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One point extensions of trees and quadratic forms

by

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Abstract. Let T be any tree with underlying graph one of the graphs D_n, $\tilde{D}_n$, $n \geqslant 4$, E_m, $\tilde{E}_m$, $m = 6, 7, 8$. Let A be one point extension of the path-algebra kT by an indecomposable preinjective kT-module M. Using methods of tilting theory and of vector space categories, we prove that A is of (infinite) tame representation type if and only if the Tits-form t_A of A is weakly non-negative.

Introduction. Let T be an oriented tree with underlying graph one of the graphs D_n, $\tilde{D}_n$, $n \geqslant 4$, E_m, $\tilde{E}_m$, $m = 6, 7, 8$. Let

$$A = \begin{bmatrix} k & M \\ 0 & kT \end{bmatrix}$$

be any one point extension of kT by an indecomposable preinjective kT-module M. The aim of the present paper is to prove the following:

THEOREM A. *The algebra A is of* (*infinite*) *tame representation type if and only if the Tits-form t_A of A is weakly non-negative.*

In representation theory of finite dimensional algebras it is common to associate to an algebra A a quadratic form in order to study the representation type of A or other invariants of mod A. We refer, for example, to [1] and also to the long list of papers cited there, which are dealing with related questions. Moreover, we refer to [11] for a detailed study of relations between quadratic forms and various module categories. Finally, we like to mention [9] received during the preparation of the present paper. In this work J. A. de la Peña proves an analogue to Theorem A for the so-called "hyperbolic algebras".

The present paper is divided as follows: In the first section we recall preliminary results. Any notion used in our paper and not defined in Section 1 can be found in [3] or [11]. In the second section we introduce sequences of triangular matrix algebras A induced by tilting functors. These sequences behave nicely in relation to the Tits-form t_A and to the representation type of A. In the third section using the above mentioned sequences we reduce the proof of Theorem A to the study of the so-called *A'-maximal tame algebras*. For any Tits-form associated to an A'-maximal tame algebra there is a convenient presentation such that one can see easily that the Tits-form is weakly non-negative. This presentation is achieved via a computer program and can be obtained from the author upon request.

There is a connection between the present work and paper [7], where Theorem A was proved for one point extensions of path algebras with underlying graph the linear quiver A_n. Here, the machinery introduced in [7] is simplified and it is applied directly to triangular matrix algebras.

1. Preliminaries.

1.1. Algebras and quivers. Throughout this paper, all algebras R are considered to be finite dimensional over a field k, associative, unitary and basic. The field k is assumed to be algebraically closed. All R-modules are finitely generated right R-modules. The corresponding module category is denoted by $\operatorname{mod} R$. Let Q be a quiver and kQ the path algebra of Q. Denote by $I(n)$ the ideal generated by all paths of length n. Any basic, finite dimensional algebra R is of the form kQ/J where Q is a uniquely determined quiver and J is an ideal with $I(n) \subseteq J \subseteq I(2)$. The ideal J is always generated by a set of relations $\{r_s/s\}$ and $\operatorname{mod} R$ can be identified with the category of the representations of Q^{op} satisfying the set of relations $\{r_s^{op}/s\}$ (cf. [5], [11]). The pair $(Q^{op}, \{r_s^{op}/s\})$ is called the *bound quiver* of R.

Any algebra R has exactly one of three possible representation types. It can have finite representation type (F), tame infinite representation type (T) or wild representation type (W) (cf. [10]). We introduce an ordering on these types $F < T < W$.

1.2. Triangular matrix algebras and vector space categories. Let C and B be two finite-dimensional algebras and M a C-B-bimodule. Consider the triangular matrix algebra

$$A = \begin{bmatrix} C & M \\ 0 & B \end{bmatrix}.$$

We will identify, as usual, the category $\operatorname{mod} A$ with the corresponding comma category (cf. [6], [10]). The A-modules will be represented as triples (X, f, Y) with X a C-module, Y a B-module and f a C-module homomorphism from X to $\operatorname{Hom}_B(M, Y)$. Let $\boldsymbol{D}$ be the full subcategory of $\operatorname{mod} B$ consisting of the modules Y with $\operatorname{Hom}(M, Y_i) \neq 0$ for any indecomposable direct summand Y_i of Y. The representation type of A depends on the representation type of B and on the representation type of the full subcategory $\boldsymbol{C}$ of $\operatorname{mod} A$ consisting of the triples (X, f, Y) with $Y \in \boldsymbol{D}$.

If A is *one point extension* of B by a B-module M, i.e.

$$A = \begin{bmatrix} k & M \\ 0 & B \end{bmatrix},$$

then the representation type of $\boldsymbol{C}$ can be determined using vector space categories as follows (cf. [10], [11]): Denote by $|-|$ the functor $\operatorname{Hom}_B(M, -)\colon \operatorname{mod} B \to \operatorname{mod} k$ and by $\boldsymbol{K}$ the vector space category $(\boldsymbol{D}, |-|)$. The category $\boldsymbol{C}$ is representation equivalent to the subspace category $U(\boldsymbol{K})$. Consequently, in order to determine the representation type of $\boldsymbol{C}$, we have to find $\boldsymbol{K}$ and the representation type of $U(\boldsymbol{K})$. Usually, it is enough to determine the full subcategory $\operatorname{Ind}(\boldsymbol{D})$ of $\boldsymbol{D}$ with objects the non-isomorphic indecomposable objects $Y \in \boldsymbol{D}$ and then the values $\dim_k |Y|$.

We recall this procedure in case B is a hereditary algebra and M is a preinjective indecomposable B-module, (cf. [3], [10]). Denote by $\Gamma(B)$ the Auslander–Reiten quiver of B. Consider the well-determined slice S_M of $\Gamma(B)$ with unique source the vertex $[M] \in \Gamma(B)$. Denote by $\Gamma(B)_0$ the set of vertices of $\Gamma(B)$. Let $\varphi\colon \Gamma(B)_0 \to \boldsymbol{Z}$ be the function defined by $\varphi([X]) = \dim_k |X|$, where X is the indecomposable module corresponding to vertex $[X]$ of $\Gamma(B)$. The function φ takes the value 1 on any vertex of S_M. Now, using the additivity of φ we determine $\operatorname{Ind}(\boldsymbol{D})$ and $\dim_k |Y|$ at any object $Y \in \operatorname{Ind}(\boldsymbol{D})$. Notice that $\boldsymbol{D}$ is usually identified with its image under $|-|$.

1.3. Tilting theory. (cf. [2], [4], [11]). Let R be an algebra and L a tilting R-module. Let E be the endomorphism ring of L over R. Consider the functors:

$$F = \operatorname{Hom}_R(L, -)\colon \operatorname{mod} R \to \operatorname{mod} E, \qquad F' = \operatorname{Ext}_R^1(L, -)\colon \operatorname{mod} R \to \operatorname{mod} E,$$

$$G = -\otimes_E L\colon \operatorname{mod} E \to \operatorname{mod} R, \qquad G' = \operatorname{Tor}_E^1(-, L)\colon \operatorname{mod} E \to \operatorname{mod} R.$$

1.3.1. (i) The functors F and G induce inverse equivalences between $\operatorname{Ker} F'$ and $\operatorname{Ker} G'$.

(ii) The functors F' and G' induce inverse equivalences between $\operatorname{Ker} F$ and $\operatorname{Ker} G$.

