. If $M\subset\operatorname{Ob}A$ is a class of objects we let $A|M$ denote the full subcategory of A with $\operatorname{Ob}(A|M)=M$. By $\operatorname{Iso}A$ we mean the subcategory of all isomorphisms in A. If A is a category such that $A=\operatorname{Iso}A$, we can identify A with its dual category A^{op}: The contravariant "dualizing" functor op: $A\to A^{\mathrm{op}}=A$ is then given by $\operatorname{op}(X)=X$ for $X\in\operatorname{Ob}A$ and $\operatorname{op}(f)=f^{-1}$ for $f\in A(X,Y)$.

We let I denote the closed unit interval $[0,1]$, J the closed interval $[-1,1]$ and $\boldsymbol{R}^n$ Euclidean n-space. The n-ball $D^n=\{(x_1,\dots,x_n)\in\boldsymbol{R}^n|\ \Sigma_{i=1}^n x_i^2\leqslant 1\}$ and the $(n-1)$-sphere $S^{n-1}=\{(x_1,\dots,x_n)\in\boldsymbol{R}^n|\ \Sigma_{i=1}^n x_i^2=1\}$ will be endowed with PL($=$ piecewise linear) structures in the natural way.

By an *operator category* we mean a system $\mathfrak{C}=(C;F_1,F_2;\mu)$ consisting of a category C, two functors F_1,F_2: $C\to C$ and a natural isomorphism μ: $F_2F_1\cong F_1F_2$ (for a much more general concept of an operator category see e.g. [1]; the definition given here should only be regarded as a convenient *terminus technicus*). Letting $\mathfrak{D}=(D;G_1,G_2;\nu)$ denote another operator category, an *operator functor* U: $\mathfrak{C}\to\mathfrak{D}$ is a functor U: $C\to D$ such that $G_1U=UF_1$, $G_2U=UF_2$ and $\nu U=U\mu$. Note that 1_C: $\mathfrak{C}\to\mathfrak{C}$ is an operator functor, and that the composition of operator functors yields again an operator functor; it is now left to the reader to give the obvious definition of an *operator category isomorphism*. $\mathfrak{C}$ is called *stable with respect to* F_i, $i=1$ or $i=2$, if F_i is a fully faithful functor; $\mathfrak{C}$ is called *stable* if it is stable with respect to both F_1 and F_2. This definition of stability possibly is somewhat naive (it would be more sophisticated to require the F_i to be auto-isomorphisms of C), but it has the advantage that *stabilizing* operator categories becomes an extremely simple procedure which does not involve any new objects (like e.g. spectra). A *stabilization* [resp. *F_i-stabilization*] of $\mathfrak{C}$ consists of an operator category $\mathfrak{C}^*=(C^*;F_1^*,F_2^*;\mu^*)$ which is stable [resp. stable with respect to F_i^*] and an operator functor λ^*: $\mathfrak{C}\to\mathfrak{C}^*$ such that the following universal property holds: For each operator category $\mathfrak{D}=(D;G_1,G_2;\nu)$ which is stable [resp. stable with respect to G_i] and for each operator functor U: $\mathfrak{C}\to\mathfrak{D}$, there is a unique operator functor U^*: $\mathfrak{C}^*\to\mathfrak{D}$ with $U=U^*\lambda^*$. By this universal property, all stabilizations [resp. F_i-stabilizations] must be isomorphic in the obvious sense. To show that F_i-stabilizations *exist*, we introduce the *canonical F_i-stabilization* $\lambda(F_i)$: $\mathfrak{C}\to\mathfrak{C}/F_i=(C/F_i;\hat{F}_1,\hat{F}_2;\hat{\mu})$ following the pattern of the stable homotopy category of Spanier and Whitehead [17]: Define $\operatorname{Ob}C/F_i=\operatorname{Ob}C$ and

$$(C/F_i)(X,Y)=\operatorname{dirlim}\{C(F_i^nX,F_i^nY)\}$$

for all X, $Y\in\operatorname{Ob}C$. Then composition of morphisms in C/F_i as well as the definitions of $\hat{F}_1,\hat{F}_2$: $C/F_i\to C/F_i$, $\hat{\mu}$: $\hat{F}_2\hat{F}_1\cong\hat{F}_1\hat{F}_2$ and $\lambda(F_i)$: $C\to C/F_i$ should be obvious. It is also left to the reader to verify the following important fact: If $\mathfrak{C}$ is already stable with respect to F_j, $j\neq i$, then $\mathfrak{C}/F_i$ is stable. As a consequence, each operator category $\mathfrak{C}$ admits a stabilization: In fact, $\lambda(\hat{F}_2)\lambda(F_1)$: $\mathfrak{C}\to(\mathfrak{C}/F_1)/\hat{F}_2$ will do, and we simply denote it by $\lambda(F_1,F_2)$: $\mathfrak{C}\to\mathfrak{C}/F_1,F_2$.

2. The basic operator categories. These are

(1) $\mathfrak{T}=(\mathrm{Ho(Top)};1,\Sigma;\mathrm{id})$: Ho(Top) denotes the ordinary homotopy category of all topological spaces; $\Sigma=\Sigma_{\mathrm{Ho(Top)}}$ denotes unreduced suspension (we adopt the convention $\Sigma\emptyset=S^0$).

(2) $\mathfrak{S}=(\mathrm{Sh};1,\Sigma;\mathrm{id})$: Sh denotes the Borsuk–Mardešić shape category of all topological spaces, given together with the shape functor S: $\mathrm{Ho(Top)}\to\mathrm{Sh}$ (cf. [2], [12]); $\Sigma=\Sigma_{\mathrm{Sh}}$ denotes the shape suspension functor which is uniquely characterized by the property $\Sigma_{\mathrm{Sh}}S=S\Sigma_{\mathrm{Ho(Top)}}$ (cf. [12]).

(3) $\mathfrak{I} = (\mathrm{Isot};\, \pi_1, \pi_2;\, \gamma)$: Isot denotes the isotopy category of locally compact spaces (whose morphisms are the isotopy classes of homeomorphisms); π_1, π_2 denote the product functors defined by

$$\pi_1(X) = X\times J,\quad \pi_2(X) = X\times(-1, 1) \text{ for the objects } X,$$
$$\pi_i(f) = f\times \mathrm{id} \text{ for the morphisms } f,\ i = 1, 2;$$

$\gamma\colon \pi_2\pi_1 \cong \pi_1\pi_2$ is induced by the rotation

$$a\colon J\times(-1, 1)\to(-1, 1)\times J,\quad a(s, t) = (t, -s).$$

(4) $\mathfrak{P} = (\mathrm{wHo}(P);\, \pi_1, \pi_2;\, \gamma)$: $\mathrm{wHo}(P)$ denotes the weak proper homotopy category of all locally compact spaces (cf. [4], [7]); π_1, π_2 denote the product functors given by the "same" definitions as above (but note that, on $\mathrm{wHo}(P)$, π_1 is naturally isomorphic to the identity); $\gamma\colon \pi_2\pi_1 \cong \pi_1\pi_2$ is again induced by the rotation a. Observe that there is a functor E: $\mathrm{Isot}\to\mathrm{wHo}(P)$ such that $E(X) = X$ for each object X and $E([h]_{\mathrm{iso}}) = [h]_{\mathrm{wp}}$ for each $[h]_{\mathrm{iso}}\in \mathrm{Isot}(X, Y)$, where $[\ ]_{\mathrm{iso}}$ denotes isotopy class and $[\ ]_{\mathrm{wp}}$ denotes weak proper homotopy class (see [6] to check that this makes sense).

