On the Pexider difference

by

K. Baron (Katowice) and **PL. Kannappan** (Waterloo)

Abstract. Several authors have studied the Cauchy difference $f(x+y)-f(x)-f(y)$, in particular in connection with stability problems. Here we are mainly interested in studying the Pexider difference $f(x+y)-g(x)-h(y)$ belonging to a certain subgroup and obtaining its representation. We also obtain a representation of the function satisfying the Cauchy difference taking values in a given subgroup under some regularity conditions.

For $f\colon \boldsymbol{R}\to\boldsymbol{R}$ (reals),

$$f(x+y)-f(x)-f(y) \tag{1}$$

is called a *Cauchy difference*. From an example of G. Godini [7, Example 2] it can be seen that it is not generally true that a function f such that (1) belongs to $\boldsymbol{Z}$ (the set of all integers) for all $x, y\in\boldsymbol{R}$ must be of the form $A+k$, where A is an additive function and k takes integer values only. However such a representation is possible under some regularity condition imposed on f. It seems that J. G. van der Corput was the first author who gave such a condition [3, on p. 64]. Making use of a theorem of M. Laczkovich [9, Theorem 5] it was possible in [1] to consider this problem assuming Lebesgue measurability of (1) as a function of two variables. Later such a representation was obtained in [2], when the domain is a real topological vector space. For more details and consideration from algebraic point of view refer to [5] and [10].

In this paper our goal is to consider the *Pexider difference*

$$f(x+y)-g(x)-h(y)\,, \tag{2}$$

where $f, g, h\colon G\to H$, G is a groupoid with identity 0 and H is a group and obtain representation of f, g, h when (2) takes values in a fixed subgroup of H. Also regarding the Cauchy difference (1), we will consider it for a function $f\colon E\to F$ continuous at zero, where E is a topological vector space and F is a topological Abelian group, assuming that (1) takes values in a fixed discrete subgroup of F and obtain its representation. The case when the function is measurable, instead of being continuous at zero will also be considered. We will consider Baire and Christensen measurable functions. For details on Christensen measurability refer to [4].

We start with the following lemma which is used in the sequel (cf. [12, Theorem 3.4]).

LEMMA 1. *Suppose E is a topological vector space (real or complex), U is a balanced neighbourhood of the origin, and H is an Abelian group. If $f\colon U\to H$ is additive, i.e.*

(3) $$f(x+y) = f(x)+f(y)$$

for all $x, y\in U$ with $x+y\in U$, then there exists a unique additive function $g\colon E\to H$ such that g is an extension of f.

Proof. Using induction it is easy to prove that

(4) $$f(nx) = nf(x)$$

for every $x\in U$ with $nx\in U$. (Recall that U is balanced implies $\alpha U\subset U$ for $|\alpha|\leqslant 1$ [11].) For x, y in U and integers m, n with $mx = ny$ we have, using (4),

(5) $$mf(x) = mnf\left(\frac{x}{n}\right) = mnf\left(\frac{y}{m}\right) = nf(y).$$

Now we introduce a function $g\colon E\to H$ by

$$g(x) = nf\left(\frac{x}{n}\right),\quad x\in E,$$

where n is any integer such that $x/n\in U$. This function g is well defined because of (5). It is easy to verify that g indeed is additive on E, is equal to f on U and is unique. This completes the proof of the lemma.

Now we prove the following theorem regarding the Pexider difference (2).

THEOREM 2. *Suppose G is a groupoid with identity 0 and H is a group. Let K be a subgroup of H. A triple (f, g, h) of functions mapping G into H fulfills the condition*

(6) $$f(x+y)-g(x)-h(y)\in K$$

for all $x, y\in G$ if, and only if, there are functions $k, l\colon G\to K$, $\varphi\colon G\to H$ and constants $a, b\in H$ such that

(7) $$\varphi(x+y)-\varphi(x)-\varphi(y)\in K$$

for all $x, y\in G$, and

$$f(x) = k(x)+\varphi(x)+a,\quad g(x) = b-a+\varphi(x)+a,\quad h(x) = l(x)+\varphi(x)+a-b,$$

for every $x\in G$.