(iii) The pair $(\operatorname{Ker} F', \operatorname{Ker} F)$ forms a torsion theory in $\operatorname{mod} R$, where $\operatorname{Ker} F'$ is the torsion part and $\operatorname{Ker} F$ is the torsion-free part. Similarly, the pair $(\operatorname{Ker} G, \operatorname{Ker} G')$ forms a torsion theory in $\operatorname{mod} E$, where $\operatorname{Ker} G$ is the torsion part and $\operatorname{Ker} G'$ is the torsion-free part.

Let

$$A = \begin{bmatrix} C & M \\ 0 & B \end{bmatrix}$$

be a triangular matrix algebra and L a tilting B-module satisfying the condition $\operatorname{Hom}_B(M, DTrL) = 0$. Denote by η the unit of the adjunction corresponding to the adjoint pair $(-\otimes_C M, \operatorname{Hom}_B(M, -))$.

1.3.2. (i) The A-module $L = (C, \eta_C, C \otimes_C M) \oplus (0, o, L)$ is a tilting module.

(ii) If the induced by L torsion theory in $\operatorname{mod} B$ splits, then the induced by L torsion theory in $\operatorname{mod} A$ splits too.

(iii) The representation type of A is less than or equal to the representation type of $\operatorname{End} L$ (cf. [6]).

1.4. Algebras and quadratic forms. Let q be an integral quadratic form. The form q is said to be *weakly positive*, if $q(x) > 0$ for any positive integral vector x. It is said to be *weakly non-negative*, if $q(x) \geqslant 0$ for any positive integral vector x and there is some positive integral vector z with $q(z) = 0$. The form q is said to be *strongly indefinite* if there is some positive integral vector z with $q(z) < 0$ (cf. [7]).

Let $R = kQ/J$ be a connected finite-dimensional algebra such that Q does not have oriented cycles. Denote by $Q(0)$ the set of vertices of Q and by $Q(1)$ the set of arrows of Q. Let α be the idempotent of kQ corresponding to the vertex $\alpha \in Q(0)$. Let W be a *system of relations* for the ideal J (cf. [1]). For any two vertices $\alpha, \beta \in Q(0)$, denote by $r(\alpha, \beta, W)$ the cardinality of $W \cap (\alpha J \beta)$. Let $t_R(x)$ be the quadratic form given by the formula

$$t_R(x) = \sum_{\alpha \in Q(0)} x_\alpha^2 - \sum_{(\alpha \to \beta \in Q(1))} x_\alpha x_\beta + \sum_{(\alpha, \beta \in Q(0))} r(\alpha, \beta, W) x_\alpha x_\beta .$$

The form t_R is called the *Tits-form* of R (cf. [1]).

1.4.1. Let R be a connected finite dimensional k-algebra of the form kQ/J, where Q is a quiver without oriented cycles. If the Auslander–Reiten quiver of R contains a preprojective component, then the following are equivalent:

(i) The algebra R is representation finite.

(ii) The Tits-form t_R is weakly positive (cf. [1], 3.3).

Given two integral quadratic forms q_i: $Q^n \to Q$, $i = 1, 2$, we recall (cf. [7], 2.1.) that q_1 is said to be of *superior type* to q_2, if for any positive integral vector x there is a natural number $m \geq 1$ and positive integral vectors $y_1, y_2, \ldots, y_m$ of Q^n such that $q_1(x) \geq \sum_{i=1}^{m} q_2(y_i)$.

1.4.2. We recall from [7] the following:

Let q_1 and q_2 be two quadratic forms such that q_1 is of superior type to q_2.

(i) If q_2 is weakly positive, then q_1 is weakly positive too.

(ii) If q_2 is weakly non-negative, then q_1 is either weakly positive or weakly non-negative.

(iii) If q_1 is strongly indefinite, then q_2 is strongly indefinite too.

1.4.3. Let $R = kQ/J$ be a connected finite-dimensional algebra such that Q does not have oriented cycles and let L be a tilting R-module. If R and $\operatorname{End} L$ have global dimension less than or equal to 2 and the induced by L torsion theory splits, then t_R is of superior type to $t_{\operatorname{End} L}$ (cf. [7], 2.3.).

1.4.4. Notice that the type of t_R is completely determined by its behaviour on the set of the dimension vectors of the indecomposable R-modules (cf. [7], 2.4.).

2. Operations on preinjective components of path-algebras. A connected quiver T is said to be a (*simple*) *tree* if T does not have parallel edges or oriented cycles. Let kT be the path algebra of a tree and $\Gamma(kT)$ be its Auslander–Reiten quiver. The quiver T coincides with the full subquiver J of $\Gamma(kT)$ consisting of the injective vertices of $\Gamma(kT)$. We call J the *injective slice of* $\Gamma(kT)$. For any vertex $x \in T$ denote by $S(x)$ the corresponding simple kT-module, by $I(x)$ its injective hull and by $P(x)$ its projective cover.

Any indecomposable preinjective kT-module M is uniquely determined by the values $\dim_k \operatorname{Hom}_B(M, I(x))$ as x runs through all vertices of T. Hence, M is uniquely determined by the values $\varphi([I(x)])$ of the function φ as $[I(x)]$ runs through all vertices of J (cf. 1.2.).

DEFINITION 2.1. A vertex $[I(x)] \in \Gamma(kT)_0$ is said to be a *right-extendable* one, or just *r.e.*, if $[I(x)]$ is a source of J. A vertex $[I(x)] \in \Gamma(kT)_0$ is said to be a *right-deletable* one, or just *r.d.*, if $[I(x)]$ is a sink of J.

DEFINITION 2.2. A pair $([M], [I(x)])$ of $\Gamma(kT)_0 \times \Gamma(kT)_0$ is said to be a *t-right extendable* pair of $\Gamma(kT)$, or just *t-r.e.*, if $[I(x)]$ is an r.e. vertex of $\Gamma(kT)$ and M is a preinjective kT-module which is not isomorphic to $P(x)$.

A pair $([M], [I(x)])$ of $\Gamma(kT)_0 \times \Gamma(kT)_0$ is said to be a *t-right deletable* pair of $\Gamma(kT)$, or just *t-r.d.*, if $[I(x)]$ is an r.d. vertex of $\Gamma(kT)$ and M is a preinjective kT-module which is not isomorphic to $I(x)$.

Given a vertex $x_i \in T$, denote by $s_i T$ the tree formed from T by reversing the direction of all arrows starting or ending at x_i. Denote by τM the Auslander–Reiten translate $DTrM$ and by $\tau^- M$ the translate $TrDM$.

Let A be the one point extension of kT by an indecomposable preinjective kT-module M and $([M], [I(x_1)])$ be a t-r.e. pair of $\Gamma(kT)$. The simple module $S(x_1)$ coincides with its projective cover $P(x_1)$ since the vertex $x_1 \in T$ is a source of T.

Let $P(x_1), P(x_2), \ldots, P(x_n)$ be a full set of representatives of the isomorphism classes of the indecomposable projective kT-modules. The module

$$L_1 = \tau^- P(x_1) \oplus P(x_2) \oplus \ldots \oplus P(x_n)$$

is a tilting kT-module, the induced by L_1 torsion theory splits and the endomorphism ring of L_1 is isomorphic to $k(s_1 T)$ (cf. [4], [10], [11]). Denote by F_1 the functor $\operatorname{Hom}_{kT}(L_1, -)$ induced by L_1 (cf. 1.3.1.). Let $\boldsymbol{L}_1$ be the corresponding A-module and $A(x_1)^+$ be the endomorphism ring $\operatorname{End} \boldsymbol{L}_1$ (cf. 1.3.2.).

