The shrinking property and the $\mathcal{B}$-property in ordered spaces

by

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Dedicated to Professor Kiyoshi Iséki on his 70th birthday

Abstract. In this paper, it will be shown that

1. Introduction. It is known from Y. Yamini [Y2] that every generalized ordered topological space has the weak $\mathcal{B}$-property. He also pointed out that $\alpha_1$ (with the order topology) does not have the $\mathcal{B}$-property. As remarked in [R], it is known that every uncountable regular cardinal with the order topology has the shrinking property, but does not have the $\mathcal{B}$-property. Note that no uncountable regular cardinal is paracompact. In this paper, it will be shown that

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First we establish our terminology. Let $\mathcal{W}$ be a collection of subsets of a space $X$ and $x$ be an infinite cardinal. A collection $\mathcal{F} = \{ F(U) : U \in \mathcal{W} \}$ of subsets of $X$ is a shrinking of $\mathcal{W}$ if $cl(F(U)) = U$ for every $U$ in $\mathcal{W}$. A space $X$ has the $\kappa$-shrinking property if for every open cover $\mathcal{W}$ of size $\leq \kappa$, there is a shrinking of $\mathcal{W}$ consisting of open sets (open shrinking) which covers $X$. A collection $\{ U_\xi : \xi < \kappa \}$ of subsets of a space $X$ by $\kappa$ is said to be increasing if $U_\xi \subseteq U_\zeta$ if $\xi < \zeta$. A space $X$ has the weak $\mathcal{A}(\alpha)$-property (the $\mathcal{A}(\alpha)$-property) if for every increasing open cover $\{ U_\xi : \xi < \alpha \}$ of $X$, there is an (increasing, respectively) open shrinking $\{ V_\xi : \xi < \alpha \}$ (i.e., $cl(V_\xi) = U_\xi$ for each $\xi < \alpha$) which covers $X$. A space has the shrinking (weak $\mathcal{A}$, $\mathcal{A}$-)property if it has the $\kappa$-shrinking (weak $\mathcal{A}(\kappa)$, $\mathcal{A}(\kappa)$) property for every infinite cardinal $\kappa$. By these definitions, it is easy to show the following implications.

Note that in a normal space, shrinking property is equivalent to that its every open cover has a shrinking which consists of closed sets (closed shrinking) and covers the space. Also note that in a normal space, for every open cover $\mathcal{W}$ which has a finite subcover, there is a closed shrinking of $\mathcal{W}$ which covers the space. A subset $S$ of a regular uncountable cardinal $\kappa$ is said to be stationary if $S$ intersects with every closed unbounded (club) subset of $\kappa$ with the order topology.

A linearly ordered topological space (LOTS) is the topological space with the topology induced by usual open intervals on a linearly ordered set. A generalized ordered topological space (GOTS) is a triple $(X, \mathcal{F}, <)$ where $<$ is a linear ordering of the set $X$ and $\mathcal{F}$ is a $T_1$ topology which has a base consisting of convex sets, see [EL]. Here a subset $C$ of a linearly ordered set $(X, <)$ is convex if $(a, b) \subseteq C$ for every $a, b \in C$ (where $C$ denotes the open interval with end points $a, b$). Note that every GOTS can be embedded in a LOTS as a closed subspace (II), every subspace of LOTS is GOTS, and every GOTS is hereditarily countable normal and hereditarily paracompact.

2. The shrinking property in GOTS. In this section, we shall show that every GOTS has the shrinking property hereditarily. First we recall some basic facts. A cut of a linearly ordered set $(X, <)$ is a pair $(A, B)$ of subsets of $X$ such that $A \cup B = X$, $A \cap B = \emptyset$, and if $x \in A$ and $y \in B$, then $x < y$. Note that $(0, X)$ and $(X, 0)$ are cuts. Let $cX$ denote the set of all cuts of $X$. Define a linear ordering $<$ on $cX$ by $(A, B) < (A', B')$ if $A \subseteq A'$ and $A \neq A'$. By identifying $x \in X$ with the cut $(\{ x \}, X \setminus \{ x \})$, we get $cX$ with the topology induced by $<$ is a compactification of the LOTS $X$ and $< x, y >$ is equivalent to $x < y$ whenever $x, y \in X$, see [E, 3.12.3(b)], [F], [K]. Note that a LOTS $X$ is compact if and only if every subset $A$ of $X$ has the least upper bound (lub), see [E, 3.12.3(a)], [HK], and every subset $A$ of a LOTS $X$ can be represented as the union of disjoint maximal convex sets, furthermore if $A$ is open in $X$, such maximal convex sets are also open, see [E, 2.7.5(b)]. In the proof of the next theorem, $(x, y)$, $(x, y, ...)$ denote intervals in $cX$.

