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On some famous examples in dimension theory

by

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Abstract. It was proved in [M–R] that for each $n \geqslant 2$ there exists an n-dimensional compactum X such that $\mathscr{E}(X, R)^{2n}$ is dense in $\mathscr{C}(X, R^{2n})$. In this note we prove that the classical examples of V. G. Boltyanskii [B] and Y. Kodama [K2], and their natural higher dimensional counterparts, have the same property.

In [M–R], D. McCullough and L. R. Rubin proved the following result:

For each $n \geqslant 2$ there exists an n-dimensional compactum X such that the space $\mathscr{E}(X, \boldsymbol{R}^{2n})$ of imbeddings from X into $\boldsymbol{R}^{2n}$ is dense in the space $\mathscr{C}(X, \boldsymbol{R}^{2n})$ of mappings to $\boldsymbol{R}^{2n}$.

The aim of this note is to show that some famous examples first studied by V. G. Boltyanskii [B] (see also [K1], [K2], [Wi]) and their straightforward n-dimensional counterparts, $n \geqslant 2$, also have this property (see Th. 5.2 and Th. 5.3).

It was shown in [Kr] that this property implies $\dim X \times X < 2n$. This gives an elementary proof of the fundamental property of the examples in which only elementary algebraic topology is needed (all previous used the Künneth formula).

Lately S. Spież [S] has proved that the latter property implies the property from the McCullough–Rubin theorem for $n \geqslant 3$. (¹)

All spaces in this note are assumed to be metric with a metric denoted by d.

1. A lemma on imbeddings. Let $\mathscr{U}$ be a cover of a space X. A mapping $f\colon X \to Y$ is said to be a *$\mathscr{U}$-mapping* provided for every $y \in f(X)$ there is a $U \in \mathscr{U}$ such that $f^{-1}(y) \subset U$. If $\mathscr{V}$ is a collection of subsets of Y then we denote

$$f^{-1}(\mathscr{V}) = \{f^{-1}(V)\colon V \in \mathscr{V}\}.$$

The following lemma is a slight generalization of Proposition 1.7 from [M–R].

1.1. *Let $X = \{X_1 \overset{\varrho_{12}}{\leftarrow} X_2 \overset{\varrho_{23}}{\leftarrow} \ldots\}$ be an inverse sequence of compacta satisfying the condition*

(*) *for each $i \geqslant 1$, for every mapping $f\colon X_i \to R^n$, for every open cover $\mathscr{U}$ of X_i and every positive number $\delta > 0$ there exist an index $j \geqslant i$ and a $\varrho_{ij}^{-1}(\mathscr{U})$-mapping $g\colon X_j \to \boldsymbol{R}^n$ such that $d(f\varrho_{ij}, g) < \delta$.*

Then $\mathscr{E}(\varprojlim X, \boldsymbol{R}^n)$ is a dense G_δ in $\mathscr{C}(\varprojlim X, \boldsymbol{R}^n)$.

(¹) Added in proof. Extended to $n = 2$ as well.

Proof. The idea is to show that for every open cover $\mathscr{V}$ of $\varprojlim X$ and every $\varepsilon>0$ the set of $\mathscr{V}$-mappings $\mathscr{C}_{\mathscr{V}}(\varprojlim X, R^n)$ is ε-dense in $\mathscr{C}(\varprojlim X, R^n)$. But the proof of this fact presents no difficulty and is left to the reader (comp. [M–R]).

2. Double mapping cylinders. By a *mapping cylinder of two mappings*

$$X \xleftarrow{f} Y \xrightarrow{g} Z$$

we mean the identification space

$$M(f,g) = (X \sqcup Y\times[0,1] \sqcup Z)/\sim$$

with identifications $f(y) = (y,1)$ and $g(y) = (y,0)$ for $y\in Y$.

As special cases we obtain two well-known objects. Namely, if $X = Y$ and f is the identity mapping then $M(f,g)$ is the mapping cylinder $M(g)$ of g. If Z is a one-point space then $M(f,g)$ is the mapping cone $C(f)$ of f.