Proof. Assume that (6) holds for all $x, y\in G$. Then setting in (6) first $y = 0$ and then $x = 0$ we have

(8) $$\mu(x) = f(x)-g(x)-h(0)\in K\quad \text{for every } x\in G$$

and

(9) $$\nu(y) = f(y)-g(0)-h(y)\in K\quad \text{for every } y\in G.$$

From (8) and (9) we get

(10) $$g(x) = -h(0)-\mu(x)+f(x),\quad h(x) = -\nu(x)+f(x)-g(0),\quad \text{for } x\in G.$$

Putting (10) into (6) and using (9) we obtain

(11) $$f(x+y)-f(x)+\mu(x)+h(0)+g(0)-f(y)\in K\quad \text{for all } x, y\in G.$$

Let $x = 0 = y$ in (6) to get $f(0)-g(0)-h(0)\in K$. Define $k, l, \varphi\colon G\to H$ by

(12) $$k(x) = \mu(x)+h(0)+g(0)-f(0),\quad l(x) = -\nu(x)+k(x),$$

(13) $$\varphi(x) = -k(x)+f(x)-f(0).$$

Indeed $k(x)$ and $l(x)$ belong to K and

$$\begin{aligned}\varphi(x+y)-\varphi(x)-\varphi(y) &= -k(x+y)+f(x+y)-f(x)+k(x)+f(0)-f(y)+k(y)\\ &= -k(x+y)+[f(x+y)-f(x)+\mu(x)+h(0)+g(0)-f(y)]+\\ &\quad +k(y)\end{aligned}$$

also belongs to K for all $x, y\in G$ because of (11). Thus (7) holds for all $x, y\in G$.

Finally, putting $a = f(0)$, $b = g(0)$, from (13), (10) and (12) we obtain the desired forms of f, g, h:

$$f(x) = k(x)+\varphi(x)+a,$$

$$g(x) = -h(0)+h(0)+b-a-k(x)+k(x)+\varphi(x)+a = b-a+\varphi(x)+a,$$

$$h(x) = -\nu(x)+k(x)+\varphi(x)+a-b = l(x)+\varphi(x)+a-b$$

for every $x\in G$. The converse is easy to check.

In the next three theorems we obtain the representation of the Cauchy difference (7).

THEOREM 3. *Let E be a topological vector space, F be a topological Abelian group and K be a discrete subgroup of F. Suppose $\varphi\colon E\to F$ satisfies (7) for all $x, y\in E$ and is continuous at the origin. Then there exists a continuous additive function $A\colon E\to F$ such that*

(14) $$\varphi(x)-A(x)\in K\quad \textit{for every } x\in E,$$

i.e.

$$\varphi(x) = A(x)+\lambda(x)\quad \textit{for every } x\in E,$$

where $\lambda\colon E\to K$ is continuous at the origin.

Proof. Since K is discrete, that is, every element of K has a neighbourhood contained in F containing only that element, there exists a neighbourhood $V(\subset F)$

of zero such that $K \cap V = \{0\}$. Further there exist a neighbourhood $W \subset F$ of zero such that

$$-W = W, \quad W+W+W \subset V$$

(see p. 10 in [11]) and a balanced neighbourhood $U \subset E$ of zero such that

$$\varphi(U) \subset \varphi(0)+W$$

(see p. 11 in [11]). Then for $x, y \in U$ with $x+y \in U$,

$$[\varphi(x+y)-\varphi(0)]-[\varphi(x)-\varphi(0)]-[\varphi(y)-\varphi(0)] \in W-W-W \subset V.$$

On the other hand, from (7), $\varphi(0) \in K$ and

$$[\varphi(x+y)-\varphi(0)]-[\varphi(x)-\varphi(0)]-[\varphi(y)-\varphi(0)] \in K.$$

Consequently,

$$\varphi(x+y)-\varphi(0) = [\varphi(x)-\varphi(0)]+[\varphi(y)-\varphi(0)]$$

for all $x, y \in U$ with $x+y \in U$. Therefore by Lemma 1 there exists an additive function $A: E \to F$ such that

(15) $$\varphi(x)-\varphi(0) = A(x) \quad \text{for every } x \in U.$$

In particular, A is continuous everywhere. Further, if $x \in E$ then $\frac{x}{n} \in U$ for some positive integer n and using (15) and (7) we get

$$\varphi(x)-A(x) = \varphi\left(n\frac{x}{n}\right)-n\varphi\left(\frac{x}{n}\right)+n\varphi\left(\frac{x}{n}\right)-nA\left(\frac{x}{n}\right)$$
$$= \left[\varphi\left(n\frac{x}{n}\right)-n\varphi\left(\frac{x}{n}\right)\right]+n\varphi(0) \in K.$$

This proves the theorem.

