Finite coverings of groups

by

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Abstract. In this paper we study finite coverings of groups by cosets. We mainly generalize three known results: Tomkinson's result which is an improvement to Neumann's theorem, Korec's result which is a generalization of Mycielski's conjecture, and Simpson's result which improves a conjecture of Znám. The main part of this paper is Section 4.

1. (M, S)-coverings of a group. A semigroup containing an identity element is called a *monoid*. In the following commutative monoids are considered as additive monoids while groups are considered as multiplicative groups.

DEFINITION 1. Let M be a commutative monoid and S a set. A system

$$\{X_i\}_{i=1}^{k} \tag{1}$$

of subsets of X is called an (M, S)-*covering of* X if there exist $m_1, \ldots, m_k \in M$ such that

$$\{\sum_{\substack{i=1\\ x\in X_i}}^{k} m_i\colon x\in X\}\subseteq S \quad \text{and} \quad \{\sum_{\substack{i\in I\\ x\in X_i}} m_i\colon x\in X\}\not\subseteq S$$

for any $I\subset\{1, \ldots, k\}$ ($\sum_{\substack{i\in I\\ x\in X_i}} m_i$ is considered as the zero element 0 of M if $x\in X_i$ for no $i\in I$).

Remark 1. Let (1) be an (M, S)-covering of X. Clearly each X_i is non-empty. Since

$$\{0\} = \{\sum_{\substack{i\in\emptyset\\ x\in X_i}} m_i\colon x\in X\}\not\subseteq S$$

we have $0\notin S$ and hence

$$\sum_{\substack{i=1\\ x\in X_i}}^{k} m_i \neq 0$$

for any $x\in X$. This implies that (1) is a covering of X.

The set of all non-negative integers forms a commutative monoid under the usual addition. We denote this monoid by $Z^+\cup\{0\}$. Clearly, (1) is a $(Z^+\cup\{0\}, \{1\})$-

covering of X if and only if it partitions X, and (1) is a $(Z^+\cup\{0\}, Z^+)$-covering of X if and only if X is irredundantly covered by (1).

Let G be a group and $G_1, \ldots, G_k$ its subgroups. If G is partitioned into cosets $a_1G_1, \ldots, a_kG_k$, then

$$\{a_iG_i\}_{i=1}^k \tag{2}$$

is called a *(left) coset decomposition of G*.

The following result is constructive.

THEOREM 1. *Let G be a group and H its subgroup of finite index. Let*

$$H_0 = H \subset H_1 \subset \ldots \subset H_n = G$$

be a chain of subgroups of G. Let $a \in G$ and

$$H_{i+1} = H_i \cup \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i, \quad i = 0, 1, \ldots, n-1.$$

Then

$$\{aH_0, ab_j^{(i)}H_i:\ j = 1, \ldots, [H_{i+1}:H_i]-1;\ i = 0, 1, \ldots, n-1\} \tag{3}$$

is a coset decomposition of G, hence it is a $(Z^+\cup\{0\}, \{1\})$-covering of G.

Proof. Let $g \in G$ and s be the smallest i such that $a^{-1}g \in H_i$ (a^{-1} denotes the inverse of a). Obviously we have

(i) $g \in aH_0$ if and only if $s = 0$,
(ii) $g \in ab_j^{(i)}H_i$ for some j (i.e. $a^{-1}g \in H_{i+1} - H_i$) if and only if $i+1 = s$,
(iii) $ab_1^{(i)}H_i, \ldots, ab_{[H_{i+1}:H_i]-1}^{(i)}$ are pairwise disjoint.

From the above, it follows that (3) is a partition of G. This completes the proof.

In the following the cardinality of a set S will be denoted by $|S|$. If H is a subgroup of G then $[G:H]$ will denote the index of H in G. If H is a normal subgroup of G then G/H will denote the factor group of G by H.

THEOREM 2. *Let G be a group and $G_1, \ldots, G_k$ its subgroups such that (2) is an (M, S)-covering of G for some commutative monoid M and some set S. Let $G_0 = G$ and H be a proper subgroup of G containing some G_i. Then*

$$\min(|\{1 \leqslant i \leqslant k: G_i \subseteq H\}|, [G:H]) \geqslant |\{a_iH: G_i \subseteq H, 1 \leqslant i \leqslant k\}| \tag{4}$$

$$\geqslant [\bigcap_{\substack{i=0\\ G_i\subseteq H, G_i \neq H}}^{k} G_i : H \cap \bigcap_{\substack{i=0\\ G_i\subseteq H, G_i \neq H}}^{k} G_i].$$

Proof. If $H \supseteq G_i$ for $i = 1, \ldots, k$ then by Remark 1 we have

$$G = \bigcup_{i=1}^k a_iG_i \subseteq \bigcup_{i=1}^k a_iH$$

and hence

$$k \geqslant |\{a_iH:\ 1 \leqslant i \leqslant k\}| = [G:H] = [G_0 : H \cap G_0]$$

which shows that (4) holds.

We now suppose that $I = \{1 \leqslant i \leqslant k:\ G_i \nsubseteq H\} \neq \emptyset$. Since (2) is an (M, S)-covering of G there exist $m_1, \ldots, m_k \in M$ such that

$$\{\sum_{\substack{i=1\\ x\in a_iG_i}}^{k} m_i:\ x \in G\} \subseteq S \tag{5}$$

and

$$\{\sum_{\substack{i\in I'\\ x\in a_iG_i}} m_i:\ x \in G\} \nsubseteq S \quad \text{for any } I' \subset \{1, \ldots, k\}. \tag{6}$$

Note that $I \subset \{1, \ldots, k\}$. For some $a \in G$ we have

$$\sum_{\substack{i\in I\\ ag\in a_iG_i}} m_i = \sum_{\substack{i\in I\\ a\in a_iG_i}} m_i \notin S \quad \text{for any } g \in \bigcap_{i\in I} G_i.$$

This together with (5) shows that for any $g \in \bigcap_{i\in I} G_i$,

$$\sum_{\substack{i\in\{1,\ldots,k\}-I\\ ag\in a_iG_i}} m_i \neq 0$$

and hence there exists an $s(g)$ $(1 \leqslant s(g) \leqslant k)$ such that $G_{s(g)} \subseteq H$ and $ag \in a_{s(g)}G_{s(g)}$. If $g, g' \in \bigcap_{i\in I} G_i$ and $g(H \cap \bigcap_{i\in I} G_i) \neq g'(H \cap \bigcap_{i\in I} G_i)$, then $g^{-1}g' \notin H$ which implies that

$$a_{s(g)}H = agH \neq ag'H = a_{s(g')}H.$$

Thus

$$|\{a_iH: G_i \subseteq H,\ 1 \leqslant i \leqslant k\}| \geqslant |\{a_{s(g)}H: g \in \bigcap_{i\in I} G_i\}| \geqslant [\bigcap_{i\in I} G_i : H \cap \bigcap_{i\in I} G_i],$$

hence (4) holds.

