[5] G. L. Forti, *On an alternative functional equation related to the Cauchy equation*, Aequationes Math. 24 (1982), 195–206.
[6] Z. Gajda, *Additive and convex functions in linear topological spaces*, Aequationes Math. 27 (1984), 214–219.
[7] G. Godoni, *Set-valued Cauchy functional equation*, Rev. Roumaine Math. Pures Appl. 20 (1975), 1113–1121.
[8] J. L. Kelley, I. Namioka et al., *Linear Topological Spaces*, Springer-Verlag, 1976.
[9] M. Laczkovich, *Functions with measurable differences*, Acta Math. Acad. Sci. Hungar. 35 (1980), 217–235.
[10] L. Paganoni, *On an alternative Cauchy equation*, Aequationes Math. 29 (1985), 214–221.
[11] W. Rudin, *Functional analysis*, Mc Graw-Hill Book Company, 1973.
[12] L. Székelyhidi, *Local polynomials and functional equations*, Publ. Math. Debrecen 30 (1983), 283–290.

INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
P-40-007 Katowice

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
Waterloo, Ontario, Canada N2L 3G1

Received 13 June 1988

The shrinking property and the $\mathscr{B}$-property in ordered spaces

by

Nobuyuki Kemoto (Okinawa)

Dedicated to Professor Kiyoski Iséki on his 70th birthday

Abstract. In this paper, it will be shown that

(1) every generalized ordered topological space has the shrinking property,

(2) a subspace of a regular uncountable cardinal $\varkappa$ with the order topology has the $\mathscr{B}(\varkappa)$-property if and only if it is not stationary in $\varkappa$.

As a corollary of (2), we shall also show that

(3) every generalized ordered topological space is paracompact if and only if it has the $\mathscr{B}$-property.

1. Introduction. It is known from Y. Yasui [Y2] that every generalized ordered topological space has the weak $\mathscr{B}$-property. He also pointed out that ω_1 (with the order topology) does not have the $\mathscr{B}$-property. As remarked in [R], it is known that every uncountable regular cardinal with the order topology has the shrinking property, but does not have the $\mathscr{B}$-property. Note that no uncountable regular cardinal is paracompact. In this paper, it will be shown that

(1) every generalized ordered topological space has the shrinking property,

(2) a subspace of a regular uncountable cardinal $\varkappa$ with the order topology has the $\mathscr{B}(\varkappa)$-property if and only if it is not stationary in $\varkappa$.

As a corollary of (2), we shall also show that

(3) every generalized ordered topological space is paracompact if and only if it has the $\mathscr{B}$-property.

In these connections, Y. Yasui asked in [Y1] if spaces having the $\mathscr{B}$-property are paracompact. Now it is known that the only known counterexample is the Navy's space, see [R], [N]. Recently S. Jiang asked in [J] if (normal) spaces having the $\mathscr{B}$-property are (sub)metacompact. Note that metacompact linearly ordered topological spaces are paracompact, see [E, 5.5.22(b)], [GH], [F].

AMS (MOS) Subject Classification (1980) 54D18, 54F05.

First we establish our terminology. Let $\mathscr{U}$ be a collection of subsets of a space X and $\varkappa$ be an infinite cardinal. A collection $\mathscr{F} = \{F(U)\colon U \in \mathscr{U}\}$ of subsets of X is a *shrinking* of $\mathscr{U}$ if $\operatorname{cl} F(U) \subset U$ for every U in $\mathscr{U}$. A space X has the *$\varkappa$-shrinking property* if for every open cover $\mathscr{U}$ of size $\leqslant \varkappa$, there is a shrinking of $\mathscr{U}$ consisting of open sets (*open shrinking*) which covers X. A collection $\{U_\alpha\colon \alpha < \varkappa\}$ of subsets of a space index by $\varkappa$ is said to be *increasing* if $U_\alpha \subset U_\beta$ if $\alpha < \beta < \varkappa$. A space X has the *weak $\mathscr{B}(\varkappa)$-property* (the *$\mathscr{B}(\varkappa)$-property*) if for every increasing open cover $\{U_\alpha\colon \alpha < \varkappa\}$ of X, there is an (increasing, respectively) open shrinking $\{V_\alpha\colon \alpha < \varkappa\}$ (i.e. $\operatorname{cl} V_\alpha \subset U_\alpha$ for each $\alpha < \varkappa$) which covers X. A space has the shrinking (weak $\mathscr{B}$-, $\mathscr{B}$-) property if it has the $\varkappa$-shrinking (weak $\mathscr{B}(\varkappa)$-, $\mathscr{B}(\varkappa)$-, respectively) property for every infinite cardinal $\varkappa$. By these definitions, it is easy to show the following implications.