(5) $\mathfrak{T}' = (\mathrm{Ho(Top)}';\, \Sigma, 1;\, \mathrm{id})$ with

$$\mathrm{Ho(Top)}' \equiv \mathrm{Ho(Top)}.$$

For our geometric purposes, however, the five categories $C = \mathrm{Ho(Top)}$, Sh, Isot, $\mathrm{wHo}(P)$, $\mathrm{Ho(Top)}'$ are too large; we are actually interested in certain *subcategories* C^+ of C. Let subsets $K_n(C)$ of $\mathrm{Ob}\, C$ be defined by $K_n(\mathrm{Ho(Top)}) = K_n(\mathrm{Sh}) = \{X\subset S^{n-1} \text{ compact}\}$, $K_n(\mathrm{Isot}) = K_n(\mathrm{wHo}(P)) = \{D^n - X\mid X\subset S^{n-1} \text{ compact}\}$ and $K_n(\mathrm{Ho(Top)}') = \{S^{n-1}-X\mid X\subset S^{n-1} \text{ compact}\}$. Then the $C_n = C|K_n(C)$ are full subcategories of C, and we take $C^+ = \bigcup_{n=1}^{\infty} C_n$. Unfortunately, none of the operator categories

$$(C;\, F_1, F_2;\, \mu) = \mathfrak{T}, \mathfrak{S}, \mathfrak{I}, \mathfrak{P}, \mathfrak{T}'$$

satisfies $F_i(C^+)\subset C^+$ for both $i = 1$ and $i = 2$. However, for each object X of C_n, we shall supply an object $F_i^+(X)$ of C_{n+1} together with a "*canonical*" C-isomorphism $\alpha_{i,X}\colon F_i^+(X)\to F_i(X)$. A straightforward construction (left to the reader) yields then functors $F_i^+\colon C^+\to C^+\subset C$ naturally isomorphic to $F_i|C^+$ as well as a natural isomorphism $\mu^+\colon F_2^+F_1^+ \cong F_1^+F_2^+$, i.e. an operator category

$$\mathfrak{C}^+ = (C^+;\, F_1^+, F_2^+;\, \mu^+)$$

which is called the *reduction of* $(C;\, F_1, F_2;\, \mu)$ *to* C^+ *by the* $\alpha_{1,X}$ and $\alpha_{2,X}$.

(1) For each object X of $\mathrm{Ho(Top)}_n$, we set

$$1^+(X) = X\times\{0\},$$
$$\Sigma^+(X) = \{(x_1, \ldots, x_{n+1})\in S^n\mid |x_{n+1}| = 1, \text{ or } |x_{n+1}|<1 \text{ and } (1-x_{n+1}^2)^{-1/2}(x_1, \ldots, x_n)\in X\}.$$

The obvious homeomorphisms $g_X\colon 1^+(X)\to X$ and $h_X\colon \Sigma^+(X)\to \Sigma X$ induce Ho(Top)-isomorphisms

$$a_X\colon 1^+(X)\to X \quad\text{and}\quad b_X\colon \Sigma^+(X)\to \Sigma X.$$

Let

$$\mathfrak{T}^+ = (\mathrm{Ho(Top)}^+;\, i = 1^+,\ \sigma = \Sigma^+;\ \delta = \mathrm{id}^+)$$

be the reduction of $\mathfrak{T}$ to $\mathrm{Ho(Top)}^+$ by the a_X and b_X.

(2) Similarly, let $\mathfrak{S}^+ = (\mathrm{Sh}^+;\, i, \sigma;\, \delta)$ be the reduction of $\mathfrak{S}$ to Sh^+ by the

$$S(a_X)\colon 1^+(X)\to X \quad\text{and}\quad S(b_X)\colon \Sigma^+(X)\to\Sigma X.$$

(3) For each object $D^n - X$ of Isot_n, we set

$$\pi_1^+(D^n-X) = D^{n+1}-iX \quad\text{and}\quad \pi_2^+(D^n-X) = D^{n+1}-\sigma X.$$

The obvious homeomorphism

$$f_X\colon D^{n+1}-\sigma X\to(D^n-X)\times(-1, 1)$$

induces an Isot-isomorphism

$$v_X\colon \pi_2^+(D^n-X)\to\pi_2(D^n-X).$$

By (2.1) below there also exists a natural *isotopy class* of homeomorphisms

$$e_X\colon D^{n+1}-iX\to(D^n-X)\times J,$$

i.e. a "canonical" Isot-isomorphism

$$u_X\colon \pi_1^+(D^n-X)\to\pi_1(D^n-X).$$

Let $\mathfrak{I}^+ = (\mathrm{Isot}^+;\, \pi_1^+, \pi_2^+;\, \gamma^+)$ be the reduction of $\mathfrak{I}$ to Isot^+ by the u_X and v_X.

(2.1) Lemma. *Let* $X\subset S^{n-1}$ *be compact,* U *be a neighbourhood of* X *in* D^n *and* $G\colon D^n\times J\to D^{n+1}$ *be a homeomorphism with*

$$G|D^n\times\{0\} = \mathrm{id} \quad\textit{and}\quad G(D^n\times I) = D_+^{n+1} = D^{n+1}\cap(\boldsymbol{R}^n\times I).$$

(a) *There is a map of triples*

$$h\colon (D^n\times J,\ X\times J,\ (D^n-X)\times J)\to(D^{n+1},\ X\times\{0\},\ D^{n+1}-X\times\{0\})$$

with the following properties:

(i) $h\colon X\times J\to X\times\{0\}$ *is the natural retraction.*
(ii) $h\colon (D^n-X)\times J\to D^{n+1}-X\times\{0\}$ *is a homeomorphism.*
(iii) $h|D^n\times\{0\} = \mathrm{id}$.
(iv) $h(D^n\times I) = D_+^{n+1}$.
(v) $h|(D^n-U)\times J = G|(D^n-U)\times J$.

(b) *For any two maps* h_0, h_1 *satisfying* (i)–(iv) *there exists an isotopy* $H\colon D^{n+1}\times I\to D^{n+1}$, *fixed on* $D^n\times\{0\}$, *such that* $H_0 = 1$ *and* $H_1h_0 = h_1$ (*with* $H_t(x) = H(x, t)$).

In particular, there exists a homeomorphism

$$e_X\colon D^{n+1}-iX\to(D^n-X)\times J$$

such that e_X^{-1} *is the restriction of a map h satisfying* (i)–(iv); e_X *is unique up to isotopy.*

Proof. (a) A simple geometric argument shows that there is a homeomorphism $g\colon D^n\times J\to D^n\times J$ such that

$$g|(D^n-U)\times J\cup D^n\times\{0\} = \mathrm{id} \quad\text{and}\quad g(x,t) = (x, t/2)$$

for each $x\in X$. Choose a map $v\colon D^n\to I$ with $X = v^{-1}(0)$ and $D^n-U\subset v^{-1}(1)$. Define $r\colon D^n\times J\to D^n\times J$ by

$$r(x,t) = \big(x, tv(x)\big) \quad\text{for}\quad |t|\leqslant\tfrac{1}{2}$$

and

$$r(x,t) = \Big(x, \operatorname{sgn}(t)\big((v(x)/2)\big)+\big(1-(v(x)/2)\big)(2|t|-1)\Big) \quad\text{for } |t|\geqslant\tfrac{1}{2}.$$

Then $h = Grg$ is the desired map.

(b) Consider the homeomorphism $k\colon D^{n+1}-X\times\{0\}\to D^{n+1}-X\times\{0\}$ defined by $k(\xi) = h_1h_0^{-1}(\xi)$. It is easy to show that k extends to a map $K\colon D^{n+1}\to D^{n+1}$. Obviously K is a homeomorphism with $K|D^n\times\{0\} = \mathrm{id}$ and $K(D_+^{n+1}) = D_+^{n+1}$. There is an isotopy $H\colon D^{n+1}\times I\to D^{n+1}$, fixed on $D^n\times\{0\}$, such that $H_0 = 1$ and $H_1 = K$. This implies $H_1h_0 = h_1$.