The spaces X and Z are naturally imbedded in $M(f,g)$ as closed subsets and we identify them with the subsets. The mapping $y\to[y,\frac{1}{2}]$ is a natural imbedding of Y in $M(f,g)$. Note that $M(f,g)$ is a union of $M(f)$ and $M(g)$ identified along Y. By the *boundary* of $M(f,g)$ we mean the set $\partial M(f,g) = X$.

If X, Y and Z are polyhedra and f, g are p.l. mappings then $M(f,g)$ is a polyhedron containing X and Y as subpolyhedra. Moreover, if f is simplicial with respect to triangulations $|K| = X$ and $|L| = Y$ then there is a triangulation T of $M(f,g)$ such that $K\cup L\subset T$ (see [Wh], p. 244).

Consider the following diagram

$$(*)\qquad \begin{array}{ccccc} X & \xleftarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_Y} & & \downarrow{\scriptstyle h_Z} \\ X' & \xleftarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

The following proposition is a simple but important property of double mapping cylinders. The proof is left to the reader.

2.1. *If the diagram* $(*)$ *commutes up to homotopy then there is a mapping* $f\colon M(f,g)\to M(f',g')$ *such that*

(1) $\qquad h(x) = h_X(x)$ *for* $x\in X$ *and* $h(z) = h_Z(z)$ *for* $z\in Z$,

(2) $\qquad h([y,\frac{1}{2}]) = [h_Y(y),\frac{1}{2}]$ *for* $y\in Y$.

If $(*)$ *strictly commutes then* (2) *can be strengthened to*

(2)′ $\qquad h([y,t]) = [h_Y(y),t]$ *for every* $(y,t)\in Y\times[0,1]$.

In particular h agrees with h_X on the boundary of $M(f,g)$. We call h an extension of h_X.

Now we limit the general situation to a special case of mappings between spheres. Fix an integer $n\geqslant 2$. For each integer $a\neq 0$ let $\mu_a\colon S^{n-1}\to S^{n-1}$ be a mapping of degree a; μ_1 is assumed to be the identity mapping (S^{n-1} is regarded as oriented with a fixed orientation). By an *n-dimensional Möbius band* $M(a,b)$ we mean either the mapping cylinder $M(\mu_a,\mu_b)$ if $b\neq 0$, or the mapping cone $C(\mu_a)$ if $b=0$ (comp. [K1]). Changing the mappings up to homotopy we obtain other mapping cylinders (cones); we distinguish them by subscripts. Below we list some mapping properties of such spaces which simply follow from 2.1.

2.2. (1) *Every homeomorphism* $\partial M_1(a,b)\to\partial M_2(a,b)$ *extends to a mapping* $M_1(a,b)\to M_2(a,b)$, $a\neq 0$.

(2) *Let* $a,b\neq 0$. *Then every homeomorphism* $\partial M(ab,a^2b)\to\partial M(1,a)$ *extends to a mapping* $M(ab,a^2b)\to M(1,a)$, *and every homeomorphism* $\partial M(ab,a^2b)\to\partial M(a,0)$ *extends to a mapping* $M(ab,a^2b)\to M(a,0)$.

(2)′ *Let* $a,b\neq 0$. *If b is divisible by a^2 then every homeomorphism* $\partial M(b/a,b)\to\partial M(1,a)$ *extends to a mapping* $M(b/a,b)\to M(1,a)$, *and every homeomorphism* $\partial M(b/a,b)\to\partial M(a,0)$ *extends to a mapping* $M(b/a,b)\to M(a,0)$.