**Theorem 2.1.** Every LOTS has the shrinking property.

**Proof.** Let $U$ be an open cover of a LOTS $X$. For every $U \in \mathcal{W}$, define $\mathcal{W}(U)$ denotes the set of all maximal convex (open) sets contained in $U$. For every $V \in \mathcal{W}$, define $V' = \bigcup \{ \mathcal{W}(U) : U \in \mathcal{W} \}$, if there is a finite set $\{ V_0, \ldots, V_n \} \subseteq \mathcal{W}$ such that $V = V_0$, $V' = V_n$, and $V_i \cap V_{i+1} \neq \emptyset$ for $0 < i < n$ (we call such a finite set finite chain), then $\mathcal{W}$ is an equivalent relation on $\mathcal{W}$. Let $\{ \mathcal{W}_\lambda : \lambda \in A \}$ be the all equivalence classes. Then $\bigcup \mathcal{W}_\lambda : \lambda \in A \} \subseteq \mathcal{W}$, and $\mathcal{W}_\lambda \cap \mathcal{W}_{\lambda'} \neq \emptyset$ for each $\lambda \neq \lambda'$, then we can construct a closed shrinking of $\mathcal{W}$ covering $\mathcal{W}_\lambda \cap \mathcal{W}_{\lambda'}$. Since each $\mathcal{W}_\lambda$ is open in $X$, the subspace topology on $\mathcal{W}_\lambda$ coincides with the topology induced on $\mathcal{W}_{\lambda'}$ by the restriction of the linear order $<$ to $\mathcal{W}_\lambda$. Therefore it suffices to show the next claim.

**Claim.** Let $\mathcal{W}$ be an open cover of a LOTS $X$ such that for every $V \in \mathcal{W}$, $V' = \{ \mathcal{W}(U) : U \in \mathcal{W} \}$, $V \subseteq V'$ holds. Then $\mathcal{W}$ has a closed shrinking which covers $X$.

**Proof of Claim.** Let $(x, <)$ be the compactification of $(X, <)$ described previously. Define $a = (0, X)$ and $b = (X, 0)$. Then $(a, b)$ is the first (last, respectively) element of $cX$. For every $V \in \mathcal{W}$, denote the set

$$\{ x \in cX : \exists y \in V (x < x' < x) \}$$

Then it is easy to show that each $V$ is open convex in $(cX, <)$, since each $V$ is open convex in $(X, <)$.

**Fact 1.** $\mathcal{W}' = \{ V' : V \in \mathcal{W} \}$ covers $cX - (a, b)$.

**Proof of Fact 1.** Let $x$ be a point in $cX - (a, b)$. Since $X$ is topologically dense in $cX$, there are $y, z \in X$ such that $y < x < x' = z$. Then $y < x' < z$ holds. Since $y, z < x$, there is a finite chain $\{ V_0, \ldots, V_n \} \subseteq \mathcal{W}$ such that $y \in V_0$ and $z \in V_n$. By induction on $n$, we shall show that $z \in V_i$ for some $i < n$. Whenever $n = 0$, $x \in V_0$ holds since $y < x < z$ and $V_0$ is convex. Assume that it is valid for $n < n$. Take a point $z_0$ in $V_{n-1} \cap V_0$. If $z_0 < z$ holds, then $x \in V'_n$ holds since $z_0, z \in V_0$. If $z_0 < z$ holds, then by the inductive assumption $x \in V'_n$ holds for some $i < n$ since $y \in V_0$, $z_0 \in V_{n-1}$, and $y < x < z_0$. Thus $x \in V'_n$ covers $cX - (a, b)$.

Note that if $X$ has the first (last) element, then $\mathcal{W}'$ also covers the point $a$ ($b$, respectively). Also note that if $X$ has no first (last) element, then $\mathcal{W}'$ does not cover the point $a$ ($b$, respectively). This completes the proof of the fact.
Fix a point \( c \in cX - \{a, b\} \). For every \( U \in \Upsilon \), define \( U' = \bigcup \{ V' : V \in \Upsilon(U) \} \). Put \( \Psi = \{ U' : U \in \Psi \} \). Whenever \( X \) has the last (first) element, since \( \Psi \) covers the compact set \([a, b']\) (\([a, b']\), respectively), \( U' \) has a shrinking \( \mathcal{F}_3 \) (\( \mathcal{F}_0 \), respectively) covering \([a, b']\) (\([a, b']\), respectively) consisting of closed subsets of \([a, b']\) (\([a, b']\), respectively). Next we shall show the following fact.