(4) Similarly, let

$$\mathfrak{P}^+ = \big(\mathrm{wHo}(P)^+;\pi_1^+,\pi_2^+;\gamma^+\big)$$

be the reduction of $\mathfrak{P}$ to $\mathrm{wHo}(P^+)$ by the

$$E(u_X)\colon \pi_1^+(D^n-X)\to\pi_1(D^n-X) \quad\text{and}\quad E(v_X)\colon \pi_2^+(D^n-X)\to\pi_2(D^n-X).$$

It is easy to verify $E(u_X) = [q_X]_{\mathrm{wp}}^{-1}[p_X]_{\mathrm{wp}}$ where $p_X\colon D^{n+1}-iX\to D^n-X$ and $q_X\colon (D^n-X)\times J\to D^n-X$ are the projections (which are proper homotopy equivalences).

(5) We begin with some notation needed below. For any nonempty space Z, we have the natural quotient map $q_Z\colon Z\times J\to\Sigma Z$; given $A\subset Z$, we set $A' = q_Z(A\times\{0\})\subset\Sigma Z$. There is a natural injection

$$j_A\colon \Sigma(Z-A)\to(\Sigma Z)-A'.$$

Now, for each object $S^{n-1}-X$ of $(\mathrm{Ho(Top)}')_n$, we set

$$\Sigma^+(S^{n-1}-X) = S^n-iX \quad\text{and}\quad 1^+(S^{n-1}-X) = S^n-\sigma X.$$

There are obvious homeomorphisms

$$\xi_X\colon S^n-iX\to(\Sigma S^{n-1})-X' \quad\text{and}\quad \eta_X\colon S^n-\sigma X\to(S^{n-1}-X)\times(-1,1).$$

The projection

$$r_X\colon (S^{n-1}-X)\times(-1,1)\to S^{n-1}-X$$

is a homotopy equivalence, and (2.2) below shows that also the injection $j_X\colon \Sigma(S^{n-1}-X)\to(\Sigma S^{n-1})-X'$ is one. (N.B.: j_X is not an embedding unless $X = \emptyset$, S^{n-1}). Hence, we have Ho(Top)-isomorphisms

$$\varphi_X = [j_X]^{-1}[\xi_X]\colon \Sigma^+(S^{n-1}-X)\to\Sigma(S^{n-1}-X) \quad\text{and}$$

$$\psi_X = [r_X\eta_X]\colon 1^+(S^{n-1}-X)\to S^{n-1}-X.$$

Let

$$(\mathfrak{T}')^+ = \big((\mathrm{Ho(Top)}')^+;\Sigma^+,1^+;\varrho = \mathrm{id}^+\big)$$

be the reduction of $\mathfrak{T}'$ to $(\mathrm{Ho(Top)}')^+$ by the φ_X and ψ_X.

(2.2) Lemma. *For each nonempty space Z and each zero-set* $A\subset Z$ (*i.e.* $A = \omega^{-1}(0)$ *for some map* $\omega\colon Z\to I$) $j_A\colon \Sigma(Z-A)\to(\Sigma Z)-A'$ *is a homotopy equivalence.*

Proof. We only consider $A\neq Z$ since $A = Z$ is trivial. Define

$$\varphi\colon J\times I-\{(0,0)\}\to J$$

by $\varphi(u,v) = u/v$ for $|u|\leqslant v$ and $\varphi(u,v) = \operatorname{sgn}(u)$ for $|u|\geqslant v$. Choose $\omega\colon Z\to I$ with $\omega^{-1}(0) = A$. Define a map

$$F\colon (Z\times J-A\times\{0\})\times I\to Z\times J-A\times\{0\}$$

$$\text{by } F(z,s,t) = \big(z,\varphi(s,(1-t)\omega(z)+t)\big).$$

Obviously F restricts to a map

$$F^*\colon (Z-A)\times J\times I\to(Z-A)\times J.$$

Define

$$\varrho\colon Z\times J-A\times\{0\}\to\Sigma(Z-A) \quad\text{by}$$

$$\varrho(z,s) = q_{Z-A}F^*(z,s,0) \quad\text{for } z\in Z-A \quad\text{and}$$

$$\varrho(z,s) = q_{Z-A}(Z-A)\times\{\operatorname{sgn}(s)\} \quad\text{for } z\in A.$$

It is readily checked that ϱ is continuous and maps both $Z\times\{1\}$, $Z\times\{-1\}$ to points. Hence, there is a unique map $r\colon \Sigma Z-A'\to\Sigma(Z-A)$ such that $\varrho = r\varrho_Z$ where

$$\varrho_Z\colon Z\times J-A\times\{0\}\to\Sigma Z-A'$$

is the quotient map induced by q_Z. The maps F and F^* induce homotopies $j_Ar\simeq 1$ and $rj_A\simeq 1$, respectively.

(2.3) Remark. In the intuitive sense, the operators $F_i^+\colon C^+\to C^+$ seem to be "less manageable" than the original $F_i\colon C\to C$. Therefore, one may ask for an F_i^+-stabilization of $\mathfrak{C}^+$ which makes *transparent* what the stabilized morphisms look like. Let

$$\lambda(F_i)\colon \mathfrak{C}\to\mathfrak{C}/F_i = (C/F_i;\hat F_1,\hat F_2;\hat\mu)$$

be the canonical F_i-stabilization. Define a subcategory of C/F_i by

$$(C/F_i^{+}) = \bigcup_{n=1}^{\infty} (C/F_n)|K_n(C)$$

and let $(\mathfrak{C}/F_i)^+$ denote the reduction of $\mathfrak{C}/F_i$ to $(C/F_i)^+$ by the $\lambda(F_i)(\alpha_{1,X})$ and $\lambda(F_i)(\alpha_{2,X})$. Then $\lambda(F_i)$ induces a functor $\lambda^+(F_i)\colon \mathfrak{C}^+ \to (\mathfrak{C}/F_i)^+$ which is readily seen to be an F_i^+-stabilization, called the *regular* one.

3. The basic operator functors. It is obvious that the shape functor S: Ho(Top) → → Sh and the functor E: Isot → wHo(P) restrict to operator functors

$$S\colon \mathfrak{T}^+ \to \mathfrak{S}^+$$

and

$$E\colon \mathfrak{J}^+ \to \mathfrak{P}^+ .$$

Let us next recall the finite dimensional category isomorphism in shape theory. We use the description given in [13], restated here in a slightly different form.

(3.1) Proposition (cf. [13]). *There exists a unique functor* T: wHo$(P)^+ \to$ Sh$^+$ *satisfying the following conditions*:

(T1) $T(D^n - X) = X$ *for each object* $D^n - X$;

(T2) *If* $f\colon D^n - X \to D^n - Y$ *is a uniformly continuous proper map, and* $f'\colon X \to Y$ *denotes the unique map that can be pieced together with* f *to a continuous* $\bar{f}\colon D^n \to D^n$, *then* $T([f]_{\mathrm{wp}}) = S([f'])$.

The functor T *is an isomorphism of categories.*

(3.2) Proposition. $T\colon \mathfrak{P}^+ \to \mathfrak{S}^+$ *is an operator functor* (*hence, an operator category isomorphism*).

Proof. $T\pi_1^+ = iT$: Since i is a full embedding, there exists a unique functor R: wHo$(P)^+ \to$ Sh$^+$ such that $iR = T\pi_1^+$. Obviously, $R(D^n - X) = X$ for each object $D^n - X$. Let $f\colon D^n - X \to D^n - Y$ be a uniformly continuous proper map. We have

$$\pi_1^+([f]_{\mathrm{wp}}) = E(u_Y)^{-1}\pi_1([f]_{\mathrm{wp}})E(u_X) = [i_Y f p_X]_{\mathrm{wp}},$$

where $i_Y\colon D^n - Y \to D^{n+1} - iY$ is given by $i_Y(y) = (y, 0)$. i_Y and p_X are uniformly continuous proper maps, and

$$(i_Y f p_X)' = (i_Y)' f' (p_X)' = g_Y^{-1} f' g_X .$$

Thus

$$iR([f]_{\mathrm{wp}}) = T\pi_1^+([f]_{\mathrm{wp}}) = S([g_Y^{-1} f' g_X]) = S(a_Y)^{-1} S([f']) S(a_X) = iS([f']),$$

i.e. $R([f]_{\mathrm{wp}}) = S([f'])$. It follows from (3.1) that $R = T$.