It is well known that

$$A(x_1)^+ \cong \begin{bmatrix} k & F_1 M \\ 0 & k(s_1 T) \end{bmatrix}.$$

Let t_A and t_{A^+} be the Tits-forms of A and $A(x_1)^+$, respectively.

PROPOSITION 2.3. (i) *The representation type of A is less than or equal to the representation type of $A(x_1)^+$.*

(ii) *The quadratic form t_A is of superior type to t_{A^+}.*

Proof. Since $([M], [I(x_1)])$ is a t-r.e. pair of $\Gamma(kT)$, it is true that $\operatorname{Hom}(M, \tau L) \cong \operatorname{Hom}(M, P(x_1)) = 0$. The first assertion is obtained by 1.3.2. Since the global dimension of $A(x_1)^+$ is less than or equal to 2, the second assertion is obtained by 1.4.3.

We recall the description of the quiver $\Gamma(k(s_1 T))$ for later reference. This quiver can be formed from $\Gamma(kT)$ as follows: For any arrow α: $[I(x_1)] \to y$ of

$\Gamma(kT)$ add to $\Gamma(kT)$ an arrow of the form $\sigma\alpha\colon y \to \tau^{-}[I(x_1)]$. Delete the vertex $[P(x_1)]$ and also all arrows starting at it.

Triangular right extendable sequences. Using t-r.s. pairs, we construct sequences of one point extensions as follows: The first term of such a sequence is A. If $([M], [I(x_1)])$ is a t-r.e. pair of $\Gamma(kT)$, then the second term is $A(x_1)^+$ as above. If there is a t-r.e. vertex $([F_1 M], [I(x_2)])$ of $\Gamma(k(s_1 T))$, then we consider a full set $\{P_1(x_1), P_1(x_2), \ldots, P_1(x_n)\}$ of representatives of the isomorphism classes of the indecomposable projective $k(s_1T)$-modules. Again, the module

$$L_2 = P_1(x_1) \oplus \tau^{-} P_1(x_2) \oplus \ldots \oplus P_1(x_n)$$

is a tilting $k(s_1T)$-module, the induced by L_2 torsion theory splits and the endomorphism ring of L_2 is isomorphic to $k(s_2 s_1 T)$. Denote by F_2 the functor induced by L_2 and construct $A(x_1)^+(x_2)^+$.

It is allowed to continue inductively forming $A(x_1)^+(x_2)^+ \ldots (x_{i-1})^+$ from $A(x_1)^+(x_2)^+ \ldots (x_i)^+$, as long as there is a t-r.e. vertex $([F_i \ldots F_1 M], [I(x_{i+1})])$ of $\Gamma(k(s_i \ldots s_1 T))$.

It is impossible to proceed further only if for some positive integer r there are no t-r.e. vertices in $\Gamma(k(s_r \ldots s_1 T))$ at all. This occurs if and only if the injective slice of $\Gamma(k(s_r \ldots s_1 T))$ contains exactly one source $[I(x)]$ and this source has the property that $F_r \ldots F_1 M$ is isomorphic to the projective cover $P(x)$ of the simple $k(s_r \ldots s_1 T)$-module $S(x)$. In this case, the algebra $\Omega = A(x_1)^+(x_2)^+ \ldots (x_r)^+$ is hereditary, because it is one point extension of a hereditary algebra by an indecomposable projective module. So, the representation type of Ω as the type of the Tits-form q_Ω are easily determined.

A sequence of one point extensions formed using t-r.e. pairs as above will be called a *t-r.e. sequence*.

PROPOSITION 2.4. (i) *If $A(i)$ and $A(j)$ are members of a t-r.e. sequence with $i \leqslant j$, then the representation type of $A(i)$ is less than or equal to the representation type of $A(j)$.*

(ii) *The Tits-form $t_{A(i)}$ is of superior type to $t_{A(j)}$.*

(iii) *If $A(i)$ is of infinite representation type and $t_{A(j)}$ is weakly non-negative, then $t_{A(i)}$ is weakly non-negative too.*

(iv) *If the Tits-form $t_{A(i)}$ is strongly indefinite, then the Tits-form $t_{A(j)}$ is strongly indefinite too.*

Proof. The assertions (i) and (ii) are immediate consequences of 2.3.

(iii) The form $t_{A(i)}$ cannot be weakly positive because $A(i)$ is of infinite representation type (cf. 1.4.1). Since $t_{A(i)}$ is of superior type to $t_{A(j)}$, we get that $t_{A(i)}$ is weakly non-negative by 1.4.2.

(iv) Since $t_{A(i)}$ is of superior type to $t_{A(j)}$ the assertion follows again from 1.4.2.

Now, we define an 'inverse procedure'. Let A be one point extension of kT by an indecomposable kT-module M and let $([M], [I(x_1)])$ be a t-r.d. pair of $\Gamma(kT)$. Denote by x_1 the corresponding sink of T and by $P(x_1)$ the projective cover of the simple kT-module $S(x_1)$. Moreover, denote by $I_1(x_1)$ the injective hull of the simple $k(s_1T)$-module $S_1(x_1)$ and by $P_1(x_1)$ its projective cover. Let L_1 be the tilting $k(s_1T)$-module constructed as before by taking the direct sum of $\tau^{-} P_1(x_1)$ and of the remaining indecomposable projective $k(s_1T)$-modules. The endomorphism ring E of L_1 is isomorphic to kT. Denote by G_1 the functor $-\otimes_E L_1$ (cf. 1.3).

Consider the one point extension

$$A(x_1)^- = \begin{bmatrix} k & G_1 M \\ 0 & k(s_1 T) \end{bmatrix}.$$

Since M is not isomorphic to $I(x_1)$, the indecomposable $k(s_1T)$-module $G_1 M$ is not isomorphic to $P_1(x_1)$. Moreover, $G_1 M$ remains a preinjective module. Hence, the pair $([G_1 M], [I_1(x_1)])$ is a t-r.e. pair of $\operatorname{mod} k(s_1T)$. So, using the tilting $k(s_1T)$-module L_1, we may construct the tilting $A(x_1)^-$-module L_1. The endomorphism ring $\operatorname{End} L_1$ is isomorphic to A.

We recall the description of the quiver $\Gamma(k(s_1T))$ for later reference. This quiver can be formed from $\Gamma(kT)$ as follows: For any arrow $\alpha\colon y \to [P(x_1)]$ of $\Gamma(kT)$ add to $\Gamma(kT)$ an arrow of the form $\sigma^{-}\alpha\colon \tau[P(x_1)] \to y$. Delete the vertex $[I(x_1)]$ and all arrows ending at it.