2.3. *Let* $f\colon \partial M(a,b)\to S^{n-1}$. *Then f can be extended to $M(a,b)$ if and only if* $\deg f = k(b/a)$ *for some integer k.*

Proof. $\Rightarrow$. Let $\bar f\colon M(a,b)\to S^{n-1}$ be an extension of f. Let $i_j\colon S^{n-1}\to M(a,b)$, $j=0,1$, be the natural imbeddings such that $\operatorname{im} i_1 = \partial M(a,b)$. Choose an integral cycle γ determining the orientation of S^{n-1}. Note that there is an n-chain c on $M(a,b)$ such that $\partial c = a(i_1\gamma)-b(i_0\gamma)$. Applying $\bar f$ to this equality it follows that $a(\deg f)\gamma$ is homologous to $bk\gamma$ for some $k\in\boldsymbol{Z}$. The implication follows.

$\Leftarrow$. Apply 2.1.

3. Cylinders and cones in cubes. Fix an integer $n\geqslant 2$ and let $\mu\colon S^{n-1}\to S^{n-1}$ be any mapping. Denote by D^n the closed unit ball in $\boldsymbol{R}^n$ so that $S^{n-1} = \partial D^n$.

3.1 (comp. [M–R]). *There exist imbeddings*

$$\varphi\colon M(\mu)\to D^n\times D^n \quad \textit{and} \quad \psi\colon C(\mu)\to D^n\times D^n$$

such that

(i) $\operatorname{im}\varphi\cap\operatorname{im}\psi = \emptyset$,

(ii) φ *carries* $\partial M(\mu)$ *homeomorphically onto* $S^{n-1}\times(0)$,

(iii) ψ *carries* $\partial C(\mu)$ *homeomorphically onto* $(0)\times S^{n-1}$,

(iv) $\varphi(M(\mu)\setminus\partial M(\mu))\cup\psi(C(\mu)\setminus\partial C(\mu))\subset\operatorname{int}(D^n\times D^n)$.

The imbeddings are defined as follows.

Recall that $M(\mu) = (S^{n-1}\times I\sqcup S^{n-1})/\sim$ with $(x,0) = \mu(x)$ for $x\in S^{n-1}$. Define φ to be the mapping induced by the following

$$S^{n-1}\times I\ni(z,t)\to(tz,\tfrac{1}{2}(t-1)\mu(z))\in D^n\times D^n,$$
$$S^{n-1}\ni z\to(0,-\tfrac{1}{2}z)\in D^n\times D^n.$$

To define ψ we need a mapping $\alpha\colon I\to I$ given by

$$\alpha(s) = \begin{cases} s & 0\leqslant s\leqslant\frac{1}{2},\\ 1-s & \frac{1}{2}\leqslant s\leqslant 1.\end{cases}$$

Note that $C(\mu) = (S^{n-1}\times I \sqcup S^{n-1})/\sim$ with $(x, 0) = (y, 0)$ and $(x, 1) = \mu(x)$ for $x, y \in S^{n-1}$. Define ψ to be the mapping induced by the following

$$S^{n-1}\times I \ni (y, s) \to (\alpha(s)y, s\mu(y)) \in D^n \times D^n$$

$$S^{n-1} \ni y \to (0, y) \in D^n \times D^n$$

One easily verifies that φ and ψ have the required properties.

3.2. Remark. Analogous imbeddings were constructed in [M–R], Lemma 2.2, generalizing the imbedding from Ex. 4.3 in [K–L]. In the same spirit we generalize the imbedding from Ex. 4.4 in [K–L]; we hope the new formulas are a little simpler.

Recall that a mapping $f\colon X\to D^n$ is said to be *essential* provided there is no mapping $g\colon X \to \partial D^n$ such that $g(x) = f(x)$ for $x \in f^{-1}(\partial D^n)$.

3.3. Corollary. *For $n \geqslant 2$ let $M = M(ab, a^2 b)$ be an n-dimensional Möbius band, with the integers a and b satisfying $|a| > 1$ and $b \neq 0$. Then for any two mappings $f, g\colon M(ab, a^2b)\to D^n$, each carrying the boundary ∂M homeomorphically onto ∂D^n and $M\setminus\partial M$ into $\operatorname{int} D^n$, their product*

$$f\times g\colon M(ab, a^2b)\times M(ab, a^2b) \to D^n\times D^n$$

is inessential though both f and g are essential.