$T\pi_2^+ = \sigma T$: Let $f\colon D^n - X \to D^n - Y$ be a uniformly continuous proper map. Computations as above yield

$$T\pi_2^+([f]_{\mathrm{wp}}) = S([h_Y^{-1}\Sigma(f')h_X]) = S(b_Y)^{-1}S(\Sigma[f'])S(b_X) = S(b_Y)^{-1}\Sigma S([f'])S(b_X) = S(b_Y)^{-1}\Sigma T([f]_{\mathrm{wp}})S(b_X) = \sigma T([f]_{\mathrm{wp}}).$$

Given a map $r\colon X \to Y$, one can easily construct a uniformly continuous proper map $R\colon D^n - X \to D^n - Y$ such that $R' = r$, i.e. $T([R]_{\mathrm{wp}}) = S([r])$ (cf. [13], [15]). This implies $S\sigma = \sigma S = T\pi_2^+ T^{-1} S$. Now let $\psi \in \mathrm{Sh}(X, Y)$. Choose a Ho(Top)-expansion $\underline{p}\colon Y \to \underline{Y}$ such that

$$\underline{Y} = \{Y_m, \lambda_m \in \mathrm{Ho(Top)}(Y_{m+1}, Y_m), m \in N\}$$

with compact polyhedra Y_m in S^{n-1} (see [12]). Note that $\underline{p}$ is a morphism in pro-Ho(Top) (cf. [7], [12]), i.e. consists of $p_m \in \mathrm{Ho(Top)}(Y, Y_m)$, $m \in N$. There is a unique $\underline{q} = \{q_m\}\colon X \to \underline{Y}$ in pro-Ho(Top) such that $S(\underline{q}) = S(\underline{p})\psi$, i.e. $S(q_m) = S(p_m)\psi$ for each m. Then $\Sigma(\psi)$ is the unique shaping such that $S\Sigma(q_m) = S\Sigma(p_m) \circ \Sigma\psi$ for all m (cf. [12]); hence $\sigma(\psi)$ is the unique shaping such that

$$S\sigma(q_m) = S\sigma(p_m) \circ \sigma(\psi) \quad \text{for all } m.$$

On the other hand,

$$S\sigma(p_m) \circ T\pi_2^+ T^{-1}(\psi) = T\pi_2^+ T^{-1} S(p_m) \circ T\pi_2^+ T^{-1}(\psi) = T\pi_2^+ T^{-1}(S(p_m)\psi) = T\pi_2^+ T^{-1}(q_m) = S\sigma(q_m),$$

implying $T\pi_2^+ T^{-1}(\psi) = \sigma(\psi)$. Thus, $T\pi_2^+ = \sigma T$.

$\partial T = T\gamma^+$: It is immediate from the construction of ∂ and γ^+ that $d_X']$ and $\gamma^+_{D^n - X} = [d_X]_{\mathrm{wp}}$, where both $d_X'\colon \sigma iX \to i\sigma X$ and $d_X\colon D^{n+2} - \sigma iX \to \Gamma$ $- \sigma X$ are given as restrictions of the rotation

$$d\colon D^{n+2} \to D^{n+2},\ d(x_1, \ldots, x_{n+2}) = (x_1, \ldots, x_n, x_{n+2}, -x_{n+1}).$$

Thus,

$$(T\gamma^+)_{D^n - X} = T(\gamma^+_{D^n - X}) = \delta_X = \delta_{T(D^n - X)} = (\delta T)_{D^n - X}.$$

Finally, we have the "boundary" functor

$$\partial\colon \mathrm{Isot}^+ \to (\mathrm{Ho(Top)}')^+ ,$$

which is given by $\partial(D^n - X) = S^{n-1} - X$ = boundary of the manifold $D^n - X$, and $\partial([h]_{\mathrm{iso}}) = [\partial h]$, where $[h]_{\mathrm{iso}} \in \mathrm{Isot}(D^n - X, D^n - Y)$ and $\partial h\colon S^{n-1} - X \to S^{n-1} - Y$ is the homeomorphism induced by h.

(3.3) Proposition. $\partial\colon \mathfrak{J}^+ \to (\mathfrak{T}')^+$ *is an operator functor.*

Proof. $\partial\pi_1^+ = \Sigma^+\partial$: For each homeomorphism $h\colon D^n - X \to D^n - Y$ we have $\partial\pi_1^+([h]_{\mathrm{iso}}) = [F]$, where $F = \partial(e_Y^{-1}(h \times 1_J)e_X)\colon S^n - iX \to S^n - iY$. Obviously F is an extension of $f = \partial h\colon S^{n-1} - X \to S^{n-1} - Y$ such that $F(S^n_\pm) \subset S^n_\pm$ (with

$S^n_{\pm} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid \operatorname{sgn}(x_{n+1}) = \pm 1\})$. It is easy to see that $\xi_Y F \xi_X^{-1} j_X \simeq j_Y \Sigma(f)$ (cf. Section 2 for j_Z and ξ_Z); this implies

$$[F] = \varphi_Y^{-1}[\Sigma(f)]\varphi_X = \Sigma^+[\partial h] = \Sigma^+\partial([h]_{\mathrm{iso}}).$$

$\partial\pi_2^+ = 1^+\partial$: For each homeomorphism $h\colon D^n - X \to D^n - Y$ we have $\partial\pi_2^+([h]_{\mathrm{iso}}) = [G]$, where

$$G = \partial(f_Y^{-1}(h \times 1_{(-1,1)})f_X)\colon S^n - \sigma X \to S^n - \sigma Y, \quad \text{and}$$

$$1^+\partial([h]_{\mathrm{iso}}) = 1^+([\partial h]) = \psi_Y^{-1}[\partial h]\psi_X.$$

It therefore suffices to show that

$$[r_Y \eta_Y G] = \psi_Y[G] = [\partial h]\psi_X = [(\partial h) r_X \eta_X].$$

The maps $r_Y\eta_Y G$, $(\partial h) r_X \eta_X\colon S^n - \sigma X \to S^{n-1} - Y$ agree on $A = (S^{n-1} - X) \times \{0\}$; since A is a strong deformation retract of $S^n - \sigma X$, they are homotopic.

$\varrho\partial = \partial\gamma^+$: It is again immediate from the construction of ϱ and γ^+ that $\varrho_{S^{n-1}-X} = [\partial d_X]$ and $\gamma^+_{D^n-X} = [d_X]_{\mathrm{iso}}$ with d_X as in the proof of (3.2). Thus,

$$(\partial\gamma^+)_{D^n-X} = \partial(\gamma^+_{D^n-X}) = \varrho_{S^{n-1}-X} = (\varrho\partial)_{D^n-X}.$$

4. Complement theorems. The operator functor $T \circ E\colon \mathfrak{I}^+ \to \mathfrak{S}^+$ induces an operator functor $\vartheta\colon \mathfrak{I}^+ \to (\mathrm{IsoSh}^+; i, \sigma; \delta) = \mathrm{Iso}\,\mathfrak{S}^+$ which gives a *functorial* version of one half of the finite-dimensional complement theorem [16]. The other half of this complement theorem can be restated as follows:

(4.1) Proposition. *Let* $X_1, X_2 \in \mathrm{Ob\,Sh}_n$, $n \geq 6$. *Assume that the fundamental dimension* $\mathrm{Fd}\,X_j$ (cf. [2]) *satisfies* $2\,\mathrm{Fd}\,X_j + 3 \leq n$, *and that* $X_j \subset S^{n-1}$ *satisfies the inessential loops condition* ILC (cf. [18]). *Then, for each* $\Phi \in \mathrm{Iso\,Sh}(X_1, X_2)$, *there exists a homeomorphism* $h\colon D^n - X_1 \to D^n - X_2$ *such that* $\vartheta([h]_{\mathrm{iso}}) = \Phi$.