Triangular right deletable sequences. Using t-r.d. pairs we construct sequences of one point extensions as follows: The first term of such a sequence is A. If $([M], [I(x_1)])$ is a t-r.d. pair of $\operatorname{mod} kT$, then the second term is $A(x_1)^-$ as above. If there is a t-r.d. pair $([G_1 M], [I(x_2)])$ of $\Gamma(k(s_1T))$, then construct the tilting $k(s_2 s_1 T)$-module L_2. Denote by $I_2(x_2)$ the injective hull of the simple $k(s_2 s_1 T)$-module $S(x_2)$. The pair $([G_2 G_1 M], [I_2(x_2)])$ is a t-r.e. pair of $\Gamma(k(s_2 s_1 T))$. Let $A(x_1)^-(x_2)^-$ be the one point extension of $k(s_2 s_1 T)$ by $G_2 G_1 M$. The module L_2 can be extended to a tilting $A(x_1)^-(x_2)^-$-module L_2 whose endomorphism ring is isomorphic to $A(x_1)^-$. It is allowed to continue inductively, forming

$$A(x_1)^-(x_2)^- \ldots (x_{i+1})^-$$

from $A(x_1)^-(x_2)^- \ldots (x_i)^-$, as long as there is a t-r.d. vertex $([G_i \ldots G_1 M], [I(x_{i+1})])$ of $\Gamma(k(s_i \ldots s_1 T))$. The algebra $A(x_1)^-(x_2)^- \ldots (x_i)^-$ is isomorphic to the endomorphism ring of the tilting $A(x_1)^-(x_2)^- \ldots (x_{i+1})^-$-module L_{i+1}, which is induced by the tilting $k(s_{i+1} \ldots s_1 T)$-module L_{i+1}. It is impossible to continue further only if for some positive integer r there are no t-r.d. vertices in $\operatorname{mod} k(s_r \ldots s_1 T)$ at all. This happens if and only if the injective slice of $\Gamma(k(s_r \ldots s_1 T))$ has exactly one sink $[I(x)]$ and $I(x)$ is isomorphic to $G_r \ldots G_1 M$. A sequence of one point extensions formed using t-r.d. pairs as above will be called a *t-r.d. sequence*.

3. The classification. From now on, we assume that

$$A = \begin{bmatrix} k & M \\ 0 & kT \end{bmatrix}$$

is the one point extension of kT by an indecomposable kT-module M and that the underlying graph of T is one of the graphs D_n, $\tilde{D}_n$, $n \geq 4$, E_m, $\tilde{E}_m$, $m = 6, 7, 8$. The case of the Dynkin diagram A_n was studied in [7].

Let T_M be the full subquiver of T with vertices the vertices x of T satisfying the property that $[I(x)]$ is a successor of $[M]$ in $\Gamma(kT)$. Clearly, M is a kT_M-module. We attach to any algebra A the algebra

$$A(M) = \begin{bmatrix} k & M \\ 0 & kT_M \end{bmatrix}.$$

The algebra $A(M)$ will be called the *m-standard algebra* attached to A.

DEFINITION 3.1. The algebra A is said to be *m-standard* if A coincides with $A(M)$.

Let M be an indecomposable preinjective kT-module. Denote by S_M the slice of $\Gamma(kT)$ with unique source the vertex $[M]$. Let $I(x)$ be the injective indecomposable kT-module belonging to the τ^--orbit of M. Let T_x be a tree equal to S_M and let $I'(x)$ be the indecomposable injective kT_x-module corresponding to the vertex x of T_x. Consider the one point extensions:

$$A = \begin{bmatrix} k & M \\ 0 & kT \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} k & I'(x) \\ 0 & kT_x \end{bmatrix}.$$

Observe that A' is an m-standard algebra and that the injective slice of $\Gamma(kT_x)$ coincides with the slice $S_{I'(x)}$ of $\Gamma(kT_x)$ which has as unique source the vertex $[I'(x)]$.

Using the above notation, we have:

PROPOSITION 3.2. *If A is an m-standard algebra, then A belongs to a t-r.e. sequence with starting term the algebra A'.*

Proof. Since A is an m-standard algebra, the slice S_M intersects the τ-orbit of any indecomposable preinjective kT-module. If M is injective, then A coincides with A'. If M is not injective, then one sees easily that there is always a t-r.d. pair $([M], [I(x_1)])$ of $\Gamma(kT)$ such that $[I(x_1)]$ does not belong to S_M. Using this pair, we construct the algebra $A(x_1)^-$. Let S'_1 be the slice of $\Gamma(k(s_1T))$ having as unique source the vertex $[G_1M]$. If there is some t-r.d. pair $([G_1M], [I(x_2)])$ of $\Gamma(k(s_1T))$ with $[I(x_2)] \notin S'_1$, then we construct the algebra $A(x_1)^-(x_2)^-$. Continuing inductively, we obtain finally an algebra

$$A(x_1)^-(x_2)^- \dots (x_n)^- = \begin{bmatrix} k & G_nG_{n-1} \dots G_1M \\ 0 & k(s_ns_{n-1} \dots s_1T) \end{bmatrix}$$

with the property that the injective slice of $\Gamma(k(s_ns_{n-1} \dots s_1T))$ coincides with the slice S'_n of $\Gamma(k(s_ns_{n-1} \dots s_1T))$ which has as unique source the vertex $[G_nG_{n-1} \dots G_1M]$. Hence, $A(x_1)^-(x_2)^- \dots (x_n)^-$ equals to A'. Now, following exactly the opposite procedure starting with A' and using the corresponding t-r.e. pairs, we obtain a t-r.e. sequence satisfying the conclusion of the proposition.

Consider the one point extension

$$A' = \begin{bmatrix} k & M \\ 0 & kT \end{bmatrix},$$

where M is an indecomposable preinjective kT-module.

DEFINITION 3.3. The algebra A' is said to be a *start algebra* if A' is m-standard and M is an indecomposable injective kT-module.

Let A' be a start algebra.

DEFINITION 3.4. The one point extension A is said to be *A'-minimal wild* if the following conditions are satisfied:

(i) A belongs to a t-r.e. sequence with first term A'.

(ii) A is of wild representation type.

(iii) Any t-r.e. sequence with first term A' and containing $A = A(i)$ as member has the property that $A(i-1)$ is not of wild representation type.

DEFINITION 3.5. The one point extension A is said to be *A'-maximal tame* if the following conditions are satisfied:

(i) A belongs to a t-r.e. sequence with first term A'.

(ii) A is of tame representation type.

(iii) Either A is a hereditary algebra or any t-r.e. sequence with first term A' which contains $A = A(i)$ as member has the property that $A(i+1)$ is of wild representation type.

In the following we determine all A'-maximal tame or A'-minimal wild algebras which belong to t-r.e. sequences with starting term some start algebra A'.

Generally we proceed as follows: Let Δ be one of the diagrams D_n, $\tilde{D}_n$, $n \geq 4$, E_m, $\tilde{E}_m$, $m = 6, 7, 8$ and let x be a fixed vertex of Δ. Choose an orientation of Δ such that x is the unique source of the tree T obtained in this way. Consider the path algebra kT and the quiver $\Gamma(kT)$. The injective slice of $\Gamma(kT)$ has as unique source the vertex $[M]$. Let A' be the one point extension of kT by M. Clearly, A' is a start algebra and any start algebra can be obtained in this way.

Since there are no projective-injective kT-modules, the pair $([M], [I(x_1)])$ with $M = I(x_1)$, is a t-r.e. pair. (It is actually the unique t-r.e. pair.) Using this pair, we construct a t-r.e. sequence of one point extensions changing first the algebra kT to $k(s_1T)$ and M to F_1M. Now, choosing another t-r.e. vertex $([F_1M], [I(x_2)])$, we continue in the same manner until appears in our t-r.e. sequence some one point extension A such that either any of the possible t-r.e. pairs gives a wild algebra or there are no t-r.e. pairs at all. If A is tame, then this implies that A is an A'-maximal tame algebra. The procedure for the A'-minimal wild algebras is similar.