The proof is now complete.

Remark 2. Theorem 1 of [9] is the special case of Theorem 2 in which G is the additive group of integers, $M = Z^+ \cup \{0\}$ and $S = Z^+$.

THEOREM 3. *Let M be a commutative monoid and S a set. Let G be a group and $G_1, \ldots, G_k$ its subgroups. If there exist $a_1, \ldots, a_k \in G$ such that (2) is an (M, S)-covering of G, then for any $I \subseteq \{1, \ldots, k\}$ we have*

$$[\bigcap_{i\in I} G_i : \bigcap_{i=1}^k G_i] \leqslant (k-|I|)! \quad (\bigcap_{i\in\emptyset} G_i \text{ refers to } G). \tag{7}$$

We can show the theorem as M. J. Tomkinson ([11]) proved his Theorem 2.2. The only thing we need to remark is the following:

If $m_1, \ldots, m_k \in M$ satisfy (5) and (6), then as in the proof of Theorem 2, for any $I \subset \{1, \ldots, k\}$ there exists an $a \in G$ such that

$$\sum_{\substack{j\notin I\\ ag\in a_jG_j}} m_j \neq 0 \quad \text{for any } g \in \bigcap_{i\in I} G_i$$

which implies $a \bigcap_{i\in I} G_i \subseteq \bigcup_{j\notin I} a_jG_j$.

In the case $M = Z^+ \cup \{0\}$, $S = \{m, m+1, \dots\}$, Theorem 3 gives

Corollary 1. *Let G be a group and $G_1, \dots, G_k$ its subgroups. If every element of G belongs at least to m ($\geqslant 1$) members of system (2), which fails to hold if (2) is replaced by one of its proper subsystems, then (7) holds for any $I \subseteq \{1, \dots, k\}$.*

Remark 3. In the case $m = 1$, the corollary is similar to Lemma 2.1 and Theorem 2.2 of [11]; we mention that it was B. H. Neumann ([5]) who first proved $[G: \bigcap_{i=1}^{k} G_i] < +\infty$. For $m > 1$, as far as I know, no one else has shown even the finiteness of $[G: \bigcap_{i=1}^{k} G_i]$.

Definition 2. Let S be a set and M be a commutative monoid. S is said to be *M-connected* if for any $m_1, m_2, m_3 \in M$, $m_1 \in S$ and $m_1+m_2+m_3 \in S$ implies $m_1+m_2 \in S$.

Obviously, both $\{m\}$ and $\{m, m+1, \dots\}$ are $Z^+ \cup \{0\}$-connected. In Section 4 we will deal with (M, S)-coverings where S is M-connected.

2. (M, S)-good classes of groups. In this section we study (M, S)-good classes of groups which will be used in Section 4.

Definition 3. Let M be a commutative monoid and S be a set. A class Γ of groups is said to be *(M, S)-good* if it has the following property:

If (2) is an (M, S)-covering of $G \in \Gamma$ with all the subgroups G_i subnormal in G then for any maximal normal subgroup H of G we have

(8) $$\{C \in G/H\colon\ C \supseteq a_i G_i \text{ for some } i = 1, \dots, k\} = \emptyset \text{ or } G/H.$$

Theorem 4. *The class of all groups is $(Z^+ \cup \{0\}, \{1\})$-good.*

Proof. Let G be a group and $G_1, \dots, G_k$ its subnormal subgroups such that (2) is a $(Z^+ \cup \{0\}, \{1\})$-covering of G (i.e. a partition of G). Let H be a maximal normal subgroup of G such that

$$\{C \in G/H\colon\ C \supseteq a_i G_i \text{ for some } i\} \neq \emptyset .$$

Suppose that $a_j G_j \subseteq aH$ for some $a \in G$ and $1 \leqslant j \leqslant k$. Since $a_j \in aH$, $a_j H = aH \supseteq a_j G_j$ and hence $G_j \subseteq H$.

Denote $\bigcap_{i=1}^{k} G_i$ by F. Obviously $F \subseteq H$. By Theorem 3 (or Corollary 1), $[G:F]$ is finite. Since G_i is subnormal and H is normal in G, by Lemma 7.19 of [7], $G_i H$ is a subnormal subgroup of G. A maximal normal subgroup is also a maximal subnormal subgroup, so we have $G_i H = H$ or G for each i. Let $x \in G$. If $G_i \nsubseteq H$, then $G_i H = G$ hence $a_i^{-1} x = g_i h$ for some $g_i \in G_i$ and $h \in H$, therefore

$$a_i G_i \cap xH = a_i G_i \cap a_i g_i hH = a_i(g_i G_i \cap g_i H) = a_i g_i (G_i \cap H)$$

consists of $[G_i \cap H: F]$ left cosets of F in G. Let I denote the set $\{1 \leqslant i \leqslant k\colon\ G_i \nsubseteq H\}$. Since $a_1 G_1, \dots, a_k G_k$ are pairwise disjoint, $xH \setminus \bigcup_{i \in I} a_i G_i = xH - \bigcup_{i \in I} (a_i G_i \cap xH)$

consists of $[H:F] - \sum_{i \in I} [G_i \cap H : F]$ left cosets of F in G. This number does not depend on x. Note that

$$a_j F \subseteq a_j G_j = a_j G_j \setminus \bigcup_{i \in I} a_i G_i \subseteq a_j H \setminus \bigcup_{i \in I} a_i G_i .$$

Hence $xH \setminus \bigcup_{i \in I} a_i G_i \neq \emptyset$ for any $x \in G$.

Suppose that $C = xH \in G/H$ does not contain any $a_i G_i$. Then

$$xH = xH \cap \Big(\bigcup_{i=1}^{k} a_i G_i\Big) = \bigcup_{i=1}^{k} (a_i G_i \cap xH) = \bigcup_{i \in I} (a_i G_i \cap xH)$$

which shows that $xH \setminus \bigcup_{i \in I} a_i G_i = \emptyset$. This contradiction implies that

$$\{C \in G/H\colon\ C \supseteq a_i G_i \text{ for some } i\} = G/H .$$

The theorem is therefore proved.

Remark 4. The proof of Theorem 4 is similar to that of Lemma 6 of [3].