$$\begin{tikzcd}
 & \text{shrinking property} \arrow[dr] & \\
\text{paracompact} \arrow[ur] \arrow[dr] & & \text{weak } \mathscr{B}\text{-property} \\
 & \mathscr{B}\text{-property} \arrow[ur] &
\end{tikzcd}$$

Note that in a normal space, shrinking property is equivalent to that its every open cover has a shrinking which consists of closed sets (*closed shrinking*) and covers the space. Also note that in a normal space, for every open cover $\mathscr{U}$ which has a finite subcover, there is a closed shrinking of $\mathscr{U}$ which covers the space.

A subset S of a regular uncountable cardinal $\varkappa$ is said to be *stationary* if S intersects with every closed unbounded (cub) subspace of $\varkappa$ with the order topology.

A *linearly ordered topological space* (LOTS) is the topological space with the topology induced by usual open intervals on a linearly ordered set. A *generalized ordered topological space* (*GOTS*) is a triple $(X, \mathscr{T}, <)$ where $<$ is a linear ordering of the set X and $\mathscr{T}$ is a T_1 topology which has a base consisting of convex sets, see [EL]. Here a subset C of a linearly ordered set $(X, <)$ is *convex* if $(a, b) \subset C$ for every a, b in C ((a, b) denotes the open interval with end points a, b). Note that every GOTS can be embedded in a LOTS as a closed subspace ([L]), every subspace of LOTS is GOTS, and every GOTS is hereditary collectionwise normal and hereditary countably paracompact.

2. The shrinking property in GOTS. In this section, we shall show that every GOTS has the shrinking property hereditarily. First we recall some basic facts. A cut of a linearly ordered set $(X, <)$ is a pair $\langle A, B\rangle$ of subsets of X such that $A \cup B = X$, $A \cap B = 0$, and if $x \in A$ and $y \in B$, then $x < y$. Note that $\langle 0, X\rangle$ and $\langle X, 0\rangle$ are cuts. Let cX denote the set of all cuts of X. Define a linear ordering $<'$ on cX by $\langle A, B\rangle <' \langle A', B'\rangle$ iff $A \subset A'$ and $A \neq A'$. By identifying $x \in X$ with the cut $\langle A_x, B_x\rangle$, where $A_x = \{y \in X\colon y \leqslant x\}$ and $B_x = X - A_x$, cX with the topology induced by $<'$ is a compactification of the LOTS X and $x <' y$ is equivalent to $x < y$ whenever $x, y \in X$, see [E, 3.12.3(b)], [F], [K]. Note that a LOTS X is compact if and only if every subset A of X has the least upper bound (lub), see [E, 3.12.3(a)], [HK], and every subset A of a LOTS X can be represented as the union of disjoint maximal convex sets, furthermore if A is open in X, such maximal convex sets are also open, see [E, 2.7.5(b)]. In the proof of the next theorem, $(x, y)'$, $[x, y)'$, ... etc. denote intervals in cX.