We present in detail the above described procedure in case of the Dynkin graph D_n, $n \geq 4$. The other cases are handled in the same way. Notice that we describe the one point extensions rather implicitly from the following data: The injective slice J of $\Gamma(kT)$ determines T and the values which takes the function φ

on the vertices of J determine the indecomposable module M. In fact we write down the part X of $\Gamma(kT)$ lying between S_M and J and calculate the values of φ at any vertex of X. By F, T and W we indicate the representation type of the corresponding vector space category. In the special case of D_n the representation type of this vector space category coincides with the representation type of the corresponding one point extension, since the path algebra kT has finite representation type. In almost all cases, the representation type of the vector space category is determined using the famous criteria of Kleiner (cf. [5], [10]) for finite and of Nazarova (cf. [8], [10]) for tame representation type.

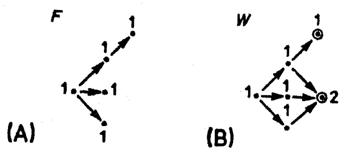
D_n, $n \geqslant 4$

```
        3
        |
1 — 2 — 4 ... n−1 — n
```

D_4. Any t-r.e. sequence with first term a start algebra A' is finite. The last term is a hereditary algebra Ω either of finite or tame representation type. If Ω is of tame representation type, then Ω is an A'-maximal tame algebra.

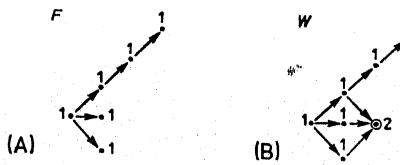
D_5. If A' is the start algebra formed using the indecomposable injective $I(i)$, $i = 1, 3, 4, 5$, then we obtain finite t-r.e. sequences with ending term some hereditary algebra Ω of finite or tame representation type. If Ω is of tame representation type, then it is an A'-maximal tame algebra.

If A' is the start algebra formed using the indecomposable injective $I(2)$, then the unique t-r.e. pair $([I(2)], [I(2)])$ gives an A'-minimal wild algebra as the second term.

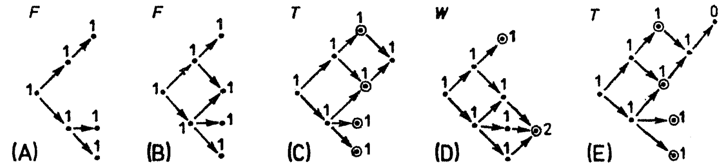


D_6. If A' is the start algebra formed using the indecomposable injective $I(i)$, $i = 1, 3, 5, 6$, then we obtain finite t-r.e. sequences with end term some hereditary algebra Ω of finite or tame representation type. If Ω is of tame representation type, then it is an A'-maximal tame algebra.

If A' is the start algebra formed using $I(2)$, then the unique t-r.e. pair $([I(2)], [I(2)])$ gives as second term an A'-minimal wild algebra.



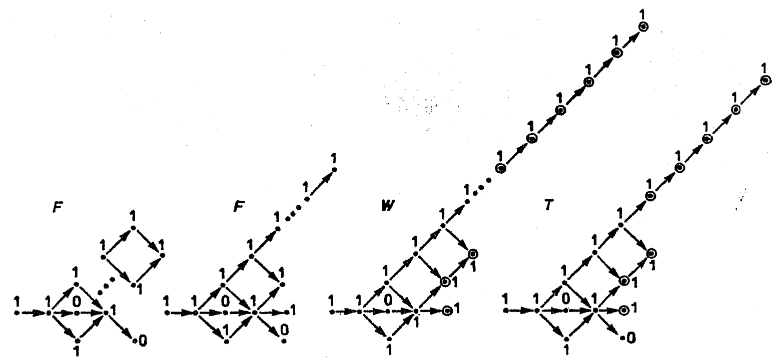
If A' is the start algebra formed using $I(4)$, then the second term of any t-r.e. sequence is term (B) below. The next term can be either (C) or (D), since there are two t-r.e. pairs. Term (D) is an A'-minimal wild algebra. In case where we consider (C) as the term next to (B), there are again two possible next terms. One of these is (E) which determines an A'-maximal tame algebra.

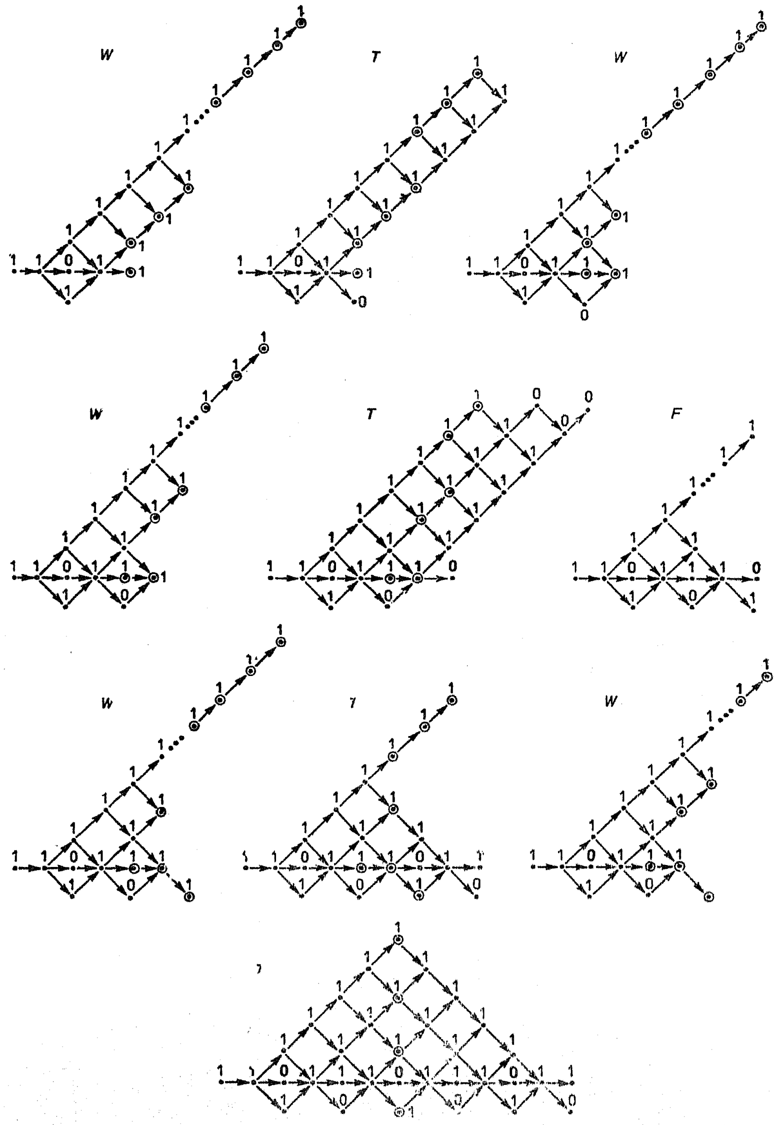


D_n, $n \geqslant 7$. We study separately the vertices with one, two or three neighbours.