Theorem 5. *Let M be a commutative monoid and S a set. Then the class of all cyclic groups is (M, S)-good.*

Proof. Let $G = (a)$ be a cyclic group and $G_1 = (a^{n_1}), \dots, G_k = (a^{n_k})$ its subgroups such that (2) is an (M, S)-covering of G. Let H be a maximal subgroup of G. If $H = G$ then (8) holds trivially. We now suppose that $H \neq G$. Obviously $H = (a^p)$ for some prime p. Let $G_0 = G$, $n_0 = 1$ and

$$I = \{0 \leqslant i \leqslant k\colon\ G_i \nsubseteq H\} .$$

Note that $0 \in I$ and $G_0 = (a^{n_0})$. If $i \in I$ then $(a^{n_i}) \nsubseteq (a^p)$, and hence n_i is relatively prime to p since p is a prime. Clearly $\bigcap_{i \in I} G_i = (a^n)$ where n is the least common multiple of those n_i for which $i \in I$. Since n is relatively prime to p we have

$$H \cap \bigcap_{i \in I} G_i = (a^p) \cap (a^n) = (a^{pn}) .$$

Suppose that the left side of (8) is non-empty. That is, $aH \supseteq a_j G_j$ for some $a \in G$ and $1 \leqslant j \leqslant k$. Clearly, $H \supseteq G_j$. By Theorem 2,

$$|\{a_i H\colon\ G_i \subseteq H,\ 1 \leqslant i \leqslant k\}| \geqslant \Big[\bigcap_{i \in I} G_i : H \cap \bigcap_{i \in I} G_i\Big] = [(a^n) : (a^{pn})] = p = [G:H] .$$

This shows that every element of G/H contains a member of (2). We are done.

3. Two functions needed. In this section we study two functions which will be needed in Section 4.

Definition 4. The (Mycielski) function $f\colon Z^+ \to Z^+ \cup \{0\}$ is defined as follows. If the canonical prime factorization of n is

(9) $$n = \prod_{i=1}^{r} p_i^{\alpha_i} ,$$

then

$$f(n) = \sum_{i=1}^{r} \alpha_i(p_i - 1).$$

Remark 5. $f(1) = 0$, $f(mn) = f(m)+f(n)$.

We mention that the function f first appeared in J. Mycielski's conjecture (cf. [4]).

Definition 5. For an arbitrary group G and its subnormal subgroup H define

$$d(G, H) = \begin{cases} [G:H] & \text{if } [G:H] \text{ is infinite}, \\ \sum_{i=1}^{n}([H_i:H_{i-1}]-1) & \text{if } [G:H] \text{ is finite}, \end{cases}$$

where $H_0 = H, H_1, \ldots, H_n = G$ is a maximal chain of subgroups of G such that H_{i-1} is normal in H_i for all $i = 1, \ldots, n$.

Remark 6. Let H be a subnormal subgroup of G. If the index of H in G is finite, then by Theorem 8.4.4 of [2] $d(G, H)$ does not depend on the choice of the chain $H_0, H_1, \ldots, H_n$. If $[G:H]$ is infinite and there exists a (finite) maximal chain $H_0 = H, \ldots, H_n = G$ of subgroups of G such that H_{i-1} is normal in H_i for all $i = 1, \ldots, n$, hen at least one of the indices $[H_i:H_{i-1}]$ is infinite since $[G:H] = \prod_{i=1}^{n} [H_i:H_{i-1}]$, and hence by the absorption law of cardinal arithmetic (cf. [1], p. 164) we have

$$\sum_{i=1}^{n}([H_i:H_{i-1}]-1) = \sum_{i=1}^{n}[H_i:H_{i-1}] - n = \sum_{i=1}^{n}[H_i:H_{i-1}]$$
$$= \max_{1 \leq i \leq n}[H_i:H_{i-1}] = \prod_{i=1}^{n}[H_i:H_{i-1}] = [G:H].$$

We mention that $d(G, H) = 0$ if and only if $H = G$. And if K is a subnormal subgroup of H then

$$d(G, H) + d(H, K) = d(G, K).$$

In the case where H is a normal subgroup of finite index in G, $d(G, H)$ was introduced by I. Korec [3]. We will define in another paper $d(H, K)$ for any two subgroups H, K of a group G.

Theorem 6. *Let G be a group and H its subnormal subgroup of finite index. Then*

(10) $$[G:H] - 1 \geq d(G, H) \geq f([G:H]) \geq \log_2[G:H].$$

Proof. We first note that

$$\prod_{i=1}^{k}(1+x_i) \geq 1 + \sum_{i=1}^{k} x_i \quad \text{for } x_1, \ldots, x_k \geq 0.$$

If (9) is the standard form of n, then

$$f(n) = \sum_{i=1}^{r}\alpha_i(p_i - 1) \leq \sum_{i=1}^{r}(p_i^{\alpha_i} - 1) \leq \prod_{i=1}^{r}(1 + p_i^{\alpha_i} - 1) - 1 = n - 1$$

and

$$2^{f(n)} = \prod_{i=1}^{r}(2^{p_i-1})^{\alpha_i} \geq \prod_{i=1}^{r} p_i^{\alpha_i} = n.$$

By the last inequality we have

$$f([G:H]) \geq \log_2[G:H].$$

Let $H_0 = H, H_1, \ldots, H_n = G$ be a maximal chain of subgroups of G such that H_{i-1} is normal in H_i for all $i = 1, \ldots, n$. Then

$$[G:H] - 1 = [H_n:H_0] - 1 = \prod_{i=1}^{n}[H_i:H_{i-1}] - 1$$
$$\geq d(G, H) = \sum_{i=1}^{n}([H_i:H_{i-1}]-1) \geq \sum_{i=1}^{n} f([H_i:H_{i-1}])$$
$$= f\Big(\prod_{i=1}^{n}[H_i:H_{i-1}]\Big) = f([H_n:H_0]) = f([G:H]).$$

The proof is now complete.

Theorem 7. *Let G be a locally nilpotent group (i.e., each finite subset of G is contained in a nilpotent subgroup of G) and let H be a subgroup of finite index in G. Then H is subnormal in G and*

$$d(G, H) = f([G:H]).$$

Proof. Let $H_0 = H \subset H_1 \subset \ldots \subset H_n = G$ be a maximal chain of subgroups of G. (The case $H = G$ is trivial.) By [6] (pp. 342–345), for a locally nilpotent group its subgroups are locally nilpotent and maximal subgroups are normal (the latter is due to Baer and McLain). Since H_{i-1} is maximal in H_i, H_{i-1} must be normal in H_i, and hence H_i/H_{i-1} is a cyclic group of prime order since it contains no proper subgroup. Thus H is subnormal in G and

$$d(G, H) = \sum_{i=1}^{n}([H_i:H_{i-1}]-1) = \sum_{i=1}^{n} f([H_i:H_{i-1}])$$
$$= f\Big(\prod_{i=1}^{n}[H_i:H_{i-1}]\Big) = f([G:H]).$$

4. Our main results. We first give a basic fact which will be often used.

Fact. *Let G be a group and H, K its subgroups.*

(a) *If $x \in G$ then $[G:x^{-1}Hx] = [G:H]$.*

(b) *If $[G:H]$ is finite then $[K:H \cap K]$ is also finite and $[K:H \cap K] \leq [G:H]$.*

(c) *If H is subnormal in G then $H \cap K$ is subnormal in K.*

(d) *If H and K are both subnormal in G then $H\cap K$ is subnormal in G.*

(e) *If $g\in G$ and H is subnormal in G then $g^{-1}Hg$ is subnormal in $g^{-1}Gg = G$.*

Proof. By [10] (p. 23), $G = HxG$ contains exactly $[G:G\cap x^{-1}Hx]$ right cosets of H, thus part (a) holds. Parts (b)–(e) can be found in [10] (p. 25, p. 124).