Proof. Since $\deg h = \pm 1$ for a homeomorphism $h\colon S^{n-1}\to S^{n-1}$, both f and g are essential by 2.3.

Let $\mu\colon S^{n-1}\to S^{n-1}$ be a mapping with $\deg\mu = a$. Let $\varphi = (\varphi_1, \varphi_2)\colon M(\mu)\to D^n\times D^n$ and $\psi = (\psi_1, \psi_2)\colon C(\mu)\to D^n\times D^n$ be the mappings from 3.1. By 2.2 of [K–L] and 3.1 (i) we infer that

(1) $\qquad \varphi_1\times\psi_2\colon M(\mu)\times C(\mu)\to D^n\times D^n$ is not essential.

Note that $\varphi_1^{-1} f$ maps ∂M homeomorphically onto $\partial M(\mu)$. Since $M(\mu) = M(1, a)$ by 2.2(2) this homeomorphism extends to a mapping $\bar f\colon M\to M(\mu)$. Thus $f|\partial M = \varphi_1\circ\bar f|\partial M$. Since $\partial M = f^{-1}(\partial D^n)$ it follows that

(2) $$f \simeq \varphi_1 \bar f \quad \text{rel.}\, f^{-1}(\partial D^n)\,.$$

Since $C(\mu) = M(a, 0)$ a similar argument shows that there is a mapping $\bar g\colon M\to C(\mu)$ such that

$$g\simeq \psi_2\bar g \quad \text{rel.}\, g^{-1}(\partial D^n)\,.$$

Put $B = D^n\times D^n$. By (2) and (3) it follows that

$$f\times g|(f\times g)^{-1}(\partial B) \simeq (\varphi_1\times\psi_2)(\bar f\times\bar g)|(f\times g)^{-1}(\partial B) \quad \text{in } \partial B\,.$$

This, by (1), implies that $f\times g$ is inessential (comp. 2.3 in [K–L]).

4. Modification of polyhedra. Let X be an n-dimensional polyhedron, $n \geqslant 2$, with a given triangulation K. Let a and b be two integers with $a \neq 0$. By an (a, b)-*modification* of K we mean a mapping

$$\varrho\colon Y\to X$$

constructed as follows (comp. [K1], [K2] and [Wi]).

For each n-simplex $s\in K$ choose a copy M_s of the n-dimensional Möbius band $M(a, b)$ such that ∂M_s is a subpolyhedron of M_s simplicially isomorphic to ∂s for some triangulation T_s of M_s. Let $h_s\colon \partial M_s\to\partial s$ be a simplicial isomorphism. Now attach M_s to $|K^{(n-1)}|$ using h_s to obtain the space Y, where s runs over the set of n-simplices of K. We may identify M_s with a subset of Y so that $\partial M_s = \partial s$. Then $Y = |K^{(n-1)}| \cup \bigcup_s M_s$ and the interiors $M_s\setminus\partial s$ are pairwise disjoint. Note that $K^{(n-1)}\cup\bigcup_s T_s$ is a triangulation of Y. We define ϱ to be any mapping such that $\varrho(M_s\setminus\partial s)\subset s\setminus\partial s$ for each s and $\varrho(x) = x$ for every $x\in |K^{(n-1)}|$.

5. The Boltyanskii–Kodama examples. Fix an integer $n\geqslant 2$. We are going to define n-dimensional counterparts of the Boltyanskii–Kodama examples (see [B] and [K2]) and to show that they have the property from the McCullough–Rubin theorem. The generalization is straightforward and we follow the description given by Y. Kodama in [K1] and [K2], p. 229, for the case $n = 2$ (comp. also [Wi]).