Vertices with one neighbor. It is enough to consider the start algebras formed using the injective modules $I(n)$ and $I(3)$. Any t-r.e. sequence with first term the start algebra A' corresponding to $I(n)$ has as last term an algebra Ω which is the path algebra of a tree with underlying graph the graph D_{n+1}. There are no t-r.e. sequences containing A'-maximal tame or A'-minimal wild algebras.

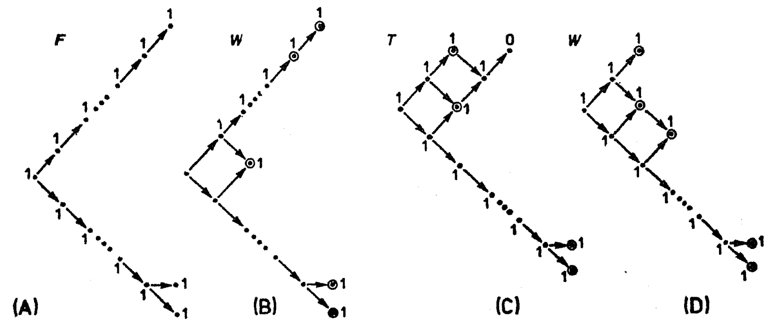
If the start algebra A' is formed using $I(3)$, we determine all A'-maximal tame or A'-minimal wild algebras following the same procedure as before. In the next list the A'-maximal tame algebras are denoted by (T) and the A'-minimal wild algebras are denoted by (W).





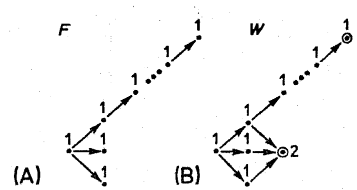
Vertices with two neighbours. Let A' be a start algebra formed using the injective $I(i)$, $4 \leqslant i \leqslant n-1$. This algebra corresponds to Figure (A) below and it is of finite representation type. For any i, $4 \leqslant i \leqslant n-3$, the next term is the one point extension corresponding to Figure (B). This algebra is A'-minimal wild.

For $i = n-2$, we obtain using t-r.e. sequences, one A'-maximal tame algebra corresponding to Figure (C) below and one A'-minimal wild algebra corresponding to Figure (D) below.



For $i = n-1$, the last term of any t-r.e. sequence is a hereditary A'-maximal tame algebra.

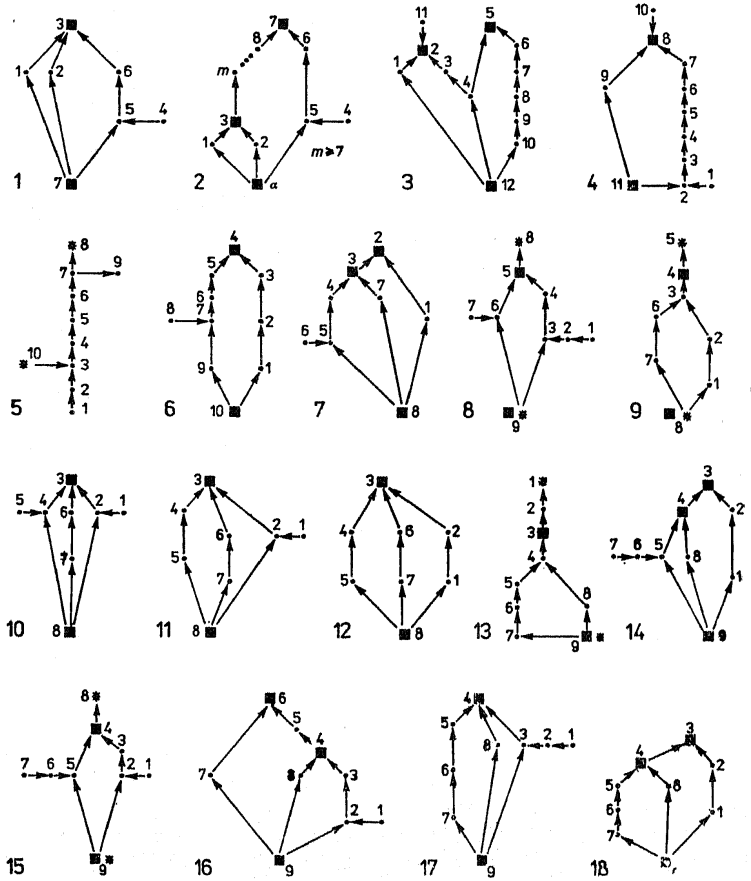
Vertices with three neighbours. Let A' be the start algebra formed using the injective $I(2)$. This algebra corresponds to Figure (A) below and it is always of finite representation type. Since there is a unique t-r.e. pair, the next term corresponds to Figure (B) below and it is always an A'-minimal wild algebra.



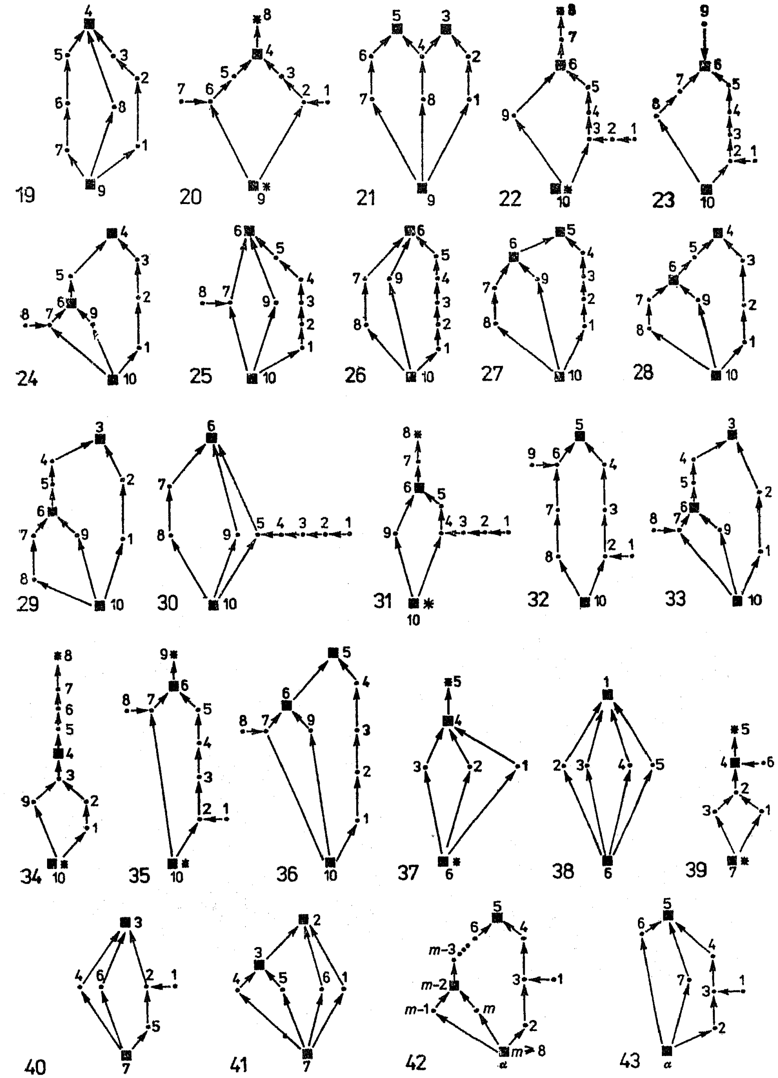
In the following two lists we describe via their bound quivers all A'-maximal tame and all A'-minimal wild algebras which are non-hereditary and belong to a t-r.e. sequence with first term some start algebra A'.