The following theorem is our central result.

THEOREM 8. *Let M be a commutative monoid and S an M-connected set. Let Γ be an (M, S)-good class of groups such that if a group belongs to Γ then so do its normal subgroups of finite indices. Suppose that $G\in\Gamma$ and (2) is an (M, S)-covering of G with all the subgroups G_i subnormal in G. Then for any subgroup K of G, either $I(K) = \{1\leqslant i\leqslant k\colon K\nsubseteq G_i\}$ is empty or there exist an $r\in I(K)$ and $x_i\in K\setminus G_i$ $(i\in I(K)-\{r\})$ such that*

$$\text{(11)}\qquad \min\Big(|(K)|-1,\ [K:K\cap\bigcap_{i=1}^{k} G_i]-1\Big) \geqslant \Big|\Big\{x_i\Big(K\cap\bigcap_{i=1}^{k} G_i\Big)\colon i\in I(K)-\{r\}\Big\}\Big| \geqslant d\Big(K, K\cap\bigcap_{i=1}^{k} G_i\Big).$$

(Note that by Fact $K\cap\bigcap_{i=1}^{k} G_i$ is subnormal in K.)

Proof. If $x_i\in K\setminus G_i$ then $x_i(K\cap\bigcap_{i=1}^{k} G_i)\neq K\cap\bigcap_{i=1}^{k} G_i$ is a left coset of $K\cap\bigcap_{i=1}^{k} G_i$ in K. So it is sufficient to prove that for any subgroup K of G either $I(K) = \varnothing$ or there exist an $r\in I(K)$ and $x_i\in K\setminus G_i$ $(i\in I(K)-\{r\})$ such that

$$\text{(12)}\qquad \Big|\Big\{x_i\Big(K\cap\bigcap_{i=1}^{k} G_i\colon i\in I(K)-\{r\}\Big\}\Big| \geqslant d\Big(K, K\cap\bigcap_{i=1}^{k} G_i\Big).$$

We now prove this by induction with respect to $[G:\bigcap_{i=1}^{k} G_i]$. (Note that by Theorem 3 the index $[G:\bigcap_{i=1}^{k} G_i]$ is finite.)

If $[G:\bigcap_{i=1}^{k} G_i] = 1$ then $\bigcap_{i=1}^{k} G_i = G$ and hence $I(K) = \varnothing$ for any subgroup K of G.

Let $[G:\bigcap_{i=1}^{k} G_i] > 1$ and K be a subgroup of G such that $I(K)$ is non-empty. Suppose that $G_j\neq G$. (Note that such a G_j exists.) Since G_j is a proper subnormal subgroup of finite index, there exists a proper maximal normal subgroup H of G containing G_j. Let $\{g_1H, \ldots, g_hH\} = G/H$ where $h = [G:H]\leqslant [G:G_j] < \infty$. Since (2) is an (M, S)-covering of G, there exist $m_1, \ldots, m_k\in M$, such that (5) and (6) hold.

We now introduce some notation. For each s $(1\leqslant s\leqslant h)$, let I_s be a minimal subset of $\{1, \ldots, k\}$ such that

$$\text{(13)}\qquad \Big\{\sum_{\substack{i\in I_s\\ x\in a_iG_i}} m_i\colon x\in g_sH\Big\}\subseteq S.$$

It is clear that I_s exists. By Remark 1, $0\notin S$ and hence I_s is non-empty. We then set

$$H_s = \bigcap_{i\in I_s}(H\cap G_i),\quad K_0 = K\cap H,\quad K_s = K_{s-1}\cap H_s,$$

$$M_s = \begin{cases}\{i\in I_s\colon K_{s-1}\nsubseteq H\cap G_i\} & \text{if } K_{s-1}\neq K_s,\\ \{1\leqslant i\leqslant k\colon a_iG_i\subseteq g_sH \text{ and } K\nsubseteq G_i\} & \text{if } K_{s-1} = K_s\end{cases}$$

for $s = 1, \ldots, h$.

We remark that:

(i) $K_h = K\cap H\cap\bigcap_{i=1}^{k}(H\cap G_i) = K\cap\bigcap_{i=1}^{k} G_i$.

This is because $I = \bigcup_{s=1}^{h} I_s$ is exactly the set $\{1, \ldots, k\}$, which follows from (5), (6), (13), the M-connectivity of S and the fact

$$\Big\{\sum_{\substack{i\in I\\ x\in a_iG_i}} m_i\colon x\in G\Big\} = \bigcup_{s=1}^{h}\Big\{\sum_{\substack{i\in I_s\\ x\in a_iG_i}} m_i + \sum_{\substack{i\in I-I_s\\ x\in a_iG_i}} m_i\colon x\in g_sH\Big\}\subseteq S.$$

(ii) $M_1, \ldots, M_h$ are pairwise disjoint. Clearly, $i\in M_s$ implies that $a_iG_i\cap g_sH\neq\varnothing$. (Note that if $i\in I_s$ then $a_iG_i\cap g_sH\neq\varnothing$ by the choice of I_s.) Suppose that $i\in M_s\cap M_t$ and $s<t$. Since $g_sH\cap g_tH = \varnothing$, we must have $K_{s-1}\neq K_s$ and $K_{t-1}\neq K_t$. Hence $i\in I_s\cap I_t$, $K_{s-1}\nsubseteq H\cap G_i$ and $K_{t-1}\nsubseteq H\cap G_i$. Since $s<t$, we have $K_{t-1}\subseteq K_s\subseteq H_s\subseteq H\cap G_i$, contrary to the fact $K_{t-1}\nsubseteq H\cap G_i$. Thus $M_s\cap M_t = \varnothing$ if $s<t$.

(iii) $M_s\subseteq I(K)$. If $K_{s-1} = K_s$ then it is obvious. Let $K_{s-1}\neq K_s$ and $i\in M_s$. Since $K_{s-1}\subseteq K_0$ and $K_{s-1}\nsubseteq H\cap G_i$, we have $H\cap K = K_0\nsubseteq H\cap G_i$ and hence $K\nsubseteq G_i$.

(iv) K_s is subnormal and has finite index in K_{s-1}, because H_s is subnormal and has finite index in G (cf. Fact).