Theorem 2.1. *Every LOTS has the shrinking property.*

Proof. Let $\mathscr{U}$ be an open cover of a LOTS X. For every U in $\mathscr{U}$, $\mathscr{U}(U)$ denotes the set of all maximal convex (open) sets contained in U. For every V, V' in

$$\mathscr{V} = \bigcup \{\mathscr{U}(U)\colon U \in \mathscr{U}\},$$

define $V \simeq V'$ if there is a finite set $\{V_0, \ldots, V_n\} \subset \mathscr{V}$ such that $V = V_0$, $V' = V_n$, and $V_i \cap V_{i+1} \neq 0$ for $i < n$ (we call such a finite set *finite chain*). Then $\simeq$ is an equivalence relation on $\mathscr{V}$. Let $\{\mathscr{V}_\lambda\colon \lambda \in \Lambda\}$ be the all equivalence classes. Then $\{\bigcup \mathscr{V}_\lambda\colon \lambda \in \Lambda\}$ decompose X into clopen sets. Thus if we can find a (closed) shrinking of $\{\bigcup \mathscr{V}_\lambda \cap U\colon U \in \mathscr{U}\}$ covering $\bigcup \mathscr{V}_\lambda$ for each $\lambda \in \Lambda$, then we can construct a closed shrinking of $\mathscr{U}$ covering X. Since each $\bigcup \mathscr{V}_\lambda$ is open in X, the subspace topology on $\bigcup \mathscr{V}_\lambda$ coincides with the topology induced on $\bigcup \mathscr{V}_\lambda$ by the restriction of the linear order $<$ to $\bigcup \mathscr{V}_\lambda$. Therefore it suffices to show the next claim.

Claim. *Let $\mathscr{U}$ be an open cover of a LOTS X such that for every V, V' in $\mathscr{V} = \{\mathscr{U}(U)\colon U \in \mathscr{U}\}$, $V \simeq V'$ holds. Then $\mathscr{U}$ has a closed shrinking which covers X.*

Proof of Claim. Let $(cX, <')$ be the compactification of $(X, <)$ described previously. Define $a = \langle 0, X\rangle$ and $b = \langle X, 0\rangle$. Then $a(b)$ is the first (last, respectively) element of cX. For every V in $\mathscr{V}$, V' denotes the set

$$\{x \in cX\colon \exists y, z \in V (y \leqslant' x \leqslant' z)\}.$$

Then it is easy to show that each V' is open convex in $(cX, <')$, since each V is open convex in $(X, <)$.

Fact 1. *$\mathscr{V}' = \{V'\colon V \in \mathscr{V}\}$ covers $cX - \{a, b\}$.*

Proof of Fact 1. Let x be a point in $cX - \{a, b\}$. Since X is topologically dense in cX, there are y, z in X such that $a \leqslant' y <' x$ and $x <' z \leqslant' b$ hold. Then $y <' x <' z$ holds. Since $y, z \in X$, there is a finite chain $\{V_0, \ldots, V_n\} \subset \mathscr{V}$ such that $y \in V_0$ and $z \in V_n$ hold. By induction on n, we shall show that $z \in V_i'$ for some $i \leqslant n$. Whenever $n = 0$, $x \in V_0'$ holds since $y, z \in V_0'$ with $y <' x <' z$ and V_0' is convex. Assume that it is valid for $\leqslant n-1$. Take a point z_0 in $V_{n-1} \cap V_n$. If $z_0 <' x$ holds, then $x \in V_n'$ holds since $z_0, z \in V_n$ and $z_0 <' x <' z$. If $x <' z_0$ holds, then by the inductive assumption $x \in V_i'$ holds for some $i \leqslant n-1$ since $y \in V_0$, $z_0 \in V_{n-1}$ and $y <' x <' z_0$. Thus $\mathscr{V}'$ covers $cX - \{a, b\}$.

Note that if X has the first (last) element, then $\mathscr{V}'$ also covers the point a (b, respectively). Also note that if X has no first (last) element, then $\mathscr{V}'$ does not cover the point a (b, respectively). This completes the proof of the fact.