The system of relations S of any bound quiver $(Q, S = \{r_s/s\})$ in the two lists below contains exactly the following linear relations: The difference of any two paths between vertices of Q denoted by black squares, and any path between vertices of Q denoted by stars. In list B below there are additional relations in S in the case where there are arrows of Q denoted by Greek letters. There are three such cases. In the first case we have arrows denoted by χ, ψ, ω, $\varkappa$ and λ. The system S contains the relations $\lambda\chi$, $\varkappa\omega$, $\varkappa\psi-\lambda\psi$ and any other relation which can be defined using squares or stars. In the second case (B, 44), S consists of the relations $\lambda\varphi\chi$, $\varkappa\varphi\psi-\lambda\varphi\psi$, $\varkappa\varphi\omega$ and the two zero relations defined by the vertices denoted by stars. In the third case (B, 45) S consists of the relations $\lambda\varphi$, $\mu\varphi$, $\varkappa\chi$, $\mu\chi$, $\varkappa\psi$, $\lambda\psi$, $\varkappa\omega-\lambda\omega$ and $\lambda\omega-\mu\omega$.

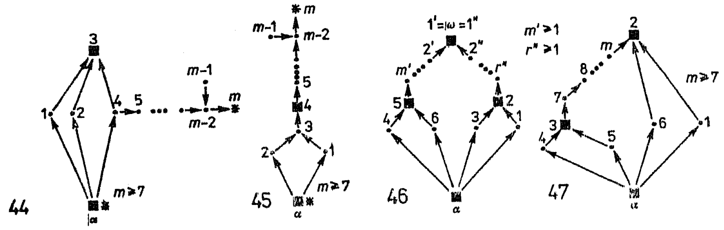
LIST A (A'-maximal tame algebras)



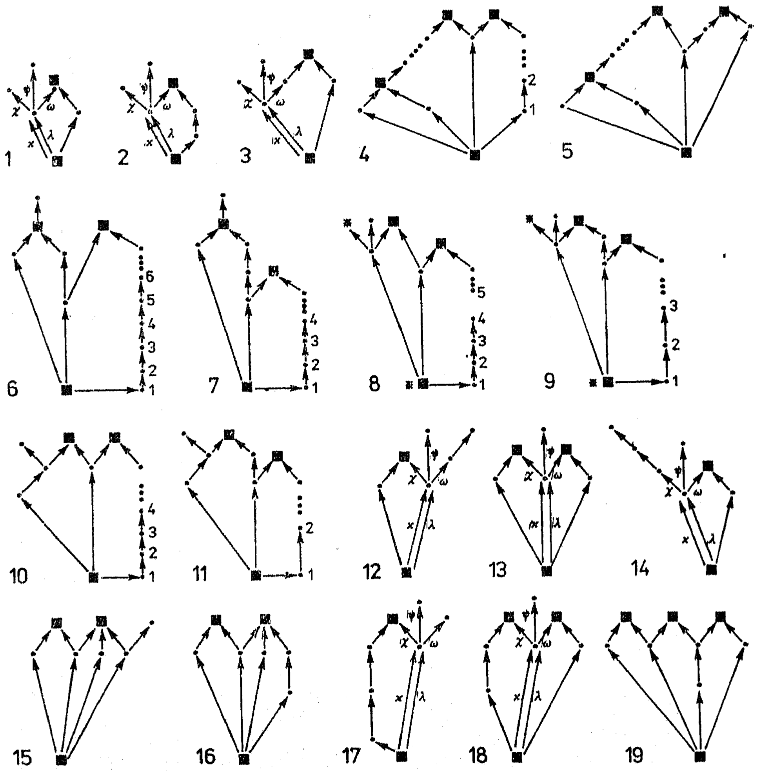
LIST A (continued)



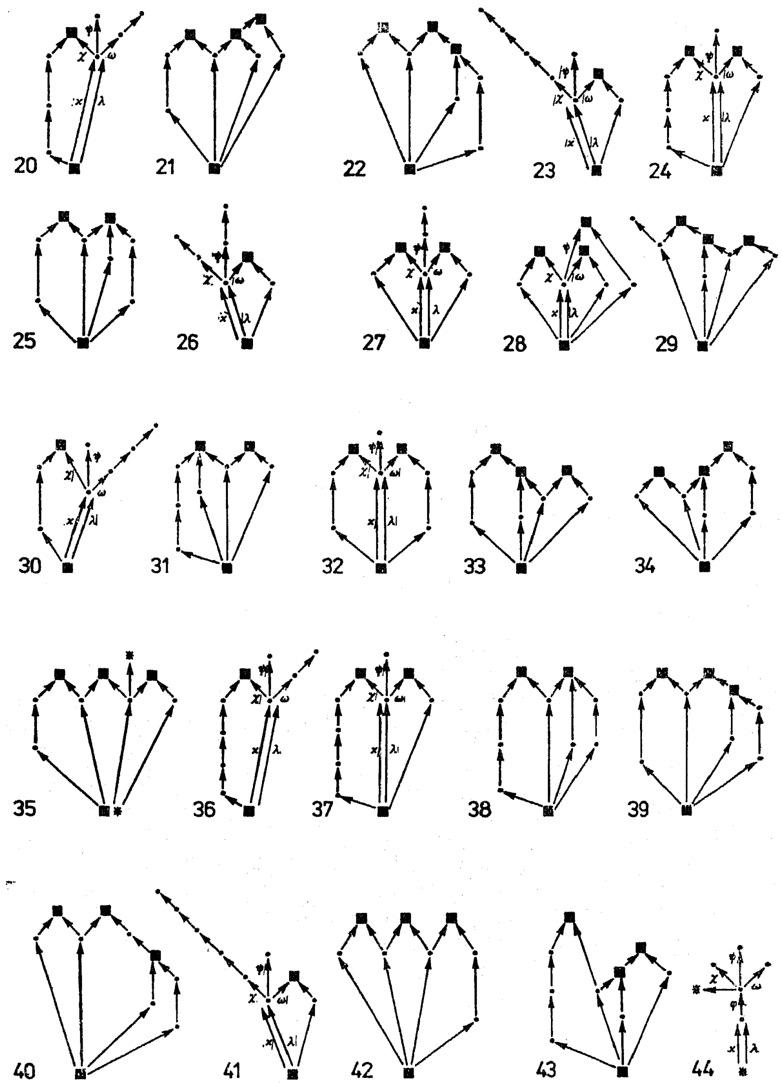
LIST A (continued)