Let $a = (a_1, a_2, \ldots)$ be a sequence of integers such that $a_i\neq 0$ and a_{i+1} is divisible by a_i for each $i\geqslant 1$. We associate with such a sequence an inverse sequence of polyhedra

$$X = \{X_1 \overset{\varrho_{12}}{\leftarrow} X_2 \overset{\varrho_{23}}{\leftarrow} \ldots\}$$

in the following way. Let X_1 be an n-simplex and let K_1 be the standard triangulation of X_1. Having defined X_l, $l\geqslant 1$, and a triangulation K_l of X_l we define $\varrho_{l l+1}\colon X_{l+1}\to X_l$ to be an $(a_{l+1}/a_l, a_{l+1})$-modification of K_l. Define K_{l+1} to be any triangulation of X_{l+1} such that $|K_l^{(n-1)}|$ is a subpolyhedron of $|K_{l+1}^{(n-1)}|$. This completes an inductive description of X. Note a homogeneity of the construction with respect to every n-simplex of every complex K_l.

Let $s\in K_i$, $i\geqslant 1$, be an n-simplex. Consider the inverse image $s(j) = \varrho_{ij}^{-1}(s)$ $j\geqslant i$. Note that $s(i) = s$. Also note that the $(n-1)$-sphere ∂s is a subset of $s(j)$, denote $\partial s(j) = \partial s$. For a special case ($i = 1$, $s = X_1$ — then $s(j) = X_j$) this defines a subset $\partial X_j\subset X_j$.

5.1. *If a mapping $f\colon \partial s(j)\to S^{n-1}$ extends to $s(j)$ then $\deg f$ is divisible by a_i. In particular, if $f\colon \partial X_j\to S^{n-1}$ extends to X_j then $a_1|\deg f$.*

Proof (by induction on $m = j-i$). If $m = 0$ then $s(j) = s$. Hence $a_i|\deg f$ since $\deg f = 0$ in this case.

Suppose $m > 0$, i.e. $j\geqslant i+1$. Let $\bar f\colon s(j)\to S^{n-1}$ be an extension of f. Let K be the restriction of K_{i+1} to $s(i+1)$. Give each n-simplex s_k of K an orientation σ_k

such that

$$(*)\qquad \sum \partial\sigma_k = (a_{i+1}/a_i)\gamma_1 - a_{i+1}\gamma_2\,,$$

where γ_1, γ_2 are cycles in $|K^{(n-1)}|$ and γ_1 represents a generator of $H_{n-1}(\partial s(j))$ (see proof of 2.3). Since $|K^{(n-1)}| \subset s(j)$ (by construction) we may apply $\bar{f}$ to both sides of $(*)$ and we get

$$\sum \bar{f}(\partial\sigma_k) = (a_{i+1}/a_i)\bar{f}(\gamma_1) - a_{i+1}\bar{f}(\gamma_2)\,.$$

By an inductive assumption applied to $j-(i+1) < m$ and $\bar{f}\,|\,\partial s_k(j)$ (since $s_k(j) \subset s(j)$ this mapping has an extension to $s_k(j)$, e.g. $\bar{f}\,|\,s_k(j)$) we have $\bar{f}(\partial\sigma_k) \sim a_{i+1} l_k \gamma$, where $l_k \in Z$ and γ represents the orientation of S^{n-1}. Passing to homology classes, with identification $H_{n-1}(S^{n-1}) = Z$, we get

$$a_{i+1} l = \pm(a_{i+1}/a_i)\deg f - a_{i+1} r\,, \qquad l, r \in Z\,.$$

Thus $a_i | \deg f$ and the proof is completed.

The spaces of interest are defined to be the inverse limits

$$X(a) = \varprojlim X\,.$$

5.2. Theorem. *If* $|a_1| > 1$ *then* $\dim X(a) = n$.

Proof. It is clear that $\dim X(a) \leqslant n$. In order to prove the other inequality note that the mapping $\varrho_{1j}\colon X_j \to X_1$ is essential for each $j \geqslant 1$. Indeed, the restriction $\partial X_j \to \partial X_1$ of ϱ_{1j} is a homeomorphism, hence its degree is equal to ± 1. Thus ϱ_{1j} is essential by 5.1. It follows that the projection $X(a) \to X_1$ is essential and therefore $\dim X(a) \geqslant n$.