Fix a point c in $cX-\{a, b\}$. For every U in $\mathscr{U}$, define $U' = \bigcup\{V': V \in \mathscr{U}(U)\}$. Put $\mathscr{U}' = \{U': U \in \mathscr{U}\}$. Whenever X has the last (first) element, since $\mathscr{U}'$ covers the compact set $[c, b]'$ ($[a, c]'$, respectively), U' has a shrinking $\mathscr{F}_1$ ($\mathscr{F}_0$, respectively) covering $[c, b]'$ ($[a, c]'$, respectively) consisting of closed subsets of $[c, b]'$ ($[a, c]'$, respectively). Next we shall show the following fact.

Fact 2. *If X does not have a last (first) element, then $\mathscr{U}'$ has a shrinking $\mathscr{F}_1$ ($\mathscr{F}_0$, respectively) covering $[c, b)'$ ($(a, c]'$, respectively) consisting of closed subsets of $[c, b)'$ ($(a, c]'$, respectively).*

Proof of Fact 2. Assume that X does not have a last element (the other case is similar). A subset A of $[c, b)'$ said to be *cofinal* in b if for every x in $[c, b)'$, there is an a in A such that $x \leqslant' a$. Let $\varkappa$ be the least cardinal $|A|$ such that A is cofinal in b. Well order such an A with $\{a_\gamma: \gamma < \varkappa\}$ by type $\varkappa$. By induction, we shall find a strict increasing cofinal (in b) sequence $\{x_\gamma: \gamma < \varkappa\}$ such that x_γ is the lub of $\{x_\beta: \beta < \gamma\}$ if γ is limit. Define $x_0 = a_0$. Assume that $\gamma < \varkappa$ and x_β has been defined for every $\beta < \gamma$. When γ is successor, since $|\{a_\beta: \beta < \gamma\} \cup \{x_\beta: \beta < \gamma\}| < \varkappa$, there is a $\lambda < \varkappa$ such that $a_\beta < a_\lambda$ and $x_\beta < a_\lambda$ for every $\beta < \gamma$. Define $x_\gamma = a_\lambda$. When γ is limit, let x_γ be the lub of $\{x_\beta: \beta < \gamma\}$. Since cX is compact and by the definition of $\varkappa$, such a x_γ is well defined and $x_\gamma <' b$. Thus we can find such a sequence. Furthermore by the definition of $\varkappa$, $\varkappa$ is a regular cardinal.

If $\varkappa = \omega$ holds, then $[c, b)' = \bigcup\{[c, x_\gamma]': \gamma < \varkappa\}$ is Lindelöf since each $[c, x_\gamma]'$ is compact. Thus we can find such an $\mathscr{F}_1$ whenever $\varkappa = \omega$. Next assume that $\varkappa$ is a regular uncountable cardinal. Since $[c, b)' = \bigcup\{[c, x_\gamma]': \gamma < \varkappa\}$ and each $[c, x_\gamma]'$ is compact, every open cover of $[c, b)'$ has a subcover of size $\leqslant \varkappa$. Thus we may assume that $|\mathscr{U}| \leqslant \varkappa$. Well order $\mathscr{U} = \{U_\alpha: \alpha < \lambda\}$ where $\lambda \leqslant \varkappa$. For every γ in $\operatorname{Lim}(\varkappa) = \{\gamma < \varkappa: \gamma \text{ is limit}\}$, fix an $\alpha(\gamma) < \lambda$ such that $x_\gamma \in U'_{\alpha(\gamma)}$. Then there is an $f(\gamma) < \gamma$ such that $[x_{f(\gamma)}, x_\gamma]' \subset U'_{\alpha(\gamma)}$ for every γ in $\operatorname{Lim}(\varkappa)$ since x_γ is the lub of $\{x_\beta: \beta < \gamma\}$ if γ is limit. Note that $\operatorname{Lim}(\varkappa)$ is cub in $\varkappa$. There are two cases.

Case 1. $\lambda < \varkappa$.