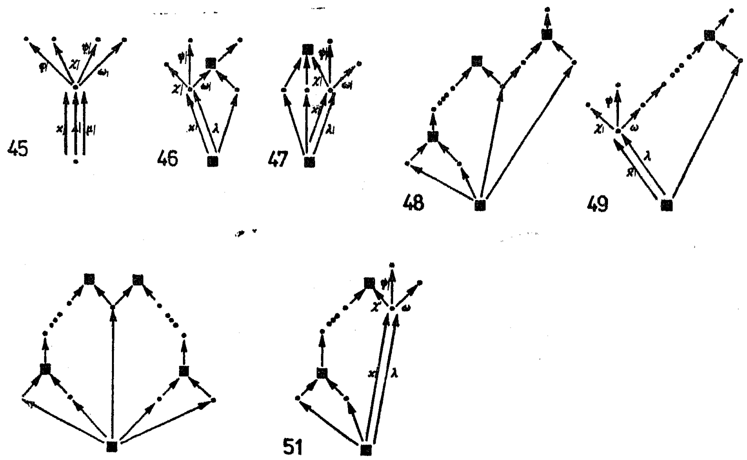
LIST B (A'-minimal wild algebras)



LIST B (continued)

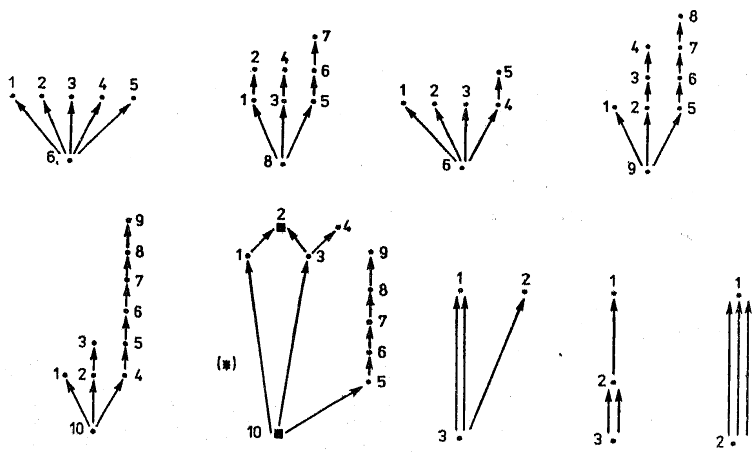


LIST B (continued)



LEMMA 3.6. *The Tits-form of any algebra with bound quiver one of the bound quivers of list* C *below is strongly indefinite.*

LIST C



Proof. The assertion of the lemma is well known for all algebras of list C whose bound quiver does not have any relation at all. The Tits-form of the algebra with

bound quiver that of (*) can be written as follows:

$$t_* = 120^{-1}(30(x_7-2x_8)^2+10(2x_6-3x_7)^2+5(3x_5-4x_6)^2 +3(4x_{10}-5x_5)^2+2(5x_2-x_{10})^2+10(3x_3-2x_{10}-2x_2)^2 +30(2x_1-x_2-x_{10})^2+30(2x_4-x_3)^2+120(x_9-x_8)x_9).$$

The form t_* evaluated at the positive integral vector (6, 2, 8, 4, 8, 6, 4, 2, 1, 10) takes the value -1.

PROPOSITION 3.7. (i) *The Tits-form of any A'-maximal tame algebra of list* A *is weakly non-negative.*

(ii) *The Tits-form of each A'-minimal wild algebra of list* B *is strongly indefinite.*

Proof. (i) The assertion is obtained checking case by case that any algebra of list A has weakly non-negative Tits-form. A convenient presentation of these forms, constructed using a computer program, can be obtained from the author upon request.

(ii) The bound quiver of each A'-minimal wild algebra contains as full subquiver one of the bound quivers of list C. Hence, the Tits-form of any A'-minimal wild algebra is strongly indefinite.

PROPOSITION 3.8. *Let A be an m-standard algebra.*

(i) *If A is of tame representation type, then the Tits-form t_A of A is weakly non-negative.*

(ii) *If A is of wild representation type, then the form t_A is strongly indefinite.*

Proof. By 3.2., there is a t-r.e. sequence having as member the algebra $A = A(i)$ and with first term some start algebra A'. If A is an A'-maximal tame algebra there is nothing to prove. Otherwise there is some t-r.e. pair which determines an algebra $A(i+1)$ of tame representation type. Continuing inductively, we obtain after finitely many steps an A'-maximal tame algebra because either the preinjective component of kT is finite or the additive function φ is unbounded on the preinjective component of $\Gamma(kT)$.

(ii) There are t-r.e. sequences having as member A and with first term a start algebra A'. Any such t-r.e. sequence can be constructed using first some t-r.d. sequence with starting term A and ending term A' and taking then the opposite direction as in 3.2. Notice that any t-r.d. sequence with last term A' is finite since we are using vertices of preinjective components. We construct a special t-r.d. sequence as follows: The first term is A. If A is A'-minimal wild, then the remaining t-r.d. pairs are chosen arbitrarily until we obtain A'. If A is not A'-minimal, then there is a t-r.d. pair $([M], [I(x_1)])$ such that $A(x_1)^-$ is again of wild representation type. If $A(x_1)^-$ is A'-minimal wild, then the remaining t-r.d. pairs are chosen again arbitrarily until A' is obtained, otherwise there is some t-r.d. pair $([G_1 M], [I(x_2)])$ of $\Gamma(k(s_1 T))$ such that $A(x_1)^-(x_2)^-$ has again wild representation type. Since A' is of finite or tame representation type and any t-r.d. sequence is finite, we obtain finally an algebra A'' which is A'-minimal wild. The remaining

t-r.d. pairs are chosen arbitrarily until we obtain A'. Hence, there is some t-r.e. sequence with members $A(k)$ and $A(i) = A$, with $k \leqslant i$ and such that $A(k)$ is A'-minimal wild. Now, the second claim of our proposition follows from 2.4 (iv).

THEOREM 3.9. *Let*

$$A = \begin{bmatrix} k & M \\ 0 & kT \end{bmatrix}$$

be a one point extension of a path algebra kT, where the underlying graph of T is one of the graphs D_n, $\tilde{D}_n$, $n \geqslant 4$, E_m, $\tilde{E}_m$, $m = 6, 7, 8$ and where M is an indecomposable preinjective kT-module. The Tits-form of A is weakly non-negative if and only if A is of tame representation type.

Proof. According to 1.4.4, it is enough to study the behaviour of t_A on the dimension vectors of the indecomposable A-modules. Let $A(M)$ be the m-standard algebra attached to A. Any indecomposable A-module U can be considered as an indecomposable module over kT or as an indecomposable module over $A(M)$. Hence, either

$$t_A(\dim U) = t_{kT}(\dim U) \quad \text{or} \quad t_A(\dim U) = t_{A(M)}(\dim U).$$

Case 1. *The algebra A is of tame representation type.* If $A(M)$ is of finite representation type, then $q_{A(M)}(\dim U) > 0$ for any indecomposable $A(M)$-module U, by 1.4.1. Since A is of tame representation type, the representation type of kT has to be tame. But in this case it is well known that for any indecomposable kT-module U we have $t_{kT}(\dim U) \geqslant 0$ and there is some U with $t_{kT}(\dim U) = 0$. So, t_A is weakly non-negative.

If $A(M)$ is of tame representation type, then by 3.8 the form $t_{A(M)}$ is weakly non-negative. The representation type of kT is either finite or tame, hence t_{kT} is either weakly positive or weakly non-negative. So, t_A is weakly non-negative.

Case 2. *The algebra A is of wild representation type.* Because kT is always of finite or tame representation type, the algebra $A(M)$ has to be of wild representation type. Since $A(M)$ is m-standard, $t_{A(M)}$ is strongly indefinite by 3.8. Hence t_A is strongly indefinite too. This closes the proof of our theorem.

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