By the choice of I_s, $\{g_s^{-1}a_iG_i\cap H\}_{i\in I_s}$ is an (M, S)-covering of H. By Remark 1 for each $i\in I_s$, $g_s^{-1}a_iG_i\cap H$ is non-empty, and hence it is a left coset of $G_i\cap H$ in H. Note that $H\in\Gamma$, $G_i\cap H$ is subnormal in H and

$$\Big[H:\bigcap_{i\in I_s}(G_i\cap H)\Big]\leqslant\Big[H:H\cap\bigcap_{i=1}^{k} G_i\Big] = \Big[H:\bigcap_{i=1}^{k} G_i\Big] < \Big[G:\bigcap_{i=1}^{k} G_i\Big].$$

We also note that K_{s-1} is a subgroup of H. If $K_{s-1}\neq K_s$ $(= K_{s-1}\cap H_s)$ then $M_s = \{i\in I_s\colon K_{s-1}\nsubseteq H\cap G_i\}\neq\varnothing$, and hence by the inductive hypothesis there exist $i_s\in M_s$ and $x_i\in K_{s-1}\setminus H\cap G_i$ for $i\in M_s-\{i_s\}$ such that

$$\big|\{x_i(K_{s-1}\cap H_s)\colon i\in M_s-\{i_s\}\}\big|\geqslant d(K_{s-1}, K_{s-1}\cap H_s),$$

i.e.,

$$\big|\{x_iK_s\colon i\in M_s-\{i_s\}\}\big|\geqslant d(K_{s-1}, K_s).$$

Suppose that $K_{s-1}\neq K_s$. If $i\in M_s-\{i_s\}$ then

$$x_i\in K_{s-1}\setminus H\cap G_i\subseteq H\cap K\setminus H\cap G_i\subseteq K\setminus G_i.$$

If $i, i' \in M_s - \{i_s\}$ and $x_i K_s \neq x_{i'} K_s$ then $x_{i'}^{-1} x_i \notin K_s$, and hence $x_{i'}^{-1} x_i \notin K_h$, $x_i K_h \neq x_{i'} K_h$. If $K_{t-1} \neq K_t$, $t > s$, $i \in M_s - \{i_s\}$ and $i' \in M_t - \{i_t\}$ then $x_i \in K_{s-1} \setminus H \cap G_i$, $x_{i'} \in K_{t-1} \setminus H \cap G_{i'}$, and hence

$$x_i \notin H \cap G_i, \quad x_{i'} K_h \subseteq K_{t-1} \subseteq K_s \subseteq H_s \subseteq H \cap G_i$$

which implies that $x_i \notin x_{i'} K_h$, i.e., $x_i K_h \neq x_{i'} K_h$. From these we have

$$\Big|\bigcup_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} \{x_i K_h:\ i \in M_s - \{i_s\}\}\Big| = \sum_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} \big|\{x_i K_h:\ i \in M_s - \{i_s\}\}\big|$$
$$\geqslant \sum_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} \big|\{x_i K_s:\ i \in M_s - \{i_s\}\}\big|$$
$$\geqslant \sum_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} d(K_{s-1}, K_s)$$
$$= \sum_{s=1}^{h} d(K_{s-1}, K_s) = d(K_0, K_h)$$
$$= d(H \cap K, K \cap \bigcap_{i=1}^{k} G_i).$$

(Note that $d(K_{s-1}, K_s) = 0$ if $K_{s-1} = K_s$.)

Case 1. $K \subseteq H$. If $K_{s-1} = K_s$ for each $s = 1, \ldots, h$ then

$$K = K_0 = K_1 = \ldots = K_h = K \cap \bigcap_{i=1}^{k} G_i \subseteq \bigcap_{i=1}^{k} G_i$$

and hence $I(K)$ is empty. We now suppose $K_{t-1} \neq K_t$. For

$$i \in I(K) - \bigcup_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} (M_s - \{i_s\})$$

we let x_i be an element of $K \setminus G_i$. Note that $i_t \in M_t \subseteq I(K)$ and $x_i \in K \setminus G_i$ for $i \in I(K) - \{i_t\}$. We have

$$\Big|\{x_i(K \cap \bigcap_{i=1}^{k} G_i):\ i \in I(K) - \{i_t\}\}\Big| \geqslant \Big|\bigcup_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} \{x_i K_h:\ i \in M_s - \{i_s\}\}\Big|$$
$$\geqslant d(H \cap K, K \cap \bigcap_{i=1}^{k} G_i) = d(K, K \cap \bigcap_{i=1}^{k} G_i)$$

which shows that (12) holds for $r = i_t$.

Case 2. $K \not\subseteq H$. In this case, $l = [K: H \cap K] > 1$. By Fact we have $l \leqslant h$. Let

$$\{H \cap K, b_1(H \cap K), \ldots, b_{l-1}(H \cap K)\}$$

partition K. If $1 \leqslant s < l$, $K_{s-1} \neq K_s$ and $b_s(H \cap K) \subseteq G_{i_s}$, then $b_s \in G_{i_s}$, $H \cap K \subseteq G_{i_s}$, and hence $K_{s-1} \subseteq K_0 = H \cap K \subseteq H \cap G_{i_s}$, contrary to the fact $i_s \in M_s$. So if $K_{s-1} \neq K_s$ and $1 \leqslant s < l$ then there exists an $x_{i_s} \in b_s(H \cap K) \setminus G_{i_s}$. If $K_{s-1} = K_s$ then

$$M_s = \{1 \leqslant i \leqslant k:\ a_i G_i \subseteq g_s H \text{ and } K \not\subseteq G_i\} = \{1 \leqslant i \leqslant k:\ a_i G_i \subseteq g_s H\},$$

for if $a_i G_i \subseteq g_s H$ then $G_i \subseteq a_i^{-1} g_s H \in G/H$, hence $G_i \subseteq H$ and it follows that $K \not\subseteq G_i$. (Note that $K \not\subseteq H$.) Since $a_j G_j \subseteq a_j H \in G/H$ and Γ is (M, S)-good, the set $\{1 \leqslant i \leqslant k:\ a_i G_i \subseteq g_s H\}$ is non-empty for each $s = 1, \ldots, h$. Hence $|M_s| \geqslant 1$ if $K_{s-1} = K_s$. In the case $K_{s-1} = K_s$, i_s is defined to be an element of M_s. We put $x_{i_s} = b_s$ if $K_{s-1} = K_s$ and $1 \leqslant s < l$. So far we have defined i_s for every $s = 1, \ldots, h$ and x_{i_s} for all $1 \leqslant s < l$. We choose x_i to be an element of $K \setminus G_i$ if

$$i \in I(K) - \Big(\bigcup_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} (M_s - \{i_s\}) \cup \{i_1, \ldots, i_{l-1}\}\Big)$$