In this case, since $\lambda < \varkappa$ and $\varkappa$ is regular uncountable, there are a stationary set $S \subset \operatorname{Lim}(\varkappa)$ and an $\alpha < \lambda$ such that $\alpha(\gamma) = \alpha$ for every $\gamma \in S$. Then by the pressing down lemma, there are a stationary set $S' \subset S$ and a $\gamma(0) < \varkappa$ such that $f(\gamma) = \gamma(0)$ for every γ in S'. This means that $[x_{\gamma(0)}, x_\gamma]' \subset U'_\alpha$ holds for every γ in S'. Thus ($(x_{\gamma(0)}, b)' \subset$) $[x_{\gamma(0)}, b)' \subset U'_\alpha$ holds. On the other hand, since $[c, x_{\gamma(0)}]'$ is compact, there is a finite subcollection of $\mathscr{U}'$ which covers $[c, x_{\gamma(0)}]'$. Therefore a finite subcollection of $\mathscr{U}'$ which covers $[c, b)'$. Thus we can take such a closed shrinking $\mathscr{F}_1$.

Case 2. $\lambda = \varkappa$.

In this case, put $C = \{\gamma \in \operatorname{Lim}(\varkappa): \forall \beta < \gamma(\alpha(\beta) < \gamma)\}$, then C is cub in $\varkappa$. There are two subcases.

Subcase 1. $\{\gamma \in C: \alpha(\gamma) < \gamma\}$ is stationary.

In this subcase, by the pressing down lemma, there are a stationary set $S \subset \{\gamma \in C: \alpha(\gamma) < \gamma\}$ and an $\alpha < \lambda = \varkappa$ such that $\alpha(\gamma) = \alpha$ for every γ in S. Then as in Case 1, we can find such a $\mathscr{F}_1$.

Subcase 2. $S = \{\gamma \in C: \gamma \leqslant \alpha(\gamma)\}$ is stationary.

In this subcase, for every γ', γ in S with $\gamma' < \gamma$, $\alpha(\gamma') < \alpha(\gamma)$ holds. Therefore elements of $\alpha''S = \{\alpha(\gamma): \gamma \in S\}$ are all distinct. By the pressing down lemma, there are a stationary set $S' \subset S$ and a $\gamma(0) < \gamma = \varkappa$ such that $f(\gamma) = \gamma(0)$ for every γ in S'. This means that $[x_{\gamma(0)}, x_\gamma]' \subset U'_{\alpha(\gamma)}$ for every γ in S'. Take a finite subcollection $\{U'_\alpha: \alpha \in T\}$ of $\mathscr{U}'$ which covers $[c, x_{\gamma(0)}]'$, where T is a finite subset of $\lambda = \varkappa$. Let $\{F'_\alpha: \alpha \in T\}$ be a collection of closed sets which covers $[c, x_{\gamma(0)}]'$ such that $F'_\alpha \subset U'_\alpha$ and $F'_\alpha \subset [c, x_{\gamma(0)}]'$ for each $\alpha \in T$. Define F_α by

$$\begin{aligned} F_\alpha &= F'_\alpha && \text{if } \alpha \in T - \alpha''S', \\ F_{\alpha(\gamma)} &= F'_{\alpha(\gamma)} \cup [x_{\gamma(0)}, x_\gamma]' && \text{if } \alpha(\gamma) \in T \cap \alpha''S', \\ F_{\alpha(\gamma)} &= [x_{\gamma(0)}, x_\gamma]' && \text{if } \alpha(\gamma) \in \alpha''S' - T, \\ F_\alpha &= 0 && \text{if } \alpha \in \varkappa - (T \cup \alpha''S'). \end{aligned}$$

Then $\mathscr{F}_1 = \{F_\alpha: \alpha < \varkappa\}$ is the desired closed shrinking of $\mathscr{U}'$ which covers $[c, b')$. Thus Fact 2 is proved.

By the above argument, we may assume that

$$\mathscr{F}_1 = \{F_1(U): U \in \mathscr{U}\} (\mathscr{F}_0 = \{F_0(U): U \in \mathscr{U}\})$$

in any case whether X has a last (first) element or not. Then

$$\{(F_0(U) \cup F_1(U)) \cap X: U \in \mathscr{U}\}$$

is a closed (in X) shrinking of $\mathscr{U}$ which covers X. Thus the proof of the claim is complete.