Proof. For $\Phi \in \mathrm{Iso\,Sh}(X_1, X_2)$, $T^{-1}(\Phi) \in \mathrm{wHo}(P)(D^n - X_1, D^n - X_2)$ is a $\mathrm{wHo}(P)$-isomorphism. By [8], $T^{-1}(\Phi) = [f]_{\mathrm{wp}}$ for some proper homotopy equivalence $f\colon D^n - X_1 \to D^n - X_2$. It follows from [18] that for each neighbourhood U of X_j in S^{n-1}, the inclusion $U - X_j \to U$ is a 1-equivalence. Careful examination of [16] shows then that we can construct a homeomorphism $h\colon D^n - X_1 \to D^n - X_2$ properly homotopic to f.

In addition to (4.1) it should be emphasized that

(4.2) ϑ *is not a full functor.*

Moreover we have

(4.3) ϑ *is not a faithful functor.*

The easiest way to check (4.3) leads to the *orientation obstruction*. Consider the category OR which has one object $*$ and two morphisms denoted by 1 and -1 (with the obvious composition). We define a functor

$$\omega\colon \mathrm{Isot}^+ \to \mathrm{OR} \text{ by } \omega([h]_{\mathrm{iso}}) = 1 \quad \text{iff} \quad h\colon D^n - X \to D^n - Y$$

is an orientation preserving homeomorphism.

For example, let X be any compact subset of S^{n-1}. Define the reflection map $R_X\colon D^{n+1} - X \times \{0\} \to D^{n+1} - X \times \{0\}$ by $R_X(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$. Then $\vartheta([R_X]_{\mathrm{iso}}) = 1$, but $\omega([R_X]_{\mathrm{iso}}) = -1$. Moreover,

(4.4) $\omega\colon \mathfrak{I}^+ \to (\mathrm{OR}; 1, 1; \mathrm{id}) = \mathfrak{OR}$ *is an operator functor.*

Another way to check (4.3) is to use the *boundary obstruction*. For example, let $X = S^{n-2} \times \{0\} \subset S^{n-1}$ $(n \geq 2)$. Then $\vartheta([R_X]_{\mathrm{iso}}) = 1$, but $\partial([R_X]_{\mathrm{iso}}) \neq 1$.

The following elementary examples show that orientation and boundary obstructions are independent.

(4.5) For $X = Y = *$ (n arbitrary), ∂ does not detect different isotopy classes but ω does.

(4.6) Define a homeomorphism τ of $\boldsymbol{R}^3 = \boldsymbol{C} \times \boldsymbol{R}$ onto itself by

$$\tau(re^{it}, s) = \begin{cases} (re^{i(t+(2s+1)\pi)}, s) & |s| \leq 1/2, \\ (re^{it}, s) & |s| \geq 1/2. \end{cases}$$

Let $X \subset S^2$ consist of the four points $\pm(0, 0, 1)$ and $\pm\sqrt{2}/2\,(0, 1, 1)$. Then τ restricts to a homeomorphism $h\colon D^3 - X \to D^3 - X$. Let $\Phi = [h]_{\mathrm{iso}}$. It is readily verified that $\vartheta(\Phi) = 1$ and $\omega(\Phi) = 1$, but $\partial(\Phi) \neq 1$. The isotopy classes

$$\Phi_r = (\pi_2^+)^r(\Phi)\colon D^{3+r} - \sigma^r X \to D^{3+r} - \sigma^r X,\ r \geq 1,$$

provide examples for the same phenomenon in higher dimensions.

The fact that ϑ is not full means that Isot^+ is too small for an "unrestricted" complement theorem: There are not enough homeomorphisms $h\colon D^n - X \to D^n - Y$. To remedy this defect, we simply stabilize $\pi_1^+\colon \mathrm{Isot}^+ \to \mathrm{Isot}^+$. Letting

$$\lambda^+(\pi_1)\colon \mathfrak{I}^+ \to (\mathfrak{I}/\pi_1)^+ = ((\mathrm{Isot}/\pi_1)^+; \pi_1^*, \pi_2^*; \gamma^*)$$

denote the regular π_1^+-stabilization, we obtain operator functors

$$\vartheta^*\colon (\mathfrak{I}/\pi_1)^+ \to \mathrm{Iso}\,\mathfrak{S}^+$$

and

$$\omega^*\colon (\mathfrak{I}/\pi_1)^+ \to \mathfrak{OR}$$

characterized by $\vartheta^*\lambda^+(\pi_1) = \vartheta$ and $\omega^*\lambda^+(\pi_1) = \omega$. Moreover, ∂ induces an operator functor

$$\partial^*\colon (\mathfrak{I}/\pi_1)^+ \to (\mathfrak{T}'/\Sigma)^+$$

characterized by $\partial^*\lambda^+(\pi_1) = \lambda^+(\Sigma)\partial$, where

$$\lambda^+(\Sigma)\colon (\mathfrak{T}')^+ \to (\mathfrak{T}'/\Sigma)^+ = ((\mathrm{Ho(Top)}'/\Sigma)^+; \Sigma^*, 1^*; \varrho^*)$$

is the regular Σ^+-stabilization. Recall that $(\mathrm{Ho(Top)}'/\Sigma)^+$ is a subcategory of the stable homotopy category $\mathrm{Ho(Top)}'/\Sigma \equiv \mathrm{Ho(Top)}/\Sigma$.

(4.8) Proposition. ϑ^* *is a full functor.*

Proof. Let $X_1, X_2 \subset S^{n-1}$ be compact and $\Phi \in \operatorname{IsoSh}(X_1, X_2)$. Consider $\overline{X}_j = i^{n+4}(X_j) \subset S^{(2n+4)-1}$. Then $2n+4 \geqslant 6$ and $2\operatorname{Fd}\overline{X}_j+3 < 2n+4$ Since $\overline{X}_j$ lies in a $(n-1)$-dimensional submanifold of S^{2n+3}, $\overline{X}_j \subset S^{2n+3}$ satisfies ILC. By (4.1), there exists $\varphi \in \operatorname{Isot}(D^{2n+4}-\overline{X}_1, D^{2n+4}-\overline{X}_2)$ such that $\vartheta(\varphi) = i^{n+4}(\Phi)$, i.e. $\vartheta^*\lambda^+(\pi_1)(\varphi) = i^{n+4}(\Phi)$. Since π_1^* is a full embedding, there exists

$$\varphi' \in (\operatorname{Isot}/\pi_1)(D^n-X_1, D^n-X_2)$$

such that $(\pi_1^*)^{n+4}(\varphi') = \lambda^+(\pi_1)(\varphi)$. Then $\vartheta^*(\varphi') = \Phi$.

Again, orientation obstruction ω^* and boundary obstruction ∂^* may be used to show

(4.9) *ϑ^* is not a faithful functor.*

With respect to (4.9), the functorial complement theorem (4.8) is not completely satisfactory — it would be more desirable to have a *section* of ϑ^* (i.e. a functor R: $\operatorname{IsoSh}^+ \to (\operatorname{Isot}/\pi_1)^+$ such that $\vartheta^* R = 1$) *controlling orientation and boundary obstruction* in the sense that the functors $\omega^* R$ and $\partial^* R$ should be explicitly known. For the purpose of this paper a partial solution of the section problem will be sufficient. Let Si denote the subcategory of Ho(Top) whose objects are the compact polyhedra and whose morphisms are the homotopy classes of simple homotopy equivalences (cf. [5], [20]), let $P(S^{n-1})$ be the set of compact subpolyhedra $X \subset S^{n-1}$, and let $\mathrm{Si}^+ = \bigcup_{n=1}^{\infty} \mathrm{Si}|P(S^{n-1})$. Then $\mathfrak{Si}^+ = (\mathrm{Si}^+; i, \sigma; \delta)$ is an operator subcategory of $\mathfrak{T}^+$, and the shape functor restricts to a faithful functor S: $\mathfrak{Si}^+ \to \operatorname{Iso}\mathfrak{S}^+$.