Note that $M_s \neq \emptyset$ for each s, $r = i_h \in M_h \subseteq I(K)$. If $K_{s-1} = K_s$ and $1 \leqslant s < l$ then we have $G_{i_s} \subseteq H$ and hence $x_{i_s} = b_s \in K - H \cap K \subseteq K \setminus G_{i_s}$. From the above it is easy to see that $x_i \in K \setminus G_i$ for each $i \in I(K) - \{r\}$. If $K_{s-1} \neq K_s$ and $i \in M_s - \{i_s\}$ then $x_i \in K_{s-1} \setminus H \cap G_i \subseteq K_{s-1} \subseteq K_0 = H \cap K$, hence $x_i \notin x_{i_t} K_h \subseteq b_t(H \cap K)$ and $x_i K_h \neq x_{i_t} K_h$ for $t = 1, \ldots, l-1$. If $1 \leqslant s < t < l$ then $x_{i_s} K_h \subseteq b_s(H \cap K)$, $x_{i_t} K_h \subseteq b_t(H \cap K)$ and hence $x_{i_s} K_h \neq x_{i_t} K_h$. Thus

$$\Big|\{x_i(K \cap \bigcap_{i=1}^{k} G_i):\ i \in I(K) - \{r\}\}\Big|$$
$$\geqslant \Big|\{x_i K_h:\ i \in \bigcup_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} (M_s - \{i_s\}) \cup \{i_1, \ldots, i_{l-1}\}\}\Big|$$
$$= \Big|\bigcup_{\substack{s=1\\K_{s-1}\neq K_s}}^{h} \{x_i K_h:\ i \in M_s - \{i_s\}\}\Big| + l - 1 \geqslant d(H \cap K, K \cap \bigcap_{i=1}^{k} G_i) + [K: H \cap K] - 1$$
$$\geqslant d(H \cap K, K \cap \bigcap_{i=1}^{k} G_i) + d(K, H \cap K) = d(K, K \cap \bigcap_{i=1}^{k} G_i).$$

By the above, Theorem 8 is proved by induction.

THEOREM 8′. *Let M be a commutative monoid and S be a non-empty M-connected set not containing the zero element of M. Let Γ be an (M, S)-good class of groups such that if a group belongs to Γ then so do its normal subgroups of finite indices. Let $G \in \Gamma$ and H its subnormal subgroup of finite index. Then $1 + d(G, H)$ is the smallest positive integer k such that there exist $a_1, \ldots, a_k \in G$ and subnormal subgroups $G_1, \ldots, G_k$ of G such that* (2) *is an (M, S)-covering of G and $\bigcap_{i=1}^{k} G_i = H$.*

Proof. In the case $H = G$ the result holds trivially.

We now suppose that $H \neq G$. Let $H_0 = H \subset H_1 \subset \ldots \subset H_n = G$ be a maximal chain of subgroups of G such that H_{i-1} is normal in H_i for all $i = 1, \ldots, n$. Obviously each H_i $(0 \leqslant i \leqslant n)$ is subnormal in G. It is clear that every $(Z^+ \cup \{0\}, \{1\})$-covering is an (M, S)-covering. Hence by Theorem 1 there exists an (M, S)-covering $\{a_i G_i\}_{i=1}^k$ of G such that all the subgroups G_i are subnormal in G, $\bigcap_{i=1}^k G_i = H_0 = H$ and

$$k = 1 + \sum_{i=0}^{n-1} ([H_{i+1} : H_i] - 1) = 1 + d(G, H).$$

If (2) is an (M, S)-covering of G such that each G_i is subnormal in G and $\bigcap_{i=1}^k G_i = H$, then $I(G) = \{1 \leqslant i \leqslant k\colon G \nsubseteq G_i\}$ is non-empty since $\bigcap_{i=1}^k G_i = H \neq G$, and hence by Theorem 8 we have

$$k \geqslant 1 + |I(G)| - 1 \geqslant 1 + d(G, \bigcap_{i=1}^k G_i) = 1 + d(G, H).$$

This completes the proof.

From Theorem 4 and Theorem 8, we immediately have

THEOREM 9. *Let G be a group and $G_1, \ldots, G_k$ its subnormal subgroups such that for some $a_1, \ldots, a_k \in G$ (2) is a coset decomposition of G. (Hence (2) is a $(Z^+ \cup \{0\}, \{1\})$-covering of G.) Then for any subgroup K of G not contained in $\bigcap_{i=1}^k G_i$, there exist a proper l-subset $\{i_1, \ldots, i_l\}$ of $\{1 \leqslant i \leqslant k\colon K \nsubseteq G_i\}$ and l distinct left cosets $C_1, \ldots, C_l$ of $K \cap \bigcap_{i=1}^k G_i$ in K such that $C_s \cap G_{i_s} = \emptyset$ for all $s = 1, \ldots, l$, where $l = d(K, K \cap \bigcap_{i=1}^k G_i)$. (By l-subset we mean a subset of l elements.)*

Remark 7. Let G be a group and (2) its coset decomposition. J. Mycielski and W. Sierpiński [4] made a conjecture that $k \geqslant 1 + f([G:G_i])$ if G is abelian and $[G:G_i]$ is finite. Š. Znám ([13]) conjectured for the infinite cyclic group G that $k \geqslant 1 + f([G: \bigcap_{i=1}^k G_i])$. In [3] Korec confirmed these two conjectures by further proving that if all the G_i are normal in G then $k \geqslant 1 + d(G, \bigcap_{i=1}^k G_i)$ holds. We remark that Korec's result follows from Theorem 9 if we put $K = G$.

Combining Theorem 4 with Theorem 8′ we obtain

THEOREM 9′. *Let G be a group and H its subnormal subgroup of finite index. Then $1 + d(G, H)$ is the smallest positive integer k such that there exists a left coset decomposition (2) of G in which all the G_i are subnormal in G and $\bigcap_{i=1}^k G_i = H$.*

Remark 8. If (2) is a left coset decomposition of G with all the G_i subnormal in G, then by Theorem 3 and Theorem 9′ (or Theorem 9) we have $k \geqslant 1 + d(G, \bigcap_{i=1}^k G_i)$, and hence by Theorem 6

$$[G: \bigcap_{i=1}^k G_i] \leqslant 2^{k-1} \leqslant k!.$$

COROLLARY 2. *Let G be a group. Suppose that it can be partitioned into k cosets C_i of subnormal subgroups G_i. Then all the indices $[G:G_i]$ are finite and*

$$k \geqslant 1 + f([n_1, \ldots, n_k]) \tag{14}$$

where $n_i = [G:G_i]$ and $[n_1, \ldots, n_k]$ denotes the least common multiple of $n_1, \ldots, n_k$.