Corollary 2.2. *Every GOTS has the shrinking property hereditarily.*

Proof. Since every subspace of a GOTS is GOTS, it suffices to show that every GOTS has the shrinking property. By using the technique of [L], we can embed a GOTS X into a LOTS Y as a closed subspace. By Theorem 2.1, Y has the shrinking property. Therefore X has the shrinking property.

3. The $\mathscr{B}$-property in GOTS.

In this section, we shall show that every GOTS having the $\mathscr{B}$-property is paracompact. First we characterize stationarity in a regular uncountable cardinal $\varkappa$ by the $\mathscr{B}(\varkappa)$-property as follows.

Theorem 3.1. *Let S be a subspace of a regular uncountable cardinal $\varkappa$ with the order topology. Then S is stationary if and only if S does not have the $\mathscr{B}(\varkappa)$-property.*

Proof. Assume that S is stationary. For every α in S, put $U_\alpha = \alpha \cap S$. Then $\{U_\alpha: \alpha \in S\}$ is an open cover of S. Furthermore by enumerating S with the increasing order, $\{U_\alpha: \alpha \in S\}$ may be considered as an increasing open cover of S. If S has the $\mathscr{B}(\varkappa)$-property, then there is an increasing (by the enumeration) open cover $\{V_\alpha: \alpha \in S\}$ such that $\operatorname{cl}_S V_\alpha \subset U_\alpha$ for each α in S. Then for each α in S, there is a $\beta(\alpha)<\alpha$ such that $(\beta(\alpha),\alpha]\cap V_\alpha=0$ since α is not a point of $\mathrm{cl}_S V_\alpha$. By the pressing down lemma, there are a stationary set $S'\subset S$ and a $\beta<\varkappa$ such that $\beta(\alpha)=\beta$ (i.e. $(\beta,\alpha]\cap V_\alpha=0$) for every α in S'. Noting the unboundedness of S, take $\alpha(0)$ in S with $\beta<\alpha(0)$. Since $\{V_\alpha\colon \alpha\in S\}$ is an increasing open cover of S, so is $\{V_\alpha\colon \alpha\in S'\}$. Therefore there is an α in S' with $\alpha(0)\in V_\alpha$. Since $V_\alpha\subset U_\alpha$, $\alpha(0)<\alpha$ holds. Thus $\alpha(0)\in(\beta,\alpha]\cap V_\alpha$ and $\alpha\in S'$. But this is a contradiction.

To prove the other direction, assume that S is not stationary. We shall show that S has the $\mathscr{B}(\varkappa)$-property. Let $\mathscr{U}=\{U_\beta\colon \beta<\varkappa\}$ be an increasing open cover of S, and C be a cub set which is disjoint from S. Enumerate C with the increasing order, say $C=\{\alpha(\gamma)\colon \gamma<\varkappa\}$. For every $\gamma<\varkappa$, put $S_\gamma=(\alpha(\gamma),\alpha(\gamma+1))\cap S$. Then S is the disjoint sum of S_γ's since C is cub, furthermore each S_γ is a clopen subset of S. By putting $\mathscr{U}_\gamma=\{U_\beta\cap S_\gamma\colon \beta<\varkappa\}$, $\mathscr{U}_\gamma$ is an increasing open cover of S_γ for each $\gamma<\varkappa$. Since $|S_\gamma|<\varkappa$ holds, there is a $\lambda(\gamma)<\varkappa$ such that $U_{\lambda(\gamma)}\cap S_\gamma=S_\gamma$ for each $\gamma<\varkappa$. For each $\gamma<\varkappa$, define $V_{\beta\gamma}=0$ if $\beta<\lambda(\gamma)$, and $V_{\beta\gamma}=S_\gamma(\subset U_{\lambda(\gamma)}\subset U_\beta)$ if $\lambda(\gamma)\leqslant\beta<\varkappa$. By putting $V_\beta=\bigcup\{V_{\beta\gamma}\colon \gamma<\varkappa\}$, $V_\beta=\mathrm{cl}_S V_\beta\subset U_\beta$ holds since V_β is the discrete sum of clopen sets $V_{\beta\gamma}$. If $\beta'<\beta<\varkappa$ holds, then we have $V_{\beta'}\subset V_\beta$ since $V_{\beta'\gamma}\subset V_{\beta\gamma}$ for every $\gamma<\varkappa$. Let x be a point of S_γ. Then $x\in U_{\lambda(\gamma)}\cap S_\gamma=S_\gamma=V_{\lambda(\gamma)\gamma}\subset V_{\lambda(\gamma)}$ holds. Thus $\{V_\beta\colon \beta<\varkappa\}$ is an increasing open cover of S such that $\mathrm{cl}_S V_\beta\subset U_\beta$ for each $\beta<\varkappa$. This shows S has the $\mathscr{B}(\varkappa)$-property.