5.3. Theorem. *If the following conditions are satisfied*:

(i) a_{i+1} *is divisible by* a_i^2 *for each* $i \geqslant 1$,

(ii) *for each* $i \geqslant 1$ *and every open cover* $\mathscr{V}$ *of* X_i *there exists an index* $l \geqslant i$ *such that* $\operatorname{st} K_l$ *refines* $\varrho_{il}^{-1}(\mathscr{V})$,

then $\mathscr{E}(X(a), R^{2n})$ *is dense in* $\mathscr{C}(X(a), R^{2n})$. *In particular* $\dim(X(a) \times X(a)) < 2n$.

5.4. Remark. The conclusion of 5.3 still holds if condition (i) is relaxed as follows:

(i)′ for each i there exists $j > i$ such that a_j is divisible by a_i^2.

Proof of 5.3. In order to prove the first assertion fix an index $i \geqslant 1$, a mapping $f\colon X_i \to R^{2n}$, an open cover $\mathscr{U}$ of X_i, and a real number $\delta > 0$. According to 1.1 it suffices to find an index $j \geqslant i$ and a mapping $g\colon X_j \to R^{2n}$ such that

$$(1)\qquad d(f\varrho_{ij}, g) < \delta \quad\text{and}\quad g \text{ is a } \varrho_{ij}^{-1}(\mathscr{U})\text{-mapping}\,.$$

There exist a subdivision L of K_i and a mapping $v\colon X_i \to R^{2n}$ linear on each simplex of L such that

$$(2)\qquad d(f, v) < \delta/3\,,$$

$$(3)\qquad \operatorname{diam} v(s) < \delta/3 \quad \text{for each } s \in L\,,$$

$$(4)\qquad \mathscr{V} = \{\operatorname{st}(x, L)\colon x \in L^{(0)}\} \text{ is a star-refinement of } \mathscr{U}\,,$$

$$(5)\qquad S(v) \subset X_i \setminus |L^{(n-1)}|\,, \qquad v(S(v)) = \{z_1, \ldots, z_t\}$$

($S(v)$ is the singular set of v; if $S(v) = \emptyset$ then $j = i$ and $g = v$ satisfy (1) and the proof is completed)

$$(6)\qquad v^{-1}(z_r) = \{x_r, y_r\} \quad \text{for each } r = 1, \ldots, t\,.$$

For $x \in X_i$, let $s(x)$ denote the simplex of L that contains x in its interior. Then $s(x_r)$ and $s(y_r)$ are n-dimensional. If L is sufficiently fine then there exist (linear) imbeddings

$$w_r\colon s(x_r) \times s(y_r) \to R^{2n}\,, \qquad r = 1, \ldots, t\,.$$

with disjoint images such that

$$(7)\qquad \operatorname{diam}(\operatorname{im} w_r) < \delta/3\,,$$

$$(8)\qquad w_r(x, y_r) = v(x)\,, \qquad w_r(x_r, x) = v(x)\,,$$

$$(9)\qquad v^{-1}(\operatorname{im} w_r) = s(x_r) \cup s(y_r)\,.$$

By (ii), there is an index $l \geqslant i$ such that $\operatorname{st} K_l$ refines $\varrho_{il}^{-1}(\mathscr{V})$. Hence there is a simplicial approximation $\varrho\colon K_l \to L$ to ϱ_{il}. Since $\operatorname{st}(\mathscr{V})$ refines $\mathscr{U}$ (by (4)) it follows that

$$(10)\qquad \varrho^{-1}(L) \text{ refines } \varrho_{il}^{-1}(\mathscr{U})\,.$$

From (3) we infer that

$$(11)\qquad d(v\varrho_{il}, v\varrho) < \delta/3\,.$$

By 3.1, for each $r = 1, \ldots, t$, there exist disjoint sets $M_r, C_r \subset \operatorname{im} w_r$ such that M_r is a copy of the n-dimensional space $M(1, a_l)$, C_r is a copy of the n-dimensional space $M(a_l, 0)$, $\partial M_r = v(\partial s(x_r))$ and $\partial C_r = v(\partial s(y_r))$.