Proof. If C_i is a left coset of G_i then we suppose that $C_i = a_i G_i$ and put $G_i^* = G_i$. If C_i is a right coset we then suppose that $C_i = G_i a_i$ and put $G_i^* = a_i^{-1} G_i a_i$. By Fact, G_i^* is subnormal in G and $[G:G_i^*] = [G:G_i]$. Note that $C_i = a_i G_i^*$ for $i = 1, \ldots, k$. By Corollary 1, the index of $H = \bigcap_{i=1}^k G_i^*$ is finite. Since

$$[G:G_i][G_i^*:H] = [G:G_i^*][G_i^*:H] = [G:H],$$

$n_i = [G:G_i]$ is finite for each i and $[n_1, \ldots, n_k]$ divides $[G:H]$. Hence by Theorem 9′ and Theorem 6 we have

$$k \geqslant 1 + d(G, H) \geqslant 1 + f([G:H]) \geqslant 1 + f([n_1, \ldots, n_k]).$$

COROLLARY 3. *Let G be a group and H its subgroup of finite index. Let H_G denote the core of H in G (i.e. the largest normal subgroup of G contained in H). If H is subnormal in G then*

$$2^{[G:H]-1} \geqslant [G:H_G] \geqslant [G:H] \geqslant 1 + d(G, H_G). \tag{15}$$

Proof. Assume that H is subnormal in G. Let $h = [G:H]$ and $\{Ha_i\}_{i=1}^h$ be a partition of G. Since H is subnormal in G, $a_i^{-1} H a_i$ is also subnormal in G (cf. Fact). By Fact,

$$[G: \bigcap_{i=1}^h a_i^{-1} H a_i] \leqslant \prod_{i=1}^h [G: a_i^{-1} H a_i] = \prod_{i=1}^h [G:H] = h^h < \infty.$$

Note that $\{a_i(a_i^{-1} H a_i)\}_{i=1}^h$ is a left coset decomposition of G. Applying Theorem 9′, we get

$$h \geqslant 1 + d(G, \bigcap_{i=1}^h a_i^{-1} H a_i).$$

We note that

$$H_G = \bigcap_{g \in G} g^{-1} H g = \bigcap_{i=1}^h \bigcap_{x \in H} (x a_i)^{-1} H (x a_i) = \bigcap_{i=1}^h \bigcap_{x \in H} a_i^{-1}(x^{-1} H x) a_i = \bigcap_{i=1}^h a_i^{-1} H a_i.$$

Thus

$$[G:H] = h \geqslant 1 + d(G, H_G).$$

Hence by Theorem 6 we have

$$2^{[G:H]-1} \geqslant 2^{d(G, H_G)} \geqslant [G:H_G] \geqslant [G:H] \geqslant 1 + d(G, H_G).$$

Remark 9. As far as I know, the only known result concerning the core of a subgroup is the following (Theorem 1.6.9 of [6]): If H is a subgroup of finite index in a group G then $[G:H_G]$ divides $[G:H]!$.

Combining Corollary 1, Theorem 7, Theorem 9 and Theorem 9′, we obtain

THEOREM 10. *Let G be a locally nilpotent group.*

(a) *If* (2) *is a left coset decomposition of G and K is a subgroup of G not contained in $\bigcap_{i=1}^{k} G_i$, then*

$$[K:K\cap \bigcap_{i=1}^{k} G_i] \leqslant [G: \bigcap_{i=1}^{k} G_i] < \infty$$

and

$$l = f([K:K\cap \bigcap_{i=1}^{k} G_i]) < |I(K)| = |\{1 \leqslant i \leqslant k:\ K \not\subseteq G_i\}|, \tag{16}$$

moreover there exist l distinct left cosets $C_1, \ldots, C_l$ of $K\cap \bigcap_{i=1}^{k} G_i$ in K and an l-subset $\{i_1, \ldots, i_l\}$ of $I(K)$ such that $C_s \cap G_{i_s} = \varnothing$ for every $s = 1, \ldots, l$.

(b) *If H is a subgroup of finite index in G then $1+f([G:H])$ is the smallest positive integer k such that there exists a left coset decomposition $\{a_i G_i\}_{i=1}^{k}$ of G satisfying $\bigcap_{i=1}^{k} G_i = H$.*

As a consequence of Theorem 5, Theorem 8 and Theorem 8′ we have

THEOREM 11. *Let M be a commutative monoid and S a non-empty M-connected set not containing the zero element of M.*

(a) *Let*

$$\{a_i(\mathrm{mod}\, n_i)\}_{i=1}^{k} \tag{17}$$

be an (M, S)-covering system of residue classes on Z, and let $[n_1, \ldots, n_k]$ denote the least common multiple of $n_1, \ldots, n_k$. If $d < N = [n_1, \ldots, n_k]$ is a divisor of N then

$$l = f(N/d) < |I(d)| = |\{1 \leqslant i \leqslant k:\ n_i \nmid d\}| \tag{18}$$

and moreover there exist an l-subset $\{i_1, \ldots, i_l\}$ of $I(d)$ and l distinct integers $m_1, \ldots, m_l$ such that $0 \leqslant m_s < N/d$ and $n_{i_s} \nmid d m_s$ for every $s = 1, \ldots, l$.

(b) *For any positive integer N, $1+f(N)$ is the smallest positive integer k such that there exists an (M, S)-covering $\{a_i(\mathrm{mod}\, n_i)\}_{i=1}^{k}$ of Z with the property that N is the least common multiple of $n_1, \ldots, n_k$.*

Remark 10. In [15] Znám conjectured that if (17) is an irredundant covering (i.e. a $(Z^+ \cup \{0\}, Z^+)$-covering of Z) then $k \geqslant 1+f([n_1, \ldots, n_k])$. In [8] R. J. Simpson proved this conjecture, in fact he proved more, his result states that (18) holds if $d < N = [n_1, \ldots, n_k]$ is a divisor of N provided that (17) is an irredundant covering of Z.

If p^α is the highest power of a prime p which divides N, then for $d = \frac{N}{p^\alpha} p^{\beta-1}$ where $0 < \beta \leqslant \alpha$, (18) is equivalent to the following:

$$|\{1 \leqslant i \leqslant k:\ p^\beta | n_i\}| \geqslant 1+(\alpha-\beta+1)(p-1).$$

5. Some open problems. In this section we will pose several unsolved problems.

It is known that $k!$ is the maximum value of the index $[G: \bigcap_{i=1}^{k} G_i]$ in a group G with an irredundant covering (2) (cf. [11]). What about $s(k)$, the maximum value of $[G: \bigcap_{i=1}^{k} G_i]$ in a group G with a partition into k left cosets of subnormal subgroups $G_1, \ldots, G_k$? By Remark 8, $s(k) \leqslant 2^{k-1}$. On the other hand, $s(k) \geqslant 2^{k-1}$ since for $k > 1$

$$\{0(\mathrm{mod}\, 2^{k-1}),\ 1(\mathrm{mod}\, 2),\ 2(\mathrm{mod}\, 2^2), \ldots, 2^{k-2}(\mathrm{mod}\, 2^{k-1})\}$$

is a coset decomposition of the additive group Z. So we have $s(k) = 2^{k-1}$.