LEMMA 3.2 [EL, 2.3]. *Let X be a GOTS. Then X is not paracompact if and only if there are a regular uncountable cardinal $\varkappa$ and a stationary subset S of $\varkappa$ such that the topological space S is homeomorphic to a closed subspace of X.*

COROLLARY 3.3. *Let X be a GOTS. Then X is paracompact if and only if X has the $\mathscr{B}$-property.*

Proof. Assume X is not paracompact. By Lemma 3.2, take a closed subspace S of X which is homeomorphic to a stationary subset of a regular uncountable cardinal $\varkappa$. If X has the $\mathscr{B}(\varkappa)$-property, then so does S. But this contradicts Theorem 3.1, thus X does not have the $\mathscr{B}(\varkappa)$-property. The other direction is obvious. This completes the proof.

Remark. Note that every well ordered set is order isomorphic to some ordinal α, and every well ordered set with the order topology has the shrinking property by Theorem 2.1. On the other hand, since every well ordered set α of uncountable cofinality $\varkappa$ has a closed subspace which is homeomorphic to $\varkappa$, it does not have the $\mathscr{B}(\varkappa)$-property ($\mathscr{B}$-property). Using this we can show that every well ordered set α is paracompact (Lindelöf, σ-compact, has the $\mathscr{B}$-property) if and only if the cofinality of $\alpha=\omega$.

References

[E] R. Engelking, *General Topology*, PWN–Polish Scientific Publishers (1977), Warszawa.
[EL] R. Engelking and D. J. Lutzer, *Paracompactness in ordered spaces*, Fund. Math. 94 (1977), 49–58.

[F] V. V. Fedorčuk, *Some questions in the theory of ordered spaces*, Siberian Math. J. 10 (1969), 124–132.
[GH] L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc. 77 (1954), 340–362.
[HK] A. Haar, D. König, *Über einfach geordnete Mengen*, J. Reine Angew. Math. 139 (1910), 16–28.
[J] S. Jiang, *On a Junnila's problem*, Q & A in General Top. 6-1 (1988), 43–48.
[K] R. Kaufman, *Ordered sets and compact spaces*, Coll. Math. 17 (1967), 35–39.
[L] D. J. Lutzer, *On generalized ordered spaces*, Diss. Math. 89 (1971).
[N] K. Navy, *ParaLindelöfness and paracompactness*, Thesis, University of Wisconsin, Madison 1981.
[R] M. E. Rudin, *Yasui's questions*, Q & A in General Top. 1-2 (1983), 122–127.
[Y1] Y. Yasui, *Some questions on the B- and weak B-property*, Q & A in General Top. 1-1 (1983), 18–24.
[Y2] — *A weak B-property of a linearly ordered topological space*, Q & A in General Top. 4-1 (1986), 77–84.

DEPARTMENT OF MATHEMATICS
COLLEGE OF EDUCATION
RYUKYU UNIVERSITY
Nishihara-cho
Okinawa
Japan

Current address:

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
OITA UNIVERSITY
Oita 870-11
Japan

Received 25 July 1988