(4.10) THEOREM. *There exists an operator functor R: $\mathfrak{Si}^+ \to (\mathfrak{I}/\pi_1)^+$ having the following properties:*

(R1) $\vartheta^* R = S$;

(R2) $\omega^* R(\varphi) = 1$ *for each morphism* φ;

(R3) *If X, Y are compact subpolyhedra of S^{n-1} with $Y \subset X$, and if the inclusion j: $Y \to X$ is a simple homotopy equivalence, then $\partial^* R([j]^{-1})$ is the stable homotopy class of the inclusion $S^{n-1}-X \to S^{n-1}-Y$.*

Proof. If a polyhedron A collapses simplicially to a subpolyhedron A_0, we let $c(A, A_0)$ denote the homotopy class of any retraction r: $A \to A_0$.

(I) Let X be a subpolyhedron of S^{n-1} which collapses simplicially to a subpolyhedron $Y \subset X$. Choose a regular neighbourhood N of X in D^n (meeting the boundary regularly, cf. [10]) and a homeomorphism h: $D^n-X \to D^n-Y$ such that $h|D^n-\operatorname{Int}N = \mathrm{id}$, where "Int" denotes the interior of subspaces (recall that $N-X \approx (\operatorname{Bd}N)\times[0, 1) \approx N-Y$, where "Bd" denotes the boundary of subspaces). The isotopy class of h does not depend on the above choices, and we define

$$\varrho(c(X, Y)) = [h]_{\mathrm{iso}} \in \operatorname{Isot}(D^n-X, D^n-Y).$$

By construction

(4.11) $\omega\varrho(c(X, Y)) = 1$.

Given $f \in c(X, Y)$, we use Borsuk's homotopy extension theorem to extend f to a map f': $N \to N$ such that $f'|\operatorname{Bd}N = \mathrm{id}$; moreover, since $Y \subset \partial N$ is a Z-set in N (cf. [13], [15]) we can assume $f'(N-X) \subset N-Y$. Next we extend f' by the identity to f'': $D^n \to D^n$. Then f'' induces a uniformly continuous proper map F: $D^n-X \to D^n-Y$ which is properly homotopic to h, i.e. $E([h]_{\mathrm{iso}}) = [F]_{\mathrm{wp}}$. By (3.1), $T([F]_{\mathrm{wp}}) = S([f])$; thus

(4.12) $\vartheta\varrho(c(X, Y)) = S(c(X, Y))$.

We have $\partial\varrho(c(X, Y)) = [\partial h] \in \operatorname{Ho}(\mathrm{Top})(S^{n-1}-X, S^{n-1}-Y)$; ∂h is homotopic to the inclusion $S^{n-1}-X \to S^{n-1}-Y$ because $N \cap S^{n-1}$ is a regular neighbourhood of X in S^{n-1}. Hence

(4.13) $\partial\varrho(c(X, Y))$ is the homotopy class of the inclusion $S^{n-1}-X \to S^{n-1}-Y$.

Let M be a regular neighbourhood of $X \times \{0\}$ in D^{n+1}. Choose a homeomorphism G: $D^n \times J \to D^{n+1}$ such that $G|D^n \times \{0\} = \mathrm{id}$, $G(D^n \times I) \subset D_+^{n+1}$ and $G(X \times J) \subset M$. Then choose a regular neighbourhood N_0 of X in D^n such that $G(N_0 \times J) \subset M$ and a homeomorphism h_0: $D^n-X \to D^n-Y$ such that $h_0|D^n-\operatorname{Int}N_0 = \mathrm{id}$. For $Z = X, Y$, we use (2.1) to represent

$$u_Z \in \operatorname{Isot}(D^{n+1}-Z\times\{0\}, (D^n-Z)\times J)$$

by a homeomorphism e_Z such that $e_Z^{-1}|(D^n-\operatorname{Int}N_0)\times J = G|(D^n-\operatorname{Int}N_0)\times J$. Then $\pi_1^+\varrho(c(X, Y)) = [H]_{\mathrm{iso}}$ where $H = e_Y^{-1}(h_0\times 1_J)e_X$. By construction $H|D^{n+1}-\operatorname{Int}M = \mathrm{id}$. Thus

(4.14) $\pi_1^+(c(X, Y)) = \varrho(c(iX, iY)) = \varrho i(c(X, Y))$.

Moreover, σX is a subpolyhedron of D^{n+1} which collapses simplicially to the subpolyhedron σY of σX; we have $c(\sigma X, \sigma Y) = \sigma c(X, Y)$. Choose regular neighbourhoods M of σX in D^{n+1} and N_0 of X in D^n such that $N_0\times(-1, 1) \subset f_X(M-\sigma\emptyset)$, and choose a homeomorphism h_0: $D^n-X \to D^n-Y$ such that $h_0|D^n-\operatorname{Int}N_0 = \mathrm{id}$. Then $\pi_2^+\varrho(c(X, Y)) = [G]_{\mathrm{iso}}$ where $G = f_Y^{-1}(h_0\times 1_{(-1,1)})f_X$. By construction, $G|D^{n+1}-\operatorname{Int}M = \mathrm{id}$. Thus

(4.15) $\pi_2^+\varrho(c(X, Y)) = \varrho(\sigma c(X, Y))$.

If also Y collapses to a subpolyhedron $Z \subset Y$, we have

(4.16) $\varrho(c(Y, Z)c(X, Y)) = \varrho(c(Y, Z))\varrho(c(X, Y))$.

(II) Let X, Y be disjoint compact subpolyhedra of S^{n-1} and $\varphi \in \mathrm{Si}(X, Y)$. We use [19] to find a sufficiently large $k \geqslant 0$ and a compact subpolyhedron P of S^{n-1+k} collapsing both to $i^k X$ and $i^k Y$ such that

$$(i^k\varphi)c(P, i^k X) = c(P, i^k Y);$$

the pair (k, P) is said to be *admissible for* φ. We write X resp. Y instead of $i^k X$ resp. $i^k Y$ and define $R(\varphi, k, P) \in (\operatorname{Isot}/\pi_1)(D^n-X, D^n-Y)$ to be the unique element such that

$$(\pi_1^*)^k R(\varphi, k, P) = \lambda^+(\pi_1)(\varrho(c(P, Y))\varrho(c(P, X))^{-1}).$$

Using (4.14), we obtain

(4.17) $R(\varphi, k+1, iP) = R(\varphi, k, P)$.

If Q is a compact subpolyhedron of S^{n-1+k} collapsing to P, then (k, Q) is also admissible, and (4.16) implies

(4.18) $R(\varphi, k, P) = R(\varphi, k, Q)$.

Consider admissible pairs (k, P) and (k, P') with "sufficiently large" k and $P \cap P' = X \cup Y$. Let $j_X\colon X \to P$, $j_Y\colon Y \to P$, $j'_X\colon X \to P'$ denote inclusions, $r_Y\colon P \to Y$, $r'_Y\colon P' \to Y$ denote collapsing maps. Then $j'_Y r_Y j_X \simeq j'_Y r'_Y j'_X \simeq j'_X$ via a homotopy $F\colon X \times I \to P'$. Define $G\colon (X \cup Y) \times I \cup P \times \{0\} \to P'$ by $G|X \times I = F$, $G|Y \times I = j'_Y p$ (with projection $p\colon Y \times I \to Y$) and $G|P \times \{0\} = j'_Y r_Y$. Use Borsuk's homotopy extension theorem to extend G to $H\colon P \times I \to P'$. Then $g\colon P \to P'$, $g(x) = H(x, 1)$, is a simple homotopy equivalence with $g|X \cup Y = \mathrm{id}$, and (for a sufficiently large k) there exists a compact subpolyhedron $Q \subset S^{n-1+k}$ collapsing both to P and P' (use again [19]).

Applying (4.18) we obtain

(4.19) $R(\varphi, k, P) = R(\varphi, k, P')$.

Finally, consider arbitrary admissible pairs (k, P) and (k', P'). Using (4.17) we may assume $k = k'$ and that k is "sufficiently large". A general position argument yields a PL embedding $f\colon P \to S^{n-1+k}$ such that $f|X \cup Y = \mathrm{id}$ and $f(P) \cap P = X \cup Y = f(P) \cap P'$. Then $(k, f(P))$ is admissible and (4.19) implies

(4.20) $R(\varphi, k, P) = R(\varphi, k', P')$.