Let $u = \varrho\varrho_{l\,l+1}$. Since $\varrho_{l\,l+1}\colon X_{l+1} \to X_l$ is an $(a_{l+1}/a_l, a_{l+1})$-modification of K_l, by applying 2.2 it is easy to construct a mapping $g\colon X_{l+1} \to R^{2n}$ satisfying the conditions

$$(12)\qquad g(x) = vu(x) \quad \text{if } u(x) \notin \bigcup_r (\operatorname{int} s(x_r) \cup \operatorname{int} s(y_r))\,,$$

$$(13)\qquad g\big(u^{-1}(s(x_r))\big) \subset M_r\,,$$

$$(14)\qquad g\big(u^{-1}(s(y_r))\big) \subset C_r\,,$$

for each $r = 1, \ldots, t$. Now we shall show that $j = l+1$ and g satisfy (1). It is clear that $d(f\varrho_{ij}, g) < \delta$. By (12), (13) and (14) it follows that for every $z \in \operatorname{im} g$ there exists a simplex $s \in L$ such that $g^{-1}(z) \subset u^{-1}(s)$. Hence by (10) g is a $\varrho_{ij}^{-1}(\mathscr{U})$-mapping. This proves (1) and completes the proof of the first assertion. The second one follows now from [Kr].

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Sur le prolongement des fonctions continues dans les complexes simpliciaux infinis

par

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Abstract. Let X be a simplicial complex which does not contain a strictly increasing infinite sequence of simplexes. It is known that, if its geometric realization $|X|$ is provided with the metric topology, it becomes an absolute neighborhood extensor for collectionwise normal spaces. This is definitely false if $|X|$ is provided with the weak topology. We study here this phenomenon in detail, and show that its only origin is the non-preservation of extension properties under formation of infinite wedges of cones.

1. Introduction et notations. Soit Q une classe d'espaces topologiques. Nous dirons qu'un espace Y a la *propriété d'extension* (locale) par rapport à Q si, pour tout espace X appartenant à Q et tout fermé A de X, toute fonction continue de A dans Y peut se prolonger à X (resp. à un voisinage de A dans X). Lorsque Q se réduit à un espace X, nous parlerons de propriété d'extension (locale) par rapport à X. Nous dirons que Y est un *rétracte absolu* de voisinage pour la classe Q (ou RAV(Q)) si Y appartient à Q et si, pour tout espace X appartenant à Q, tout fermé de X homéomorphe à Y est un rétracte de voisinage de X.

Si K est un complexe simplicial, nous noterons $|K|$ sa réalisation géométrique munie de la topologie faible, et $|K|_m$ cette même réalisation géométrique munie de la topologie métrique. Par un simplexe infini, nous entendrons un complexe simplicial infini dont tout ensemble fini de sommets détermine un simplexe.

Il est connu que, si K est un complexe simplicial, $|K|_m$ a la propriété d'extension locale par rapport aux espaces collectivement normaux si, et seulement si, K ne contient aucun simplexe infini. Ceci découle des trois résultats suivants:

(a) $|K|_m$ est un RAV(métrique) (voir [11], p. 106).

(b) un RAV(métrique) a la propriété d'extension locale par rapport aux espaces collectivement normaux si, et seulement si, c'est un G_δ absolu [7],

(c) $|K|_m$ est un G_δ absolu si, et seulement si, K ne contient aucun simplexe infini ([11], p. 107).

Avec la topologie faible, la situation change complètement. Un exemple de van Douwen et Pol ([6], exemple 2; voir la note finale) montre l'existence d'un complexe K de dimension un, d'un espace régulier dénombrable Π, d'un fermé A