In the next theorem we assume a measurability condition on φ, instead of the continuity at the origin.

THEOREM 4. *Let E and F be topological vector spaces and K be a countable, discrete subgroup of F. Suppose φ: $E \to F$ satisfies* (7) *for all $x, y \in E$. Then whether E is a Baire space and φ is Baire measurable, or E is a separable F-space and φ is Christensen measurable, there always exists an additive function A: $E \to F$ such that* (14) *holds.*

Proof. It is possible to choose neighbourhoods V_0, V, $W(\subset F)$ of zero such that

$$K \cap V_0 = \{0\}; \quad -V = V; \quad V+V+V \subset V_0; \quad W-W \subset V.$$

Since

$$E = \varphi^{-1}(F) = \varphi^{-1}\left(\bigcup_{n=1}^{\infty} nW\right) = \bigcup_{n=1}^{\infty} \varphi^{-1}(nW)$$

and E is of the second category (or E is not Christensen zero set) we infer that there exists a positive integer n, such that $\varphi^{-1}(nW)$ is of the second category (or it is not a Christensen zero set).

First we show that $\varphi^{-1}(nW) \subset n\varphi^{-1}\left(W+\frac{1}{n}K\right)$.

Let $x \in \varphi^{-1}(nW)$. Then $\varphi(x) \in nW$ and, in view of (7), $\varphi(x)-n\varphi\left(\frac{x}{n}\right) \in K$. Consequently,

$$n\varphi\left(\frac{x}{n}\right) \in K+\varphi(x) \subset K+nW.$$

Hence

$$\varphi\left(\frac{x}{n}\right) \in W+\frac{1}{n}K,$$

which means that $\frac{x}{n} \in \varphi^{-1}\left(W+\frac{1}{n}K\right)$, that is, $\varphi^{-1}(nW) \subset n\varphi^{-1}\left(W+\frac{1}{n}K\right)$. This implies,

$$\varphi^{-1}(nW) \subset n\varphi^{-1}\left(W+\frac{1}{n}K\right) = \bigcup_{y \in K} n\varphi^{-1}\left(W+\frac{y}{n}\right).$$

Hence, since K is countable and $\varphi^{-1}(nW)$ is of the second category (or it is not a Christensen zero set), there exists a $y \in \frac{1}{n}K$ such that $n\varphi^{-1}(W+y)$ and so also $\varphi^{-1}(W+y)$ is of the second category (or it is not a Christensen zero set). Moreover, this set satisfies the condition of Baire (or it is Christensen measurable). Consequently

$$0 \in \operatorname{Int}[\varphi^{-1}(W+y)-\varphi^{-1}(W+y)]$$

(see Difference Theorem 10.4 in [8] and [4]) and there exists a balanced neighbourhood $U \subset E$ of zero such that

$$U \subset \varphi^{-1}(W+y)-\varphi^{-1}(W+y).$$

Suppose $x \in U$. Then $x = x_1-x_2$ for some $x_1, x_2 \in \varphi^{-1}(W+y)$. Consequently $\varphi(x_1), \varphi(x_2) \in W+y$ and using (7) we have

$$\varphi(x) = \varphi(x_1-x_2) = [\varphi(x_1-x_2)-\varphi(x_1)+\varphi(x_2)]+[\varphi(x_1)-\varphi(x_2)]$$
$$\in K+(W+y)-(W+y) = K+W-W \subset K+V.$$

That is, $\varphi(U) \subset K+V$ which shows that there exists a function f: $U \to V$ such that