**Fact 2.** If \( X \) does not have a last (first) element, then \( \Psi \) has a shrinking \( \mathcal{F}_0 \) covering \([c, b']\) (\([a, c]\), respectively) consisting of closed subsets of \([c, b']\) (\([a, c]\), respectively).

**Proof of Fact 2.** Assume that \( X \) does not have a last element (the other case is similar). A subset \( A \) of \([c, b']\) said to be cofinal in \( b \) if for every \( x \in [c, b'] \), there is an \( a \in A \) such that \( x \leq a' \). Let \( x \in b \). We order such an \( A \) with \( (a_i : \gamma < x) \) by type \( x \). By induction, we shall find a strict increasing cofinal (in \( b \)) sequence \( (x_i : \gamma < x) \) such that \( x_i \) is the lub of \([x_i : \beta < \gamma] \). Assume that \( \gamma = 0 \) and \( x_0 \) has been defined for every \( \beta < \gamma \). When \( \gamma \) is successor, since \( [a_0 : \beta < \gamma] \cup [x_0 : \beta < \gamma] \) is a \( \lambda < x \) such that \( \lambda < x \), we have \( x_\lambda < x_0 \). \( x_\lambda \) is defined as \( \gamma \). When \( \gamma \) is limit, let \( x_\gamma \) be the lub of \([x_\beta : \beta < \gamma] \). Since \( cX \) is compact and by the definition of \( x \), each \( x_\gamma \) is well defined and \( x_0 < b \). Thus we can find such a sequence. Furthermore, by the definition of \( x \), \( x \) is a regular cardinal.

If \( x = \alpha \) holds, then \( [c, b'] = \bigcup \{ \{c, x_\gamma\} : \gamma < x \} \) is Lindelöf since each \( \{c, x_\gamma\} \) is compact. Thus we can find such an \( \mathcal{F}_0 \) whenever \( x = \alpha \). Next assume that \( x \) is a regular uncountable cardinal. Since \([c, b'] = \bigcup \{ \{c, x_\gamma\} : \gamma < x \} \) and each \( \{c, x_\gamma\} \) is compact, every open cover of \([c, b']\) has a subcover of size \( < \kappa \). Thus we may assume that \( \|\Psi\| < \kappa \). Well order \( \Psi = \{ U_\gamma : \beta < \xi \} \) where \( \lambda < \kappa \). For every \( \gamma \) in \( \text{Lim}(\kappa) = \{ \gamma : \gamma < \xi \} \), fix an \( \alpha(\gamma) < \lambda \) such that \( x_{\alpha(\gamma)} \in U_\gamma \). Then there is an \( \alpha(\gamma) < \lambda \) such that \( [x_{\alpha(\gamma)}, x_\gamma] \) is \( U_\gamma \) for every \( \gamma \) in \( \text{Lim}(\kappa) \). Note that \( \text{Lim}(\kappa) \) is cub in \( x \). There are two cases.

**Case 1.** \( \lambda < \kappa \).

In this case, since \( \lambda < \kappa \) and \( \kappa \) is regular uncountable, there is a stationary set \( S \subseteq \text{Lim}(\kappa) \) and an \( \alpha < \lambda \) such that \( \alpha(\gamma) = \alpha \) for every \( \gamma \in S \). Then by the pressing down lemma, there is a stationary set \( S' \subseteq S \) and a \( \gamma(0) < \kappa \) such that \( f(\gamma) = \gamma(0) \) for every \( \gamma < S' \). This means that \( [x_{\alpha(\gamma)}, x_\gamma] \subseteq U_\gamma \) holds for every \( \gamma < S' \). Thus \( (x_{\alpha(\gamma)}, b'] \subseteq [x_{\alpha(\gamma)}, b'] \subseteq U_\gamma \) holds. On the other hand, since \( \{c, x_{\alpha(\gamma)}\} \) is compact, there is a finite subcollection of \( \Psi \) which covers \([c, x_{\alpha(\gamma)}]\). Therefore a finite subcollection of \( \Psi \) which covers \([a, b']\). Thus we can find such a closed shrinking \( \mathcal{F}_0 \).

**Case 2.** \( \lambda = \kappa \).

In this case, put \( C = \{ \gamma \in \text{Lim}(\kappa) : \forall \beta < \gamma(\alpha(\beta) < \gamma) \} \), then \( C \) is cub in \( x \). There are two subcases.