The above discussion, together with Theorem 1, Theorem 3, Theorem 8′, Theorem 9′, the latter parts of Theorem 10 and Theorem 11, suggests

PROBLEM 1. Let G be a group and H its subgroup of finite index. Let $1+c_1(G, H)$ $(1+c_2(G, H))$ be the smallest positive integer k such that there exists an irredundant covering (a left coset decomposition) (2) of G with the property $\bigcap_{i=1}^{k} G_i = H$. Does $c_1(G, H) = c_2(G, H)$ hold? If we denote by $1+c_{(M,S)}(G, H)$ the smallest $k \in Z^+$ such that there exists an (M, S)-covering (2) of G which satisfies $\bigcap_{i=1}^{k} G_i = H$ (where M is a commutative monoid and S is a non-empty set not containing the zero element of M), then

$$c_1(G, H) = c_{(Z^+\cup\{0\}, Z^+)}(G, H), \qquad c_2(G, H) = c_{(Z^+\cup\{0\}, \{1\})}(G, H)$$

and $(1+c_{(M,S)}(G, H))! \geqslant [G:H]$ (cf. Theorem 3). What is the precise value of $c_{(M,S)}(G, H)$? Is the following conjecture true?

Let (2) be an (M, S)-covering of a group G. Then for any subgroup K of G, either $I(K) = \{1 \leqslant i \leqslant k:\ G_i \not\supseteq K\}$ is empty or there exist an $r \in I(K)$ and $x_i \in K \setminus G_i$ $(i \in I(K)-\{r\})$ such that

$$|\{x_i(K\cap \bigcap_{i=1}^{k} G_i):\ i \in I(K)-\{r\}\}| \geqslant c_{(M,S)}(K, K\cap \bigcap_{i=1}^{k} G_i).$$

(Note that $[K:K\cap \bigcap_{i=1}^{k} G_i] \leqslant [G: \bigcap_{i=1}^{k} G_i] \leqslant k! < \infty$.)

The following problem seems easier than the first one.

PROBLEM 2. Let M be a commutative monoid and S, X two arbitrary sets. Is it true that for any (M, S)-covering $\mathscr{A}$ of X there exists an M-connected set S' such that $\mathscr{A}$ is an (M, S')-covering of X? Is the condition that S is M-connected unnecessary for Theorem 8 to hold?

Let H be a subgroup of G with finite index. If H is not subnormal in G, $d(G, H)$ is defined to be $d(G, H_G) - d(H, H_G)$. (Notice that $[G:H_G] \leqslant [G:H]! < \infty$.)

PROBLEM 3. Is it true that if H is a subgroup of finite index in a group G then $d(G, H) \geqslant d(K, H \cap K)$ for any subgroup K of G?

I conjecture that this problem has a positive answer.

PROBLEM 4. Prove or disprove the following conjecture:

Let G be a finite group and $G_1, \ldots, G_k$ its subgroups. If $a_1, \ldots, a_k \in G$ then

$$\Big|\bigcup_{i=1}^{k} a_i G_i\Big| \geqslant \Big|\bigcup_{i=1}^{k} G_i\Big| .$$

PROBLEM 5. If we replace (2) by $\{a_i G_i b_i\}_{i=1}^{k} (a_i, b_i \in G)$, will Theorem 8′ still hold? Can we omit the word "subnormal" from Corollary 2?

We give one more problem.

PROBLEM 6. Does the following conjecture hold?

Let (17) be an irredundant covering of Z with $1 < n_1 < n_2 < \ldots < n_k$. Suppose that $[n_1, \ldots, n_k]$ has the standard form $p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ $(p_1 < \ldots < p_r)$. Let $n_0 = 1$ and

$$\{\alpha\colon p_s^{\alpha} \| n_i \text{ for some } i = 0, 1, \ldots, k\} = \{\alpha_{s0}, \alpha_{s1}, \ldots, \alpha_{si_s}\}$$

where $\alpha_{s0} = 0 < \alpha_{s1} < \ldots < \alpha_{si_s} = \alpha_s$, $s = 1, \ldots, r$. (By $p^{\alpha} \| n$ we mean $p^{\alpha} | n$ and $p^{\alpha+1} \nmid n$). Then for any s $(1 \leqslant s \leqslant r)$ and t $(1 \leqslant t \leqslant i_s)$

$$|\{1 \leqslant i \leqslant k\colon n_i \text{ is of the form } p_1^{\beta_1} \ldots p_{s-1}^{\beta_{s-1}} p_s^{\alpha_{st}} \ (0 \leqslant \beta_j \leqslant \alpha_j)\}| \geqslant (\alpha_{st} - \alpha_{s(t-1)})(p_s - 1) .$$

(We remark that $f([n_1, \ldots, n_k]) = \sum_{s=1}^{r} \sum_{t=1}^{i_s} (\alpha_{st} - \alpha_{s(t-1)})(p_s - 1)$).

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References

[1] H. B. Enderton, *Elements of Set Theory*, Academic Press, Inc., New York 1977.
[2] M. Hall, Jr., *The Theory of Groups*, New York 1959.
[3] I. Korec, *On a generalization of Mycielski's and Znám's conjectures about coset decomposition of Abelian groups*, Fund. Math. 85 (1974), 41–48.
[4] J. Mycielski et W. Sierpiński, *Sur une propriété des ensembles linéaires*, Fund. Math. 58 (1966), 143–147.
[5] B. H. Neumann, *Groups covered by finitely many cosets*, Publ. Math. Debrecen 3 (1954), 227–242.
[6] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag New York Inc., 1982.
[7] J. S. Rose, *A Course on Group Theory*, Cambridge University Press, 1978.
[8] R. J. Simpson, *Regular coverings of the integers by arithmetic progressions*, Acta Arith. 45 (1985), 145–152.
[9] Zhi-Wei Sun and Zhi-Hong Sun, *Some results on covering systems of congruences*, J. Southwest-China Teachers Univ., 1987, no. 1, 10–15.

[10] M. Suzuki, *Group Theory* I (Grundlehren der mathematischen Wissenschaften; 247), Springer-Verlag, New York 1982.
[11] M. J. Tomkinson, *Groups covered by finitely many cosets or subgroups*, Comm. Algebra, 15 (1987), 845–859.
[12] Š. Znám, *On Mycielski's problem on systems of arithmetic sequences*, Colloq. Math. 15 (1966), 201–204.
[13] — *On exactly covering systems of arithmetic sequences*, Colloquia Math. Soc. János Bolyai, 2. Number Theory, Debrecen 1968.
[14] — *A remark to a problem of J. Mycielski on arithmetic sequences*, Colloq. Math. 20 (1969), 69–70.
[15] — *On properties of systems of arithmetic sequences*, Acta Arith. 26 (1975), 279–283.

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