(16) $$\varphi(x)-f(x) \in K \quad \text{for every } x \in U.$$

For $x, y \in U$ with $x+y \in U$ we have

$$f(x+y)-f(x)-f(y) = [f(x+y)-\varphi(x+y)]+[\varphi(x+y)-\varphi(x)-\varphi(y)] \\ +[\varphi(x)-f(x)]+[\varphi(y)-f(y)] \in K$$

and, on the other hand,

$$f(x+y)-f(x)-f(y) \in V-V-V \subset V_0 .$$

Consequently (3) holds and by Lemma 1 there exists an additive function $A: E \to F$ such that

$$f(x) = A(x) \quad \text{for every } x \in U .$$

If $x \in E$ then $\frac{x}{n} \in U$ for some positive integer n and using (7) and (16) we get

$$\varphi(x)-A(x) = \varphi\left(n\frac{x}{n}\right)-n\varphi\left(\frac{x}{n}\right)+n\varphi\left(\frac{x}{n}\right)-nA\left(\frac{x}{n}\right)$$
$$= \left[\varphi\left(n\frac{x}{n}\right)-n\varphi\left(\frac{x}{n}\right)\right]+n\left[\varphi\left(\frac{x}{n}\right)-f\left(\frac{x}{n}\right)\right] \in K .$$

This proves the theorem.

Unfortunately, the above Theorem 4 guarantees only the existence of an additive function $A: E \to F$ satisfying (14) but no regularity of it is guaranteed. However, if K has a special form then we can use an idea of Z. Gajda from [6] to get more information about A.

THEOREM 5. *Let E and F be real topological vector spaces and assume that $L: F \to \mathbf{R}$ is a continuous linear functional. Suppose $\varphi: E \to F$ satisfies*

$$\varphi(x+y)-\varphi(x)-\varphi(y) \in L^{-1}(\mathbf{Z}) \quad \textit{for all } x, y \in E .$$

Whether E is a Baire space and φ is Baire measurable or E is a separable F-space and φ is Christensen measurable, there exists a continuous linear operator $M: E \to F$ such that

$$\varphi(x)-M(x) \in L^{-1}(Z) \quad \textit{for every } x \in E .$$

Proof. If $L = 0$, then it is enough to take $M = 0$. So we can assume that $L \neq 0$. Since $L \circ \varphi$ is Baire (or Christensen) measurable and

$$L[\varphi(x+y)]-L[\varphi(x)]-L[\varphi(y)] \in \mathbf{Z} \quad \text{for all } x, y \in E$$

we may use Proposition 1 and 2 from [2] to get a continuous linear functional $N: E \to \mathbf{R}$ such that

$$L[\varphi(x)]-N(x) \in Z \quad \text{for every } x \in E .$$

Fix an $x_0 \in F$ such that $L(x_0) \neq 0$ and define $M: E \to F$ by the formula

$$M(x) = \frac{N(x)}{L(x_0)} x_0 .$$

This function M is continuous and linear. Moreover,

$$L[\varphi(x)-M(x)] = L[\varphi(x)]-L[M(x)] = L[\varphi(x)]-N(x) \in \mathbf{Z} ,$$

for every $x \in E$ which ends the proof.

Returning to the Pexider difference (2) we have the following corollaries.

COROLLARY 6. *Let E be a topological vector space, F be a topological Abelian group, K be a discrete subgroup of F and $f, g, h: E \to F$ satisfy (6) for all $x, y \in E$.*

If any one of f, g, h is continuous at the origin, then there exist a continuous additive function $A: E \to F$, functions $k, l, m: E \to K$ and constants $a, b \in F$ such that

$$\text{(17)} \qquad \begin{cases} f(x) = k(x)+A(x)+a+b , \\ g(x) = l(x)+A(x)+a , \quad \textit{for every } x \in E . \\ h(x) = m(x)+A(x)+b , \end{cases}$$

Proof. This follows from Theorems 2 and 3 and from the fact that E and F are commutative groups.

COROLLARY 7. *Let E and F be topological vector spaces, K be a countable and discrete subgroup of F and $f, g, h: E \to F$ satisfy (6) for all $x, y \in E$.*

If either E is a Baire space and any one of f, g, h is Baire measurable, or E is a separable F-space and any one of f, g, h is Christensen measurable, then there exist an additive function $A: E \to F$, functions $k, l, m: E \to K$ and constants $a, b \in F$ such that (17) holds.