**Subcase 1.** \( \gamma \in C : \alpha(\gamma) < \gamma \)

In this subcase, by the pressing down lemma, there is a stationary set \( S \subseteq \{ \gamma \in \text{Lim}(\kappa) : \forall \beta < \gamma(\alpha(\beta) < \gamma) \} \) and an \( \alpha < \lambda = x \) such that \( \alpha(\gamma) = \alpha \) for every \( \gamma \in S \). Then as in Case 1, we can find such a \( \mathcal{F}_1 \).

**Subcase 2.** \( S = \{ \gamma \in C : \alpha(\gamma) < \gamma \} \) is stationary.

In this subcase, for every \( \gamma', \gamma \in S \) with \( \gamma' < \gamma \), \( \alpha(\gamma') < \alpha(\gamma) \) holds. Therefore elements of \( \alpha'.S = \{ \alpha(\gamma) : \gamma \in S \} \) are all distinct. By the pressing down lemma, there are a stationary set \( S' \subseteq S \) and a \( \gamma(0) < \gamma = x \) such that \( f(\gamma) = \gamma(0) \) for every \( \gamma \in S' \). This means that \( [x_{\alpha(\gamma)}, x_\gamma] \subseteq U_\gamma \) for every \( \gamma \in S' \). Take a finite subcollection \( \{ U_\alpha : \alpha \in \mathcal{G} \} \) of \( \Psi' \) which covers \([c, x_{\alpha(\gamma)}]\), where \( \mathcal{G} \) is a finite set of \( \lambda = \kappa \). Let \( \{ F_\alpha : \alpha \in \mathcal{G} \} \) be a collection of closed sets which covers \([c, x_{\alpha(\gamma)}]\) such that \( F_\alpha \subseteq U_\alpha \) and \( F_\alpha \subseteq \{ \epsilon : x_{\alpha(\gamma)} \} \) for each \( \alpha \in \mathcal{G} \). Define \( F_\alpha \) by

\[
F_\alpha = F_\alpha \\
F_\alpha \subseteq F'_\alpha \cup [x_{\alpha(\gamma)}, x_\gamma] \\
F_\alpha \subseteq [x_{\alpha(\gamma)}, x_\gamma] \\
F_\alpha = 0
\]

Then \( \mathcal{F}_1 = \{ F_\alpha : x < x \} \) is the desired closed shrinking of \( \Psi' \) which covers \([c, b']\).

Thus Fact 2 is proved.

By the above argument, we may assume that

\[
\mathcal{F}_1 = \{ F_1(U) : U \subseteq \Psi \}(\mathcal{F}_0 = \{ F_0(U) : U \subseteq \Psi \})
\]

in any case whether \( X \) has a last (first) element or not. Then

\[
\{ F_0(U) \cup F_1(U) : U \subseteq \Psi \}
\]

is a closed (in \( X \)) shrinking of \( \Psi \) which covers \( X \). Thus the proof of the claim is complete.

**Corollary 2.2.** Every GOTS has the shrinking property hereditarily.

**Proof.** Since every subspace of a GOTS is GOTS, it suffices to show that every GOTS has the shrinking property. By using the technique of [1], we can embed a GOTS \( X \) into a LOT's \( Y \) as a closed subspace. By Theorem 2.1, \( Y \) has the shrinking property. Therefore \( X \) has the shrinking property.

3. The \( \beta \)-property in GOTS. In this section, we shall show that every GOTS having the \( \beta \)-property is paracompact. First we characterize stationarity in a regular uncountable cardinal \( x \) by the \( \beta \)-property as follows.

**Theorem 3.1.** Let \( S \) be a subspace of a regular uncountable cardinal \( x \) with the order topology. Then \( S \) is stationary if and only if \( S \) does not have the \( \beta(x) \)-property.