Hence, $R(\varphi, k, P)$ does not depend on the admissible pair (k, P), and we simply write $R(\varphi)$.

The following properties are immediate consequences of the definition.

(4.21) $\omega^* R(\varphi) = 1$;

(4.22) $\vartheta^* R(\varphi) = S(\varphi)$;

(4.23) $R(i\varphi) = \pi_1^* R(\varphi)$;

(4.24) $R(\varphi^{-1}) = R(\varphi)^{-1}$.

Let X, Y, Z be pairwise disjoint compact subpolyhedra of S^{n-1} and $\psi \in \mathrm{Si}(Y, Z)$. Choose admissible pairs (k, P_1) for φ and (k, P_2) for ψ. For a sufficiently large k, general position arguments allow to assume $P_1 \cap P_2 = Y$. Then $(k, P_1 \cup P_2)$ is admissible for φ, ψ and $\psi\varphi$. This implies

(4.25) $R(\psi\varphi) = R(\psi) R(\varphi)$.

(III) Let X, Y be arbitrary compact subpolyhedra of S^{n-1} and $\varphi \in \mathrm{Si}(X, Y)$. Choose any $\alpha \in \mathrm{Si}(iY, Y')$, where Y' is a suitable compact subpolyhedron of S^n such that $Y' \cap (iX \cup iY) = \emptyset$. Using (II), we can define

$$R(\varphi; \alpha) \in (\mathrm{Isot}/\pi_1)(D^n - X, D^n - Y)$$

to be the unique element such that $\pi_1^*(\varphi; \alpha) = R(\alpha^{-1}) R(\alpha i\varphi)$. Using (4.24) and (4.25) one verifies that $R(\varphi; \alpha)$ actually does not depend on the choice of α and that, in case $X \cap Y = \emptyset$, $R(\varphi; \alpha) = R(\varphi)$. We therefore always write $R(\varphi) = R(\varphi; \alpha)$. Again (4.24) and (4.25) imply that R is a functor from Si^+ to $(\mathrm{Isot}/\pi_1)^+$ (with $R(X) = D^n - X$ for $X \subset S^{n-1}$). The following properties are immediate from the definition.

(4.26) $\omega^* R(\varphi) = 1$ for each morphism φ;

(4.27) $\vartheta^* R = S$;

(4.28) $Ri = \pi_1^* R$.

Let $X \subset S^{n-1}$ collapse to the subpolyhedron $Y \subset X$. Choose a PL embedding $f\colon X \times I \to S^n$ such that $f|X \times \{0\} = \mathrm{id}$. Define $Y' = f(Y \times \{1\})$, $P = f(X \times I)$ and $\alpha = c(P, Y')c(P, iY)^{-1}$. Then $\pi_1^* R(c(X, Y)) = R(\alpha^{-1}) R(\alpha c(X, Y))$, and straightforward computations using (4.16) and (4.17) show that the latter is equal to

$$\lambda^+(\pi_1)\varrho(c(iX, iY)) = \pi_1^* \lambda^+(\pi_1)\varrho(c(X, Y)).$$

Hence

(4.29) $R(c(X, Y)) = \lambda^+(\pi_1)\varrho(c(X, Y))$.

We know that $\delta_X \in \mathrm{Si}(\sigma iX, i\sigma X)$ is represented by the restriction $d'_X\colon \sigma iX \to i\sigma X$ of the rotation $d\colon D^{n+2} \to D^{n+2}$ considered in the proof of (3.2). Define

$$P = \{(x_1, \ldots, x_n, x_{n+1}\cos t, x_{n+1}\sin t)|\ (x_1, \ldots, x_{n+1}) \in \sigma X, t \in [0, \pi/2]\}.$$

Then P is a compact subpolyhedron of S^{n+1} collapsing both to σiX and $i\sigma X$; moreover, it is easy to see that $\delta_X c(P, \sigma iX) = c(P, i\sigma X)$. (4.29) implies

(4.30) $R(\delta_X) = \lambda^+(\pi_1)(\gamma^+_{D^n - X}) = \gamma^*_{D^n - X}$ (cf. Proof of (3.3)).

Let X, Y be disjoint compact subpolyhedra of S^{n-1} and $\varphi \in \mathrm{Si}(X, Y)$. Choose an admissible pair (k, P) for φ. Then $(\sigma i^k\varphi)c(\sigma P, \sigma X) = c(\sigma P, \sigma Y)$. (4.29), (4.15) and (4.28) imply that $R(\sigma i^k\varphi) = \pi_2^* R(i^k\varphi) = \pi_2^*(\pi_1^*)^k R(\varphi)$. Using (4.28) and (4.30) we obtain $(\pi_1^*)^k R(\sigma\varphi) = R(i^k\sigma\varphi) = (\pi_1^*)^k \pi_2^* R(\varphi)$, i.e. $R(\sigma\varphi) = \pi_2^* R(\varphi)$. We easily generalize this to

(4.31) $R\sigma = \pi_2^* R$.

We now know that R is an operator functor satisfying (R1) and (R2). (R3) follows from an obvious argument using [19] and [4.29].

5. S-duality. Let Pol be the full subcategory of Ho(Top) whose objects are the compact polyhedra, and let $\mathrm{Pol}^+ = \bigcup_{n=1}^{\infty} \mathrm{Pol}|P(S^{n-1})$. Then $\mathfrak{Pol}^+ = (\mathrm{Pol}^+; i, \sigma; \delta)$ is an operator subcategory of $\mathfrak{T}^+$.

(5.1) Theorem (cf. [17]). (a) *There exists a unique contravariant operator functor $d\colon \mathfrak{Pol}^+ \to (\mathfrak{T}'/\Sigma)^+$ having the following properties:*

(d1) *$d(X) = S^{n-1} - X$ for each object $X \in P(S^{n-1})$.*

(d2) *If $\iota_{X,Y}$ is the homotopy class of an inclusion $X \to Y$, then $d(\iota_{X,Y})$ is the stable homotopy class $I_{X,Y}$ of the inclusion $S^{n-1} - Y \to S^{n-1} - X$.*

(b) *The functor d satisfies $d|\mathrm{Si}^+ = \partial^* R(\mathrm{op})$ where $\mathrm{op}\colon \mathrm{Si}^+ \to \mathrm{Si}^+$ is the "dualizing" functor.*

Proof. Let X, Y be compact subpolyhedra of S^{n-1} and $\alpha \in \mathrm{Ho(Top)}(X, Y)$. There exist $k \geqslant 0$, a compact subpolyhedron $P \subset S^{n-1+k}$ containing $X = i^k X$ and $\varphi \in \mathrm{Si}(P, Y)$ such that $\alpha = \varphi\iota_{X,P}$ (choose a PL representative $f\colon X \to Y$ of α and

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et P be the polyhedral mapping cylinder of f, suitably embedded in S^{n-1+k}). Define $d(\alpha) \in (\mathrm{Ho(Top)}/\Sigma)(S^{n-1}-Y, S^{n-1}-X)$ to be the unique element such that $(\Sigma^*)^k d(\alpha) = I_{X,P}(\partial^* R(\varphi^{-1}))$. Similarly as in the proof of (4.10) one checks that $d(\alpha)$ does not depend on the choices of k, P, φ and that d is a contravariant operator functor satisfying (d1), (d2) and (b). The uniqueness of d follows from (d2) and the fact that for each $\alpha \in \mathrm{Ho(Pol)}(X, Y)$ there exists a $k \geqslant 0$ such that $i^k\alpha = \beta_r \dots \beta_2\beta_1$ where each $\beta_i \in \mathrm{Ho(Pol)}(X_i, X_{i+1})$ is the homotopy class of either an inclusion or of a homotopy equivalence whose inverse is an inclusion.

The regular stabilizations of $\mathfrak{Pol}^+$, $\mathfrak{S}^+$, $\mathfrak{J}^+$ are

$$\lambda^+(\Sigma)\colon \mathfrak{Pol}^+ \to (\mathfrak{Pol}/\Sigma)^+ = ((\mathrm{Pol}/\Sigma)^+; i^*, \sigma^*; \partial^*),$$

$$\lambda^+(\Sigma)\colon \mathfrak{S}^+ \to (\mathfrak{S}/\Sigma)^+ = ((\mathrm{Sh}/\Sigma)^+; i^*, \sigma^*; \partial^*),$$

$$\lambda^+(\pi_1, \pi_2)\colon \mathfrak{J}^+ \to (\mathfrak{J}/\pi_1, \pi_2)^+ = ((\mathrm{Isot}/\pi_1, \pi_2)^+; \pi_1^*, \pi_2^*; \gamma^*)$$

where $(\mathrm{Pol}/\Sigma)^+$ and $(\mathrm{Sh}/\Sigma)^+$ are subcategories of the stable homotopy category $\mathrm{Ho(Top)}/\Sigma$ and the stable *shape category* Sh/Σ. Moreover, if $j\colon \mathfrak{Si}^+ \to \mathrm{Iso}\,\mathfrak{Pol}^+$ denotes inclusion, then $\lambda(\sigma) = (\mathrm{Iso}\,\lambda^+(\Sigma)) \circ j\colon \mathfrak{Si}^+ \to \mathrm{Iso}(\mathfrak{Pol}/\Sigma)^+$ is a stabilization of $\mathfrak{Si}^+$ because double suspension makes everything simply connected. The stabilizing construction yields operator functors

$$S^{\#}\colon \mathrm{Iso}(\mathfrak{Pol}/\Sigma)^+ \to \mathrm{Iso}(\mathfrak{S}/\Sigma)^+,$$

$$\vartheta^{\#}\colon (\mathfrak{J}/\pi_1, \pi_2)^+ \to \mathrm{Iso}(\mathfrak{S}/\Sigma)^+,$$

$$\partial^{\#}\colon (\mathfrak{T}/\pi_1, \pi_2)^+ \to (\mathfrak{T}'/\Sigma)^+,$$

$$R^{\#}\colon \mathrm{Iso}(\mathfrak{Pol}/\Sigma)^+ \to (\mathfrak{J}/\pi_1, \pi_2)^+.$$

Observe that $S^{\#}$ is a full embedding. Moreover,

(5.2) $\vartheta^{\#} R^{\#} = S^{\#}$.

From (4.8) we obtain a functorial complement theorem in stable shape theory:

(5.3) PROPOSITION. *$\vartheta^{\#}$ is a full functor.*

(5.4) THEOREM (cf. [17]). (a) *There exists a unique contravariant operator functor $D\colon (\mathfrak{Pol}/\Sigma)^+ \to (\mathfrak{T}'/\Sigma)^+$ satisfying the following conditions:*

(D1) *$D(X) = S^{n-1} - X$ for each $X \in P(S^{n-1})$.*

(D2) *If $\iota_{X,Y}$ is the stable homotopy class of an inclusion $X \to Y$, then $D(\iota_{X,Y}) = I_{X,Y}$.*

(b) *The functor D satisfies $D|\mathrm{Iso}(\mathrm{Pol}/\Sigma)^+ = \partial^{\#} R^{\#}\mathrm{op}$.*

Proof. Let D be defined by $D\lambda^+(\Sigma) = d$. Then (D1), (D2) and (b) are satisfied. Uniqueness follows from 5.1.

The functor D is the *duality functor* of Spanier and Whitehead [17] (the duality property follows from (D1) and (D2)). *Property* (b) *and* (5.2) reflect the intimate relationship between S-duality and the complement and category isomorphism theorems in shape theory.

We conclude with

(5.5) THEOREM (cf. [11], [14]). *Let $X, Y \subset S^n$ be compact. Then the following are equivalent.*

(a) *X and Y have the same stable shape* (*i.e. are isomorphic in the stable shape category* Sh/Σ).

(b) *$S^n - X$ and $S^n - Y$ have the same stable homotopy type.*

(c) *$S^n - X$ and $S^n - Y$ have the same stable shape.*

Proof. The equivalence of (b) and (c) follows from (2.2): All suspensions $\Sigma^k(S^n - X)$, $\Sigma^k(S^n - Y)$ are homotopy equivalent to polyhedra. By (5.3), (a) implies (b). To show that (b) implies (a), one can use Lima's extension of Spanier–Whitehead duality: The functor D of (5.4) extends to a contravariant operator category isomorphism $D\colon (\mathfrak{S}/\Sigma)^+ \to (\mathfrak{CS}/\Sigma)^+$ in which the operator category

$$\mathfrak{CS} = (\mathrm{CoSh}; 1, \Sigma; \mathrm{id})$$

is based on the *coshape category* CoSh of finite-dimensional σ-compact polyhedra. The reader is urged to consult [11] or [14] (in [11], the shape and coshape constructions are performed on the level of stable homotopy theory; cf. also [9]).

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Mapping approximate inverse systems of compacta

by

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Abstract. Recently, T. Watanabe has extensively studied approximate mappings of inverse systems of spaces $f\colon X \dashrightarrow Y$. He showed if $p\colon X \to X$ and $q\colon Y \to Y$ are ANR-resolutions of topologically complete spaces X, Y, then $f\colon X \dashrightarrow Y$ induces a mapping $f\colon X \to Y$ and conversely, every mapping $f\colon X \to Y$ is obtainable in this way. In this paper it is shown that the basic results of Watanabe's theory are valid also for approximate mappings of approximate inverse systems $\boldsymbol{X}$, $\boldsymbol{Y}$ of compact ANR's and compact Hausdorff spaces $X = \lim \boldsymbol{X}$, $Y = \lim \boldsymbol{Y}$. Approximate systems, newly introduced by S. Mardešić and L. R. Rubin, have bonding maps $p_{aa'}$, $a \leqslant a'$, where in general $p_{a_1a_2}p_{a_2a_3}$ differs from $p_{a_1a_3}$, but in a controlled way.

1. Introduction. An inverse system of spaces $\boldsymbol{X} = (X_a, p_{aa'}, A)$ (in the usual sense) consists of a directed set $(A, \leqslant)$, spaces X_a, $a \in A$, and maps $p_{aa'}\colon X_{a'} \to X_a$, $a \leqslant a'$, such that $p_{aa} = \mathrm{id}$ and

$$p_{a_1a_2}p_{a_2a_3} = p_{a_1a_3}, \quad a_1 \leqslant a_2 \leqslant a_3. \tag{1}$$

The (usual) inverse limit $X = \lim \boldsymbol{X}$ is the subspace $X \subseteq \Pi X_a$, which consists of all points $x = (x_a) \in \Pi X_a$ such that $p_{aa'}(x_{a'}) = x_a$, $a \leqslant a'$. Projections $p_a\colon X \to X_a$ are restrictions to X of the projections $\pi_a\colon \Pi X_a \to X_a$, $a \in A$.

A mapping of systems $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y} = (Y_b, q_{bb'}, B)$ consists of a function $f\colon B \to A$ and of mappings $f_b\colon X_{f(b)} \to Y_b$, $b \in B$, such that whenever $b_1 \leqslant b_2$, there exists an index $a \in A$, $a \geqslant f(b_1), f(b_2)$, such that

$$f_{b_1}p_{f(b_1)a} = q_{b_1b_2}f_{b_2}p_{f(b_2)a} \tag{2}$$

(see, e.g., [8], I, § 1.1). It is well known that for any mapping of systems $\boldsymbol{f} = (f, f_b)\colon \boldsymbol{X} \to \boldsymbol{Y}$ there is a unique mapping $f\colon X \to Y$ of the limits $X = \lim \boldsymbol{X}$, $Y = \lim \boldsymbol{Y}$ such that

$$f_b p_{f(b)} = q_b f, \quad b \in B. \tag{3}$$

This mapping is called the *limit of* $\boldsymbol{f}$ and is denoted by $f = \lim \boldsymbol{f}$ (see, e.g., [8], I, § 5.1).

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