COROLLARY 8. *Let E and F be real topological vector spaces and assume that $L: F \to \mathbf{R}$ is a continuous linear functional. Suppose $f, g, h: E \to F$ satisfy*

$$f(x+y)-g(x)-h(y) \in L^{-1}(\mathbf{Z}) \quad \textit{for all } x, y \in E .$$

If either E is a Baire space and any one of f, g, h is Baire measurable, or E is a separable F-space and any one of f, g, h is Christensen measurable, then there exist a continuous linear operator $A: E \to F$, functions $k, l, m: E \to L^{-1}(\mathbf{Z})$ and constants $a, b \in F$ such that (17) holds.

Acknowledgement. The authors acknowledge support provided by NSERC of Canada.

References

[1] K. Baron, *A remark on the stability of the Cauchy equation*, Rocznik Naukowo-Dydaktyczny WSP w Krakowie. Prace Matematyczne 11 (1985), 7–12.

[2] K. Baron and P. Volkmann, *On the Cauchy equation modulo* $\mathbf{Z}$, Fund. Math., 131 (1988), 143–148.

[3] J. G. van der Corput, *Goniometrische functies gekarakteriseerd door een functionaalbetrekking*, Euclides 17 (1940), 55–75.

[4] P. Fischer and Z. Slodkowski, *Christensen zero sets and measurable convex functions*, Proc. Amer. Math. Soc. 79 (1980), 449–453.

[5] G. L. Forti, *On an alternative functional equation related to the Cauchy equation*, Aequationes Math. 24 (1982), 195–206.
[6] Z. Gajda, *Additive and convex functions in linear topological spaces*, Aequationes Math. 27 (1984), 214–219.
[7] G. Godoni, *Set-valued Cauchy functional equation*, Rev. Roumaine Math. Pures Appl. 20 (1975), 1113–1121.
[8] J. L. Kelley, I. Namioka et al., *Linear Topological Spaces*, Springer-Verlag, 1976.
[9] M. Laczkovich, *Functions with measurable differences*, Acta Math. Acad. Sci. Hungar. 35 (1980), 217–235.
[10] L. Paganoni, *On an alternative Cauchy equation*, Aequationes Math. 29 (1985), 214–221.
[11] W. Rudin, *Functional analysis*, Mc Graw-Hill Book Company, 1973.
[12] L. Székelyhidi, *Local polynomials and functional equations*, Publ. Math. Debrecen 30 (1983), 283–290.

INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
P-40-007 Katowice

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
Waterloo, Ontario, Canada N2L 3G1

Received 13 June 1988

The shrinking property and the $\mathscr{B}$-property in ordered spaces

by

Nobuyuki Kemoto (Okinawa)

Dedicated to Professor Kiyoshi Iséki on his 70th birthday

Abstract. In this paper, it will be shown that

(1) every generalized ordered topological space has the shrinking property,

(2) a subspace of a regular uncountable cardinal $\varkappa$ with the order topology has the $\mathscr{B}(\varkappa)$-property if and only if it is not stationary in $\varkappa$.

As a corollary of (2), we shall also show that

(3) every generalized ordered topological space is paracompact if and only if it has the $\mathscr{B}$-property.

1. Introduction. It is known from Y. Yasui [Y2] that every generalized ordered topological space has the weak $\mathscr{B}$-property. He also pointed out that ω_1 (with the order topology) does not have the $\mathscr{B}$-property. As remarked in [R], it is known that every uncountable regular cardinal with the order topology has the shrinking property, but does not have the $\mathscr{B}$-property. Note that no uncountable regular cardinal is paracompact. In this paper, it will be shown that

(1) every generalized ordered topological space has the shrinking property,

(2) a subspace of a regular uncountable cardinal $\varkappa$ with the order topology has the $\mathscr{B}(\varkappa)$-property if and only if it is not stationary in $\varkappa$.

As a corollary of (2), we shall also show that

(3) every generalized ordered topological space is paracompact if and only if it has the $\mathscr{B}$-property.

In these connections, Y. Yasui asked in [Y1] if spaces having the $\mathscr{B}$-property are paracompact. Now it is known that the only known counterexample is the Navy's space, see [R], [N]. Recently S. Jiang asked in [J] if (normal) spaces having the $\mathscr{B}$-property are (sub)metacompact. Note that metacompact linearly ordered topological spaces are paracompact, see [E, 5.5.22(b)], [GH], [F].

AMS (MOS) Subject Classification (1980) 54D18, 54F05.