**Proof.** Assume that \( S \) is stationary. For every \( x \in S \), let \( U_\alpha = x \cap S \). Then \( \{ U_\alpha : \alpha \in S \} \) is an open cover of \( S \). Furthermore by enumerating \( S \) with the increasing order, \( \{ U_\alpha : \alpha \in S \} \) may be considered as an increasing open cover of \( S \). If \( S \) has the \( \beta(x) \)-property, then there is an increasing (by the enumeration) open cover \( \{ V_\alpha : \alpha \in S \} \) such that \( \bigcup \{ V_\alpha : \alpha \in x \} \) for each \( x \in S \). Then for each \( x \in S \),
There is a $\beta(x) < x$ such that $(\beta(x), x] \cap V_x = 0$ since $x$ is not a point of $c_0 V_x$. By the pressing down lemma, there are a stationary set $S' \subseteq S$ and a $\beta < x$ such that $\beta(x) = \beta \ (i.e. \ (\beta, x] \cap V_x = 0)$ for every $x \in S'$. Noting the unboundedness of $S$, take $\alpha(0)$ in $S$ with $\beta < \alpha(0)$. Since $\{V_x : x \in S\}$ is an increasing open cover of $S$, so is $\{V_x : x \in S'\}$. Therefore there is an $x$ in $S'$ with $\alpha(0) \in V_x$. Since $V_x \subseteq U_x$, $\alpha(0) < \alpha(x)$ holds. Thus $\alpha(0) \alpha(1) < V_x$ and $x \in S'$. But this is a contradiction.

To prove the other direction, assume that $S$ is not stationary. We shall show that $S$ has the $\mathcal{A}(x)$-property. Let $\Psi = \{U_x : \beta < x\}$ be an increasing open cover of $S$, and $C$ be a cub set which is disjoint from $S$. Enumerate $C$ with the increasing order, say $C = \{\alpha(y) : \gamma < \gamma\}$. For every $\gamma < x$, put $S_\gamma = (\alpha(y), \alpha(y+1)] \cap S$. Then $S$ is the disjoint sum of $S_\gamma$'s since $C$ is cub, furthermore each $S_\gamma$ is a clopen subset of $S$.

By putting $\Psi_\gamma = \{U_x \cap S_\gamma : \beta < x\}$, $\Psi_\gamma$ is an increasing open cover of $S_\gamma$ for each $\gamma < x$. Since $|S_\gamma| < \kappa$ holds, there is a $\lambda(y) < \kappa$ such that $U_{\lambda(y)} \cap S_\gamma = S_\gamma$ for each $\gamma < \kappa$. For each $\gamma < \kappa$, define $V_\gamma^y = 0$ if $\beta < \lambda(y)$ and $V_\gamma = S_\gamma \cap U_{\lambda(y)} = U_\gamma$ if $\lambda(y) \beta < x$. By putting $V_\gamma = \bigcup \{V_\gamma^y : \gamma < \kappa\}$, $V_\gamma = c_0 V_\gamma \subseteq U_\gamma$ holds since $V_\gamma$ is the discrete sum of clopen sets $V_\gamma^y$. If $\beta < x$ holds, then we have $V_\gamma \subseteq V_\gamma^y$ since $V_\gamma^y \subseteq V_\gamma$ for every $\gamma < x$. Let $x$ be a point of $S$. Then $x \in U_{\lambda(y)} \cap S_\gamma$, $S_\gamma = V_\gamma \cap U_{\lambda(y)} \subseteq U_{\lambda(y)}$ holds. Thus $\{V_\gamma : \beta < x\}$ is an increasing open cover of $S$ such that $c_0 V_\gamma \subseteq U_\gamma$ for each $\beta < x$. This shows $S$ has the $\mathcal{A}(x)$-property.

**Lemma 3.2 [EL, 2.3].** Let $X$ be a GOTS. Then $X$ is not paracompact if and only if there is a regular uncountable cardinal $\kappa$ and a stationary subset $S$ of $\kappa$ such that the topological space $S$ is homeomorphic to a closed subspace of $X$.

**Corollary 3.3.** Let $X$ be a GOTS. Then $X$ is paracompact if and only if $X$ has the $\mathcal{A}$-property.

**Proof.** Assume $X$ is not paracompact. By Lemma 3.2, take a closed subspace $S$ of $X$ which is homeomorphic to a stationary subset of a regular uncountable cardinal $\kappa$. If $X$ has the $\mathcal{A}(x)$-property, then so does $S$. But this contradicts Theorem 3.1, thus $X$ does not have the $\mathcal{A}(x)$-property. The other direction is obvious. This completes the proof.

**Remark.** Note that every well ordered set is order isomorphic to some ordinal $\alpha$, and every well ordered set with the order topology has the shrinking property by Theorem 2.1. On the other hand, since every well ordered set $\alpha$ of uncountable cofinality $\chi$ has a closed subspace which is homeomorphic to $\chi$, it does not have the $\mathcal{A}(x)$-property ($\mathcal{A}$-property). Using this we can show that every well ordered set $\alpha$ is paracompact (Lindelöf, $\sigma$-compact, has the $\mathcal{A}$-property) if and only if the cofinality of $\alpha = \omega$.

References: