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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

The Nielsen number product formula for coincidences

by

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Abstract. In [24] Cheng-Ye You gave a condition equivalent to the Nielsen number product formula for fibre maps. In this paper we present a similar condition for coincidences of fibre maps. As an application we generalize the Anosov theorem:

If f and g are self-maps of a nilmanifold then

$$N(f, g) = |L(f, g)|.$$

Introduction. In [3] Robert Brown raised the problem when for a self fibre map of a compact connected ANR the product formula for Nielsen numbers $N(f) = N(f_b)N(\bar{f})$ holds. After a series of papers [3, 4, 8, 10, 15, 18] where only sufficient conditions were given Cheng-Ye You [24] found a condition which is also necessary (see also [14], Chapter 4, and [12]).

In this paper we present a similar condition for coincidences of fibre maps. We consider the commutative diagram

$$(*)\qquad \begin{array}{ccc} E & \xrightarrow{(f,g)} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{(\bar{f},\bar{g})} & B' \end{array}$$

where (E, p, B) and (E', p', B') are locally trivial bundles with total spaces, base spaces and fibres compact connected closed oriented topological manifolds of respectively equal dimensions. We find a necessary and sufficient condition for the formula

$$N(f, g) = N(f_b, g_b)N(\bar{f}, \bar{g})$$

to hold. It generalizes [24] in the case of manifolds. The method used here follows that of You. As an application we generalize the Anosov theorem:

If f and g are self-maps of a nilmanifold then $N(f, g) = |L(f, g)|$ (see [0], [9]).

In §§1–3 some preliminary results are given. Then in § 4 we consider the diagram (*). Let b_0, b_1 be coincidence points of $\bar{f}$, $\bar{g}$ and let $\bar{u}$ be a path joining these points and establishing the Nielsen relation between them. Then the diagram

$$\begin{array}{ccc} E_{b_0} & \xrightarrow{(f_{b_0},\, g_{b_0})} & E'_{fb_0} \\ {\scriptstyle \tau_u}\downarrow & & \downarrow{\scriptstyle \tau'_{fu}} \\ E_{b_1} & \xrightarrow{(f_{b_1},\, g_{b_1})} & E'_{fb_1} \end{array}$$

is homotopy commutative and we show that it gives rise to a bijective map $T_{\bar{u}}$ between the sets of Reidemeister classes $\nabla_K(f_{b_0}, g_{b_0})$ and $\nabla_K(f_{b_1}, g_{b_1})$. In particular, for $b_0 = b_1 = b$ this gives an action on the set $\nabla_K(f_b, g_b)$. This action turns out to be the obstruction for the product formula of Nielsen numbers to hold (Theorem (6.5)). More exactly, the product formula is equivalent to two conditions: one of them, $N(f_b, g_b) = N_K(f_b, g_b)$, is connected with the inclusion of the fibre into the total space and the other says that the above action is trivial.

1. H-Nielsen number for coincidences. Let Y be a path-connected space and let u, v be two paths in Y such that $u(1) = v(0)$. We denote by $u+v$ the composition of these paths and by $-u$ the path opposite to u.

Let H be a normal subgroup of $\Pi_1 Y$ (this means that for each $v \in Y$ a normal subgroup $H(y) \subset \Pi_1(Y, y)$ is given such that for any path w from v to y' and any loop a based at y and representing an element of $H(y)$ we have $\langle -w+a+w\rangle \in H(y')$. Two paths u, v are said to be *H-homotopic* if $u(0) = v(0)$, $u(1) = v(1)$ and $\langle u-v\rangle \in H'(u(0))$. We then write $u \overset{H}{\sim} v$. It is easy to see that $u \overset{H}{\sim} v$ implies $v \overset{H}{\sim} u$, $-u \overset{H}{\sim} -v$, $u+w \overset{H}{\sim} v+w$, $r+u \overset{H}{\sim} r+v$ if $u+w$, $r+u$ are well defined. H is an equivalence relation and the class of the path a will be denoted by $\langle a\rangle_H$. This relation confined to the loops based at a point $y \in Y$ gives the quotient group $\Pi_1(Y, y)/H(y)$.

Let X be another path-connected space and let $f, g\colon X \to Y$ be continuous maps. We denote the set of coincidence points of the pair (f, g) by

$$\Phi(f, g) = \{x \in X\colon fx = gx\}.$$

Two points $x, x' \in \Phi(f, g)$ are said to be *H-Nielsen equivalent* (briefly *H-equivalent*) if there is a path c in X from x to x' such that $fc \overset{H}{\sim} gc$ in Y. We denote the quotient set by $\Phi'_H(f, g)$ and call its elements *H-Nielsen classes* of the pair (f, g).

Fix $x \in X$ and a path r in Y from fx to gx. Let $\langle d\rangle \in \Pi_1(X, x)$ and

$$\langle a\rangle_H \in \Pi_1(Y, fx)/H(fx).$$

Then the formula

$$(\langle d\rangle, \langle a\rangle_H) \to \langle fd+a+r-gd-r\rangle_H$$

defines a left action of $\Pi_1(X, x)$ on $\Pi_1(Y, fx)/H(fx)$. We denote the orbit space by $\nabla_H(f, g\colon x, r)$ and the orbit of a loop a by $[\langle a\rangle_H]$.

(1.1) Remark. The above definition of $\nabla_H(f, g\colon x, r)$ is not symmetric with the respect to f and g. Nevertheless one can check that the correspondence

$$\nabla_H(f, g\colon x, r) \ni [\langle a\rangle_H] \to [\langle -r-a+r\rangle_H] \in \nabla_H(g, f\colon x, -r)$$

is bijective.

(1.2) Lemma. *For each $x_0 \in \Phi(f, g)$ the set $\{\langle fc-gc-r\rangle_H\colon c$ is a path from x to $x_0\}$ is an orbit of the above action and so x_0 determines an element of $\nabla_H(f, g\colon x, r)$. Two coincidence points determine the same element iff they are H-Nielsen equivalent.*

Proof. Let c, c' be paths from x to x_0. Then $c-c'$ is a loop based at x and so

$$\langle fc-gc-r\rangle_H = \langle f(c-c')+(fc'-gc'-r)+r-g(c-c')-r\rangle_H$$

lies in the same orbit as $\langle fc'-gc'-r\rangle_H$.

Let now c be a path from x to x_0 and let $\langle u\rangle_H$ lie in the same orbit as $\langle fc-gc-r\rangle_H$. Then

$$\langle u\rangle_H = \langle fd+(fc-gc-r)+r-gd-r\rangle_H = \langle f(d+c)-g(d+c)-r\rangle_H$$

for some loop d based at x. Thus we get the first conclusion.

Let $x_0, x_1 \in \Phi(f, g)$ be H-equivalent. Take a path d from x_0 to x_1 such that $fd \sim gd$. Then $c+d$ is a path from x to x_1 and the equality

$$\langle f(c+d)-g(c+d)-r\rangle_H = \langle fc+fd-gd-gc-r\rangle_H = \langle fc-gc-r\rangle_H$$

shows that x_0 and x_1 determine the same orbit.

Let now $x_0, x_1 \in \Phi(f, g)$ determine the same orbit. Then according to the first conclusion for any path c from x to x_0 we can choose a path c' from x to x_1 such that

$$\langle fc-gc-r\rangle_H = \langle fc'-gc'-r\rangle_H.$$

This implies $f(-c+c') \overset{H}{\sim} g(-c+c')$ and since $-c+c'$ connects the points x_0 and x_1, they are H-equivalent. ∎

The above lemma defines an injective map

$$\varrho(x, r)\colon \Phi'_H(f, g) \to \nabla_H(f, g\colon x, r)$$

which assigns to the class of $x_0 \in \Phi(f, g)$ the class $[\langle fc-gc-r\rangle_H]$ where c is a path from x to x_0.

Let $x, x' \in X$ and let r, r' be paths in Y from fx to gx and from fx' to gx' respectively. The next lemma establishes a relationship between the sets $\nabla_H(f, g\colon x, r)$ and $\nabla_H(f, g\colon x', r')$.

(1.3) Lemma. *Let u be a path from x to x'*

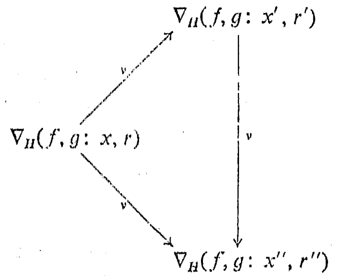
(a) *If $[\langle a\rangle_H] = [\langle a'\rangle_H] \in \nabla_H(f, g\colon x, r)$ then*

$$[\langle -fu+a+r+gu-r'\rangle_H] = [\langle -fu+a'+r+gu-r'\rangle_H] \in \nabla_H(f, g\colon x', r').$$

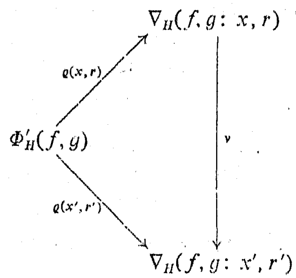
Thus we may define a transformation $v\colon \nabla_H(f, g\colon x, r) \to \nabla_H(f, g\colon x', r')$ by the formula

$$v[\langle a\rangle_H] = [\langle fu+a+r+gu-r'\rangle_H].$$

(b) *v is independent of the choice of the path u.*
(c) *If $x = x'$ and $r \overset{H}{\sim} r'$ then v is the identity.*
(d) *If (x'', r'') is another reference pair then the diagram*



commutes.
(e) *The diagram*



commutes. ∎

It follows from (c) and (d) that v is bijective, so we may identify all sets $\nabla_H(f, g: x, r)$ by means of v. We denote the obtained quotient set by $\nabla_H(f, g)$. We will call its elements *H-Reidemeister classes* of (f, g).

(1.4) Remark. Lemmas (1.2) and (1.3) give us an injective map $\varrho: \Phi'_H(f, g) \to \nabla_H(f, g)$. Thus each H-Nielsen class may be identified with an H-Reidemeister class. On the other hand, we say that a class $A \in \nabla_H(f, g)$ *equals* $A' \in \Phi'_H(f, g)$ *as a set* if $\varrho(A') = A$, and that A is *empty* if it does not lie in the image of ϱ.

We will omit the sign H if $H = 0$.

Let (F, G): $(f_0, g_0) \sim (f_1, g_1)$: $X \to Y$ be a homotopy, i.e. F is a homotopy from f_0 to f_1 and G is a homotopy from g_0 to g_1.

(1.5) Lemma. *Let (x_i, r_i) be a reference pair for (f_i, g_i) $(i = 0, 1)$ and let u be a path from x_0 to x_1.*

(a) *If $[\langle a\rangle_H] = [\langle a'\rangle_H] \in \nabla_H(f_0, g_0: x_0, r_0)$ then*

$$[\langle -\Delta(F, u) + a + r_0 + \Delta(G, u) - r_1\rangle_H] = [\langle -\Delta(F, u) + a' + r_0 + \Delta(G, u) - r_1\rangle_H]$$

in $\nabla_H(f_1, g_1: x_1, r_1)$. ($\Delta(F, u)$ and $\Delta(G, u)$ stand for the diagonal paths: $\Delta(F, u)(t) = F(u(t), t)$).

Thus we get a map

$$\mu_{(F,G)}: \nabla_H(f_0, g_0: x_0, r_0) \to \nabla_H(f_1, g_1: x_1, r_1)$$

which determines a transformation

$$\mu_{(F,G)}: \nabla_H(f_0, g_0) \to \nabla_H(f_1, g_1).$$

(b) *$\mu_{(F,G)}$ does not depend on the choice of (x_0, r_0), (x_1, r_1) and the path u.*
(c) *$\mu_{(F,G)}$ is bijective.*
(d) *Let $x_i \in A_i \in \Phi'_H(f_i, g_i)$ $(i = 0, 1)$. Then $\mu_{(F,G)}A_0 = A_1$ iff there is a path c rom x_0 to x_1 such that*

$$\Delta(F, c) \overset{H}{\sim} \Delta(G, c).$$

(e) *If (F', G'): $(f_1, g_1) \sim (f_2, g_2)$: $X \to Y$ is another homotopy then $\mu_{(F*F', G*G')} = \mu_{(F',G')}\mu_{(F,G)}$. Here $F * F'$ denotes the concatenation of the homotopies F and F':*

$$F * F'(x, t) = \begin{cases} F(x, 2t) & \text{for } t \leqslant 1/2, \\ F'(x, 2t-1) & \text{for } t \geqslant 1/2. \end{cases}$$ ∎

(1.6) Remark. If $f_0 = f_1$, $g_0 = g_1$ and the homotopies F, G are constant then the maps

$$v, \mu_{(F,G)}: \nabla_H(f_0, g_0) \to \nabla_H(f_0, g_0)$$

are equal.

Let M, N be compact connected oriented n-dimensional topological manifolds without boundary. Let $U \subseteq M$ be an open subset and let $f, g: U \to N$ be a pair of maps such that $\Phi(f, g)$ is compact. Now we recall the definition of the coincidence index of such a pair [21].

Let $z_M \in H_n M$ be the fundamental class and let $\mu_N \in H^n(N \times N, N \times N - \Delta N)$ be the Thom class corresponding to the chosen orientations. All homology and cohomology groups are taken with rational coefficients. We define the coincidence index of (f, g) as the image of the fundamental class z_M in the sequence of homomorphisms

$$H_n M \to H_n(M, M - \Phi) \xrightarrow{\text{exc}} H_n(U, U - \Phi) \xrightarrow{(f,g)} H_n(N \times N, N \times N - \Delta N) \xrightarrow{[\mu_N, \cdot]} Q$$

(exc denotes the excision isomorphism, [...] the Kronecker index and Q the field of rational numbers). We denote it by $\operatorname{ind}(f, g: U)$. For its properties see [21]; we recall only that it is an integer and that $\operatorname{ind}(f, g: U) \neq 0$ implies the existence of coincidence points of (f, g) in U. If $U = M$ then $\operatorname{ind}(f, g: M)$ equals the coincidence Lefschetz number $L(f, g)$ [21].

Let now again $f, g: M \to N$ and let H be a normal subgroup of $\Pi_1 N$. Let $A \in \nabla_H(f, g)$. We define the index of the class A as

$$\operatorname{ind}(f, g: A) = \operatorname{ind}(f, g: U)$$

where U is an arbitrary open subset of M containing A and disjoint from the other classes of (f, g) (here we treat A as a subset of M; compare (1.4). Obviously, if A is not the Nielsen class we may take $U = \emptyset$ so in this case $\operatorname{ind}(f, g\colon A) = 0$.

A class $A \in \nabla_H(f, g)$ is called *essential* if its index is not zero, otherwise it is called *unessential*. We define the *H-Nielsen number* of (f, g) as the number of its essential classes and we denote it by $N_H(f, g)$. Notice that since the manifolds are locally contractible and compact, the set of all Nielsen classes is finite and hence $N_H(f, g)$ is a natural number or zero.

Let $(F, G)\colon (f_0, g_0) \sim (f_1, g_1)\colon M \to N$ be homotopies. One can modify [19] to get

(1.7) THEOREM. *$\mu_{(F,G)}$ preserves index (more precisely, if $A \in \nabla_H(f_0, g_0)$ then* $\operatorname{ind}(f_0, g_0\colon A) = \operatorname{ind}(f_1, g_1\colon \mu_{(F,G)} A)$). *This implies the equality* $N_H(f_0, g_0) = N_H(f_1, g_1)$. ■

2. Homotopy commuting diagrams. In this section we define transformations between the sets of Reidemeister (or Nielsen) classes of pairs of maps appearing in a homotopy commutative diagram. Later we will use the obtained formulae to compare the Nielsen numbers of fibre maps restricted to different fibres.

Consider a commutative diagram of path-connected topological spaces

$$\begin{array}{ccc} M & \xrightarrow{(f,g)} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M' & \xrightarrow{(f',g')} & N' \end{array}$$

Let (x, r) be a reference pair for (f, g) and let $H \subset \Pi_1 N$, $H' \subset \Pi_1 N'$ be normal subgroups such that $K_\# H \subset H'$. Then the map $\varkappa\colon \nabla_H(f, g\colon x, r) \to \nabla_{H'}(f', g'\colon hx, kr)$ given by $\varkappa[\langle a\rangle_H] = [\langle ka\rangle_{H'}]$ defines a transformation $\varkappa\colon \nabla_H(f, g) \to \nabla_H(f', g')$ which does not depend on the choice of (x, r).

Assume now that the above diagram is only homotopy commutative and fix a homotopy $(F, G)\colon (kf, kg) \sim (f' h, g' h)$. Moreover, assume that h and k are homeomorphisms. Then our diagram may be considered as the composition of two squares

$$\begin{array}{ccc} M & \xrightarrow{(f,g)} & N \\ \| & & \| \\ M & \xrightarrow{(k^{-1}f'h,\, k^{-1}g'h)} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M' & \xrightarrow{(f',g')} & N' \end{array}$$

where the upper square is homotopy commutative (with homotopy $(k^{-1}F, k^{-1}G)$) and the lower is commutative. In this way we get the composition

$$\nabla_H(f, g) \xrightarrow{\mu_{(k^{-1}F,\, k^{-1}G)}} \nabla_H(k^{-1}f' h, k^{-1}g' h) \xrightarrow{\varkappa} \nabla_{H'}(f', g')$$

which will be denoted by $\eta_{(F,G)}$.

Fix reference pairs (x_0, r_0) for (f, g), (x_1, r_1) for $(k^{-1}f' h, k^{-1}g' h)$ and a path u from x_0 to x_1. Then we may represent $\eta_{(F,G)}$ as the composition

$$\nabla_H(f, g\colon x_0, r_0) \to \nabla_H(k^{-1}f' h, k^{-1}g' h\colon x_1, r_1) \to \nabla_{H'}(f', g'\colon hx_1, kr_1),$$

$$\begin{aligned} \varkappa\mu_{(k^{-1}F, k^{-1}G)}[\langle a\rangle_H] &= \varkappa[\langle -\Delta(k^{-1}F, u) + a + r_0 + \Delta(k^{-1}G, u) - r_1\rangle_H] \\ &= [\langle -k\Delta(k^{-1}F, u) + ka + kr_0 + k\Delta(k^{-1}G, u) - kr_1\rangle_{H'}] \\ &= [\langle -\Delta(F, u) + ka + kr_0 + \Delta(G, u) - kr_1\rangle_{H'}]. \end{aligned}$$

In particular, if we choose $x_0 = x_1 = x$, $r_0 = r$, $r_1 = -k^{-1}F(x, \cdot) + r + k^{-1}G(x, \cdot)$ and u a constant path based at x then

$$\eta_{(F,G)}\colon \nabla_H(f, g\colon x, r) \to \nabla_{H'}(f', g'\colon hx, -F(x, \cdot) + kr + G(x, \cdot))$$

is represented by $\eta_{(F,G)}[\langle a\rangle_H] = [\langle -F(x, \cdot) + ka + F(x, \cdot)\rangle_{H'}]$.

We sum up the above formulae:

(2.1) THEOREM. *Consider the diagram of path-connected topological spaces*

$$\begin{array}{ccc} M & \xrightarrow{(f,g)} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M' & \xrightarrow{(f',g')} & N' \end{array}$$

Let $H \subset \Pi_1 N$, $H' \subset \Pi_1 N'$ be normal subgroups such that $k_\# H \subset H'$.

(a) *If our diagram is commutative then it determines a map $\varkappa\colon \nabla_H(f, g) \to \nabla_{H'}(f', g')$ given by*

$$\varkappa\colon \nabla_H(f, g\colon x, r) \to \nabla_{H'}(f', g'\colon hx, kr), \qquad \varkappa[\langle a\rangle_H] = [\langle ka\rangle_{H'}],$$

where (x, r) is a reference pair for (f, g).

(b) *If the diagram is only homotopy commutative (with some homotopies $(F, G)\colon (kf, kg) \sim (f' h, g' h)$) but $k_\# H = H'$ and k, h are homeomorphisms then it determines a bijective transformation $\eta\colon \nabla_H(f, g) \to \nabla_{H'}(f', g')$ which may be represented by*

$$\eta\colon \nabla_H(f, g\colon r) \to \nabla_{H'}(f', g'\colon hx, -F(x, \cdot) + kr + G(x, \cdot)),$$

$$\eta[\langle a\rangle_H] = [\langle -F(x, \cdot) + ka + F(x, \cdot)\rangle_{H'}].$$

(c) *If the assumptions of* (a) *and* (b) *hold and the considered homotopies are constant then $\varkappa = \eta$.*

(d) *Let the diagram be homotopy commutative and let $k_\# H = H'$. Assume that all spaces involved are compact closed connected oriented manifolds of the same dimension and let h, k be orientation-preserving homeomorphisms. Then the transformation η preserves index, i.e. if $A \in \nabla_H(f, g)$ then $\operatorname{ind}(f, g\colon A) = \operatorname{ind}(f', g'\colon \eta A)$.*

Proof. (d) follows from (1.7).

The transformation $\eta_{(F,G)}$ depends on the choice of the homotopies (F, G). Nevertheless the following lemma shows that it does not distinguish between homotopic homotopies.

(2.2) LEMMA. *Let $h_t\colon X\to X'$, $k_t\colon Y\to Y'$ denote two continuous families of maps and assume that the diagrams*

$$\begin{array}{ccc} X & \xrightarrow{(f,g)} & Y \\ \downarrow{\scriptstyle h_t} & & \downarrow{\scriptstyle k_t} \\ X' & \xrightarrow{(f',g')} & Y' \end{array}$$

are homotopy commutative ($t\in[0,1]$), *i.e. there exist maps $F,G\colon X\times I\times I\to Y'$ such that $F(x,0,t)=k_tfx$, $G(x,0,t)=k_tgx$, $F(x,1,t)=f'h_tx$, $G(x,1,t)=g'h_tx$.*

Then

$$\eta_{(F_0,G_0)}=\eta_{(F_1,G_1)}$$

there $F_t(x,s)=F(x,s,t)$, $G_t(x,s)=G(x,s,t)$).

Proof. We have to show the commutativity of the diagram

$$\begin{array}{ccc} \nabla_H(f,g\colon x,r) & \xrightarrow{\eta(F_0,G_0)} & \nabla_{H'}(f',g'\colon h_0x,\,-F_0(x,\cdot)+k_0r+G_0(x,\cdot)) \\ & {\scriptstyle \eta(F_1,G_1)}\searrow \quad \swarrow{\scriptstyle v} & \\ & \nabla_{H'}(f',g'\colon h_1x,\,-F_1(x,\cdot)+k_1r+G_1(x,\cdot)) & \end{array}$$

To define v we will use the path $h_{(\,)}X$ from h_0x to h_1x given by the formula $t\to h_tx$ (see (1.3b)). Then

$$\eta_{(F_1,G_1)}[\langle a\rangle_H]=v[\langle -F_1(x,\cdot)+k_1a+F_1(x,\cdot)\rangle_{H'}].$$

On the other hand,

$$\begin{aligned} v\eta_{(F_0,G_0)}[\langle a\rangle_H] &= v[\langle -F_0(x,\cdot)+k_0a+F_0(x,\cdot)\rangle_{H'}] \\ &= [\langle -f'h_{(\cdot)}x-F_0(x,\cdot)+k_0a+F_0(x,\cdot) \\ &\quad -F_0(x,\cdot)+k_0r+G_0(x,\cdot)+g'h_{(\cdot)}x-G_1(x,\cdot)-k_1r+F_1(x,\cdot)\rangle_H] \\ &= [\langle \underbrace{-F(x,1,\cdot)-F(x,\cdot,0)}_{(1)}+k_0a+k_0r \\ &\quad +\underbrace{G(x,\cdot,0)+G(x,1,\cdot)-G(x,\cdot,1)}_{(2)}-k_1r+F(x,\cdot,1)\rangle_{H'}] \\ &= [\langle \underbrace{-F(x,\cdot,1)-F(x,0,\cdot)}_{(1)}\underbrace{+k_0a+F(x,0,\cdot)}_{(3)} \\ &\quad \underbrace{-F(x,0,\cdot)+k_0r+\underbrace{G(x,0,\cdot)}_{(2)}-k_1r}_{(4)}+F(x,\cdot,1)\rangle_{H'}] \\ &= [\langle -F(x,\cdot,1)+\underbrace{k_1a}_{(3)}+F(x,\cdot,1)\rangle_{H'}] \\ &= [\langle -F_1(x,\cdot)+k_1a+F_1(x,\cdot)\rangle_{H'}]=\eta_{(F_1,G_1)}[\langle a\rangle_H] \end{aligned}$$

(we apply the homotopies $\Phi_1(s,t)=F(x,s,t)$, $\Phi_2(s,t)=G(x,s,t)$, $\Phi_3(s,t)=k_sa(t)$, $\Phi_4(s,t)=k_sr(t)$).

3. Reduction to $\Phi(f,g)$ finite. In this section we prove the following

(3.1) THEOREM. *Let M, N be compact connected closed topological manifolds of the same dimension. Let d denote a metric on N and let $\varepsilon>0$. Then for any $(f,g)\colon M\to N$ there exists $(f_\varepsilon,g_\varepsilon)$ such that $d(f,f_\varepsilon)<\varepsilon$, $d(g,g_\varepsilon)<\varepsilon$ and $\Phi(f_\varepsilon,g_\varepsilon)$ is finite* (here $d(f,f_\varepsilon)=\sup_{x\in M}d(fx,f_\varepsilon x)$).

For the fixed point case of this theorem see [22, 23, 6] and for the coincidences on triangulable manifolds see [5].

The proof of the theorem is preceded by two lemmas. We follow the method used in [6]. The first lemma is evident:

(3.2) LEMMA. *Let $f,g\colon \mathbf{R}^n\to\mathbf{R}^n$ be two continuous maps and let $\eta>0$. Then there exist $\bar f,\bar g\colon \mathbf{R}^n\to\mathbf{R}^n$ such that $d(f,\bar f)<\eta$, $d(g,\bar g)<\eta$ and $\Phi(\bar f,\bar g)$ is discrete.* ∎

The next lemma is an easy consequence of the Theorem (3.1), 103 of [1].

(3.3) LEMMA. *If Y is a compact ANR and $\varepsilon>0$ then there is a $\delta>0$ such that for every metric space X, its closed subspace A and maps $\varphi\colon X\to Y$, $\psi\colon A\to Y$ with $d(\varphi(x),\psi(x))<\delta$ there exists an extension $\bar\psi\colon X\to Y$ of ψ satisfying $d(\varphi,\bar\psi)<\varepsilon$.*

Proof of Theorem (3.1). Fix metrics on M and N. For any metric space (X,d) and a subset $A\subset X$ set $B(A,\eta)=\{x\in X\colon d(x,A)\leqslant\eta\}$,

$$U(A,\eta)=\{x\in X\colon d(x,A)<\eta\}.$$

For every $x\in\Phi(f,g)$ we find euclidean neigbourhoods $V,W\subset M$ and $U\subset N$ satisfying $x\in V\subset\bar V\subset W$, $fx=gx\in U$, $f(\bar W)\cup g(\bar W)\subset U$. Since $\Phi(f,g)$ is compact, $\Phi(f,g)\subset V_1\cup\ldots\cup V_r$. Let $Y=M-(V_1\cup\ldots\cup V_r)$. There is no coincidence point of (f,g) in Y. We will construct inductively two sequences of maps

$$f=f_0,f_1,\ldots,f_r,\ \text{and}\ g=g_0,g_1,\ldots,g_r,$$

such that:

(i) $d(f_i,f_{i-1})<\varepsilon/r$, $d(g_i,g_{i-1})<\varepsilon/r$.

(ii) $f_ix=fx$, $g_ix=gx$ for $x\in Y$.

(iii) (f_i,g_i) has on $Y_i=Y\cup\bar V_1\cup\ldots\cup\bar V_i$ only a finite number of coincidence points, each of them isolated.

(iv) $f_i\bar W_j\cup g_i\bar W_j\subset U_j$, $i,j=1,\ldots,r$.

We then obtain the desired maps by putting $f_\varepsilon=f_r$, $g_\varepsilon=g_r$.

The maps $f_0=f$, $g_0=g$ are given. Suppose that (f_i,g_i) satisfies (i)–(iv). Then $\Phi(f_i,g_i)\cap Y_i$ is finite and all its points are isolated, hence there is no coincidence point of (f_i,g_i) on $B(Y_i,\varepsilon')-Y_i$ for some $\varepsilon'>0$. Choose $\eta<\varepsilon'/2$ such that $B(V_{i+1},\eta)\subset W_{i+1}$. Since W_{i+1}, U_{i+1} are homeomorphic to $\mathbf{R}^n$ and $f_i(W_{i+1})\subset U_{i+1}$, $g_i(W_{i+1})\subset U_{i+1}$, by (3.2) the restrictions $f_{i|W_{i+1}}$, $g_{i|W_{i+1}}$ can be approximated by $\bar f_i,\bar g_i\colon W_{i+1}\to U_{i+1}$ such that $\Phi(\bar f_i,\bar g_i)$ is discrete in W_{i+1}, hence $\Phi(\bar f_i,\bar g_i)\cap$

$\cap B(V_{i+1}, \eta)$ is finite. We define $Z = V_{i+1} - U(Y_i, \varepsilon')$. Then $Z \cap U(Y_i, \varepsilon') = \emptyset$ and $B(Z, \eta) \cap B(Y_i, \varepsilon' - 2\eta) = \emptyset$.

Now we apply (3.3) to $X = M$, $Y = N$, $A = B(Z, \eta) \cup B(Y_i, \varepsilon' - 2\eta)$, $\bar{\varepsilon} < \min(\varepsilon/r, \beta, \alpha_1, \ldots, \alpha_r)$, $\varphi = f_i$ and

$$\psi x = \begin{cases} \bar{f}_i x & \text{for } x \in B(Z, \eta), \\ f_i x & \text{for } x \in B(Y_i, \varepsilon' - 2\eta), \end{cases}$$

where $\alpha_j = \operatorname{dist}(f_i \overline{W}_j \cup g_i \overline{W}_j, N - U_j)$,

$$\beta = \tfrac{1}{2}\min\{d(f_i x, g_i x) \colon x \in B(Y_i, \varepsilon') - U(Y_i, \varepsilon' - 2\eta)\}.$$

By (3.2) we can require $d(f_i x, \bar{f}_i x) < \delta$ for all $x \in W_{i+1}$, hence for $x \in B(Z, \eta) \subset B(V_{i+1}, \eta) \subset W_{i+1}$. Thus the assumptions of (3.3) are satisfied and we get f_{i+1}, an extension of ψ satisfying $d(f_i, f_{i+1}) < \bar{\varepsilon}$.

In the same way we can find $g_{i+1} \colon M \to N$ such that

$$g_{i+1} x = \begin{cases} g_i x & \text{for } x \in B(Y_i, \varepsilon - 2\eta), \\ \bar{g}_i x & \text{for } x \in B(Z, \eta), \end{cases}$$

and $d(g_i, g_{i+1}) < \bar{\varepsilon}$.

Now (f_{i+1}, g_{i+1}) satisfies (i) since $\bar{\varepsilon} < \varepsilon/r$, (ii) since $f_{i+1} x = \psi x = f_i x$ for $x \in Y \subset Y_i$, and (iv) since $\bar{\varepsilon} < \alpha_j$ $(j = 1, \ldots, r)$.

As to (iii) we notice that:

(a) $\Phi(f_{i+1}, g_{i+1}) \cap B(Z, \eta)$ is finite so all coincidence points from Z are isolated.

(b) (f_{i+1}, g_{i+1}) coincides with (f_i, g_i) on $B(Y_i, \varepsilon' - 2\eta)$ and the latter has only a finite number of coincidence points, each of them in Y_i.

(c) If $x \in Y_{i+1}$ and x lies neither in $B(Y_i, \varepsilon' - 2\eta)$ nor in Z then it belongs to $B(Y_i, \varepsilon') - U(Y_i, \varepsilon' - 2\eta)$. Now, $\bar{\varepsilon} < \beta$ implies $f_{i+1} x \neq g_{i+1} x$.

4. The transformation $T_{\bar{u}}$. Let $p \colon E \to B$, $p' \colon E' \to B'$ be locally trivial bundles such that the total spaces, base spaces and fibres are path-connected. We assume that B and B' are paracompact. Then the above bundles are Hurewicz fibrations; denote by λ, λ' their lifting functions [20]. Let $E_b = p^{-1} b$ be the fibre over $b \in B$. For a path $\bar{u}$ in B we denote by $\tau_{\bar{u}} \colon E_{\bar{u}(0)} \to E_{\bar{u}(1)}$ the map given by the formula

$$\tau_{\bar{u}} x = \lambda(\bar{u}, x)\ (1).$$

Since the considered bundles are locally trivial we may assume that the maps $\tau_{\bar{u}} \colon E_{\bar{u}(0)} \to E_{\bar{u}(1)}$ (and the similar maps, for the second bundle) are homeomorphisms. For a path u we denote by u_r^s the path given by the formula $u_r^s(t) = u(r(1-t) + st)$.

Suppose we are given two fibre maps, i.e. a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{(f, g)} & E' \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{(\bar{f}, \bar{g})} & B' \end{array}$$

In this section we show how a path joining points from one Nielsen class of $\Phi(\bar{f}, \bar{g})$ induces a map between the Nielsen classes on the corresponding fibres.

For $b \in B$ we denote by $f_b \colon E_b \to E'_{\bar{f}b}$, $g_b \colon E_b \to E'_{\bar{g}b}$ the restrictions of f and g. For a fixed $b' \in B'$ we denote by K the normal subgroup of $\Pi_1 E'_{b'}$ given by the formula

$$K(x') = \ker(\Pi_1(E'_{b'}, x') \to \Pi_1(E', x')).$$

Let $b \in \Phi(\bar{f}, \bar{g})$. Then $(f_b, g_b) \colon E_b \to E'_{\bar{f}b = \bar{g}b}$, so we get the set $\nabla_K(f_b, g_b)$. Let $b_0, b_1 \in \Phi(\bar{f}, \bar{g})$ be Nielsen equivalent and let $\bar{u}$ be a path joining them such that $\bar{f}\bar{u} \sim \bar{g}\bar{u}$. We are going to define a bijective transformation between the sets $\nabla_K(f_{b_0}, g_{b_0})$ and $\nabla_K(f_{b_1}, g_{b_1})$ induced by $\bar{u}$.

Consider the diagram

$$\text{(4.1)} \qquad \begin{array}{ccc} E_{b_0} & \xrightarrow{(f_{b_0}, g_{b_0})} & E'_{\bar{f}b_0} \\ \big\downarrow{\scriptstyle \tau_{\bar{u}}} & & \big\downarrow{\scriptstyle \tau'_{\bar{f}\bar{u}}} \\ E_{b_1} & \xrightarrow{(f_{b_1}, g_{b_1})} & E'_{\bar{f}b_1} \end{array}$$

The diagrams

$$\begin{array}{ccc} E_{b_0} & \xrightarrow{f_{b_0}} & E'_{\bar{f}b_0} \\ \big\downarrow{\scriptstyle \tau_{\bar{u}}} & & \big\downarrow{\scriptstyle \tau'_{\bar{f}\bar{u}}} \\ E_{b_1} & \xrightarrow{f_{b_1}} & E'_{\bar{f}b_1} \end{array} \qquad \begin{array}{ccc} E_{b_0} & \xrightarrow{g_{b_0}} & E'_{\bar{g}b_0} \\ \big\downarrow{\scriptstyle \tau_{\bar{u}}} & & \big\downarrow{\scriptstyle \tau'_{\bar{g}\bar{u}}} \\ E_{b_1} & \xrightarrow{g_{b_1}} & E'_{\bar{g}n_1} \end{array}$$

are homotopy commutative by means of

$$F(x, t) = \tau'_{\bar{f}\bar{u}_t^1} f_{\bar{u}(t)} \tau_{\bar{u}_0^t}(x) \quad \text{and} \quad G_1(x, t) = \tau'_{\bar{g}\bar{u}_t^1} g_{\bar{u}(t)} \tau_{\bar{u}_0^t}(x)$$

respectively.

Let D_t be a homotopy between the paths $\bar{f}\bar{u}$ and $\bar{g}\bar{u}$. We define $G_2 \colon E_{b_0} \times I \to E_{\bar{f}b_1}$ by $G_2(x, t) = \tau'_{D_t} g_{b_0}(x)$. One can check that the diagram (4.1) is homotopy commutative with homotopies F and $G = G_2 * G_1$.

Now we may apply Definition (2.1 b) to (4.1) and we denote the obtained map by

$$T_{\bar{u}} \colon \nabla_K(f_{b_0}, g_{b_0}) \to \nabla_K(f_{b_1}, g_{b_1}).$$

The formula from (2.1) representing this map takes the form

$$\text{(4.2)} \quad T_{\bar{u}} \colon \nabla_K(f_{b_0}, g_{b_0} \colon x, r) \to \nabla_K(f_{b_1}, g_{b_1} \colon \tau_{\bar{u}}(x), (*)),$$
$$(*) = \left(-\tau'_{\bar{f}\bar{u}^1_{(\cdot)}} f_{\bar{u}(\cdot)} \tau_{\bar{u}_0^{(\cdot)}}(x) + \tau'_{\bar{f}\bar{u}} r + \tau'_{D_{(\cdot)}} g_{b_0}(x) + \tau'_{\bar{g}\bar{u}^1_{(\cdot)}} g_{\bar{u}(\cdot)} \tau_{\bar{u}_0^{(\cdot)}}(x)\right),$$

$$\text{(4.3)} \quad T_{\bar{u}}[\langle a \rangle_K][\langle (**) \rangle_K] = [\langle -\tau'_{\bar{f}\bar{u}^1_{(\cdot)}} f_{\bar{u}(\cdot)} \tau_{\bar{u}_0^{(\cdot)}}(x) + \tau'_{\bar{f}\bar{u}} a + \tau'_{\bar{f}\bar{u}^1_{(\cdot)}} f_{\bar{u}(\cdot)} \tau_{\bar{u}_0^{(\cdot)}}(x) \rangle_K].$$

Now we are going to simplify the above formulae. Recall first

(4.4) Lemma [24]. *Let v be a path in E such that $pv(0) = pv(1) = b$ and let pv be contractible in B. Then there is a path l in E_b from $v(0)$ to $v(1)$ such that l is homo-*

topic to v in E. For any other path l' in E_b the conditions $l' \sim v$ in E and $l' \overset{K}{\sim} l$ are equivalent.

In particular, any loop u in E based at $x \in E_b$ such that pu is contractible determines an element of the set $\Pi_1(E_b, x)/K(x)$.

Now we show that the path $(*)$ from (4.2) is homotopic in E' to $-fu+r+gu$ where $u = \lambda(\bar{u}, x)$.

The homotopies

$$H_1(t, s) = \lambda'(f\bar{u}_t^1, f_{\bar{u}(t)}\tau_{\bar{u}_0^t}(x))(s),$$
$$H_2(t, s) = \lambda'(f\bar{u}, r(t))(s),$$
$$H_3(t, s) = \lambda'(D_t, g_{b_0}(x))(s),$$
$$H_4(t, s) = \lambda'(g\bar{u}_t^1, g_{\bar{u}(t)}\tau_{\bar{u}_0^t}(x))(s)$$

may be joined to get

$$H(t, s) = \begin{cases} H_1(1-4t, s), & 0 \leqslant t \leqslant 1/4, \\ H_2(4t-1, s), & 1/4 \leqslant t \leqslant 2/4, \\ H_3(4t-2, s), & 2/4 \leqslant t \leqslant 3/4, \\ H_4(4t-3, s), & 3/4 \leqslant t \leqslant 1. \end{cases}$$

Then $H(0, s) = H_1(1, s) = fu(1)$, $H(1, s) = H_4(1, s) = gu(1)$. Moreover, $H_1(t, 0) = f\tau_{\bar{u}_0^t}(x)$ and since the paths $u(s) = \lambda(\bar{u}_0^1, x)(s)$ and $\tau_{\bar{u}_0^t}(x) = \lambda(\bar{u}_0^t, x)(1)$ are homotopic via $\tilde{H}(t, s) = \lambda(\bar{u}_0^t, x)(s)$, we get

$$H_1(\cdot, 0) \sim fu, \quad H_2(t, 0) = r(t), \quad H_3(t, 0) = g(x) \text{ and}$$
$$H_4(t, 0) = g\tau_{\bar{u}_0^t}(x), \text{ so as above } H_4(\cdot, 0) \sim gu.$$

Thus $H(\cdot, 0) \sim -fu+r+gu$.

On the other hand,

$$H_1(t, 1) = \tau'_{f\bar{u}_t^1} f_{\bar{u}(t)}\tau_{\bar{u}_0^t}(x), \quad H_2(t, 1) = \tau'_{f\bar{u}} r(t),$$
$$H_3(t, 1) = \tau'_{D_t} g_b(x), \quad H_4(t, 1) = \tau'_{g\bar{u}_t^1} g_{\bar{u}(t)}\tau_{\bar{u}_0^t}(x),$$

so the path $(*)$ from (4.2) equals $H(\cdot, 1)$ which homotopic in E' to $H(\cdot, 0)$, hence to $-fu+r+gu$.

A similar calculation shows that the path from (4.3) is homotopic in E' to $-fu+a+fu$.

Thus the map $T_{\bar{u}}\colon \nabla_K(f_{b_0}, g_{b_0}\colon x, r) \to \nabla_K(f_{b_1}, g_{b_1}\colon x', r')$ is given by

$$T_{\bar{u}}[\langle a\rangle_K] = [\langle a'\rangle_K]$$

where $x' = u(1)$ and r', a' are paths in $E'_{f\bar{b}_1}$ homotopic in E' to $-fu+r+gu$ and $-fu+a+fu$ respectively.

(4.5) **Remark.** It follows from the above that $T_{\bar{u}}$ does not depend on the choice of the homotopy D_t used in its construction. Nor does it depend on the lifting functions λ, λ'. For suppose that λ_1, λ'_1 is another pair of lifting functions. Now λ is homotopic to λ_1 and λ' is homotopic to λ'_1, so there exist two families of lifting functions between them. They give rise to a continuous family of diagrams of the form (4.1). We apply Lemma (2.2) to them.

(4.6) Definition. We denote by $p_\nabla\colon \nabla(f, g) \to \nabla(\bar{f}, \bar{g})$ the map induced by the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{(f,g)} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow{(\bar{f},\bar{g})} & B' \end{array}$$

with constant homotopies (see (2.1 a)). Let $b \in \Phi(\bar{f}, \bar{g})$. Then $(i_b)_\nabla\colon \nabla_K(f_b, g_b) \to \nabla(f, g)$ will stand for the map induced by the commutative diagram

$$\begin{array}{ccc} E_b & \xrightarrow{(f_b,g_b)} & E'_{\bar{f}b} \\ \cap\downarrow & & \cap\downarrow \\ E & \xrightarrow{(f,g)} & E' \end{array}$$

with constant homotopies.

(4.7) Lemma. *Let $b_0, b_1 \in \Phi(\bar{f}, \bar{g})$ and let $\bar{u}$ be a path joining them such that $\bar{f}\bar{u} \sim \bar{g}\bar{u}$. Then the diagram*

$$\begin{array}{ccc} \nabla_K(f_{b_0}, g_{b_0}) & \xrightarrow{T_{\bar{u}}} & \nabla_K(f_{b_1}, g_{b_1}) \\ & {\scriptstyle (i_{b_0})_\nabla}\searrow \quad \swarrow{\scriptstyle (i_{b_1})_\nabla} & \\ & \nabla(f, g) & \end{array}$$

is commutative.

Proof. Choose a reference pair (x_0, r_0) for (f_{b_0}, g_{b_0}). Then

$$(i_{b_0})_\nabla[\langle a\rangle_K] = [\langle a\rangle] \in \nabla(f, g\colon x_0, r_0),$$
$$(i_{b_1})_\nabla T_{\bar{u}}[\langle a\rangle_K] = [\langle a'\rangle] \in \nabla(f, g\colon x_1, r_1),$$

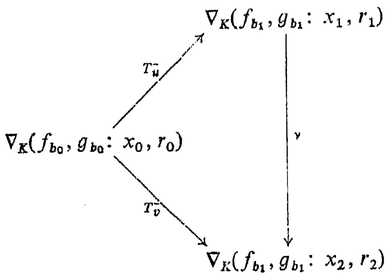
(where $x_1 = u(1)$, $r_1 = -fu+r_0+gu$, $a' = -fu+a+fu$ and $u = \lambda(\bar{u}, x_0)$).

But the above elements represent the same element in $\nabla(f, g)$ since

$$v\colon \nabla(f, g\colon x_0, r_0) \to \nabla(f, g\colon x_1, r_1),$$
$$v[\langle a\rangle] = [\langle -fu+a+r_0+gu-r_1\rangle] = [\langle -fu+a+r_0+gu-gu-r_0+fu\rangle]$$
$$= [\langle -fu+a+fu\rangle] = [\langle a'\rangle]. \quad ■$$

(4.8) LEMMA. *If the paths $\bar{u}$, $\bar{v}$ joining the points $b_0, b_1 \in \Phi(\bar{f}, \bar{g})$ are homotopic and so are $\bar{f}\bar{u}$ and $\bar{g}\bar{u}$ then $T_{\bar{u}} = T_{\bar{v}}$.*

Proof. Let (x_0, r_0) be a reference pair for (f_{b_0}, g_{b_0}), and let (x_1, r_1), (x_2, r_2) be reference pairs for (f_{b_1}, g_{b_1}) induced by (x_0, r_0) with $\bar{u}$ and by (x_0, r_0) with $\bar{v}$ respectively, i.e. if $u = \lambda(\bar{u}, x_0)$, $v = \lambda(\bar{v}, x_0)$ then $x_1 = u(1)$, $x_2 = v(1)$ and r_1, r_2 are paths in $E_{\bar{f}b_1}$ homotopic in E' to $-fu+r_0+gu$ and $-fv+r_0+gv$ respectively. We have to prove that the following diagram is commutative:



Since $p(-u+v)$ is contractible in B, there exists a path l in E_{b_1} from $u(1) = x_1$ to $v(1) = x_2$ homotopic in E to $-u+v$.

Now $T_{\bar{u}}[\langle a\rangle_K] = [\langle a'\rangle_K]$, $T_{\bar{v}}[\langle a\rangle_K] = [\langle a''\rangle_K]$ where a' is homotopic in E' to $-fu+a+fu$ and a'' is homotopic in E' to $-fv+a+fv$, so

$$vT_{\bar{u}}[\langle a\rangle_K] = v[\langle a'\rangle_K] = [\langle -fl+a'+r_1+gl-r_2\rangle_K].$$

But in E'

$$\begin{aligned}\langle -fl+a'+r_1+gl-r_2\rangle \\ &= \langle -(-fu+fv)+(-fu+a+fu)+(-fu+r_0+gu)+(-gu+gv) \\ &\quad -(-fv+r_0+gv)\rangle = \langle -fv+a+fv\rangle = \langle a''\rangle.\end{aligned}$$

Thus $vT_{\bar{u}}[\langle a\rangle_K] = [\langle a''\rangle_K] = T_{\bar{v}}[\langle a\rangle_K]$. ■

(4.9) LEMMA. *Let $b_0, b_1, b_2 \in \Phi(\bar{f}, \bar{g})$, let $\bar{u}$ and $\bar{v}$ be paths from b_0 to b_1 and from b_1 to b_2 respectively satisfying $\bar{f}\bar{u} \sim \bar{g}\bar{u}$ and $\bar{f}\bar{v} \sim \bar{g}\bar{v}$. Then $T_{\bar{u}+\bar{v}} = T_{\bar{v}}T_{\bar{u}}$.*

Proof. This may be proved by calculations similar to those in (4.8). Notice that the maps $\tau_{\bar{u}+\bar{v}}$ and $\tau_{\bar{v}}\tau_{\bar{u}}$ are homotopic. ■

(4.10) LEMMA. *Let $b_0, b_1 \in \Phi(\bar{f}, \bar{g})$ and let $\bar{u}$ be a path from b_0 to b_1 such that $\bar{f}\bar{u} \sim \bar{g}\bar{u}$. Let $A_i \in \Phi'(f_{b_i}, g_{b_i})$ $(i = 0, 1)$. Then $T_{\bar{u}}A_0 = A_1$ iff for any points $x_i \in A_i$ $(i = 0, 1)$ there is a path u_1 from x_0 to x_1 such that $pu_1 \sim \bar{u}$ in B and $fu_1 \sim gu_1$ in E'.*

Proof. Let $x_i \in A_i \in \nabla_K(f_{b_i}, g_{b_i})$ and let e_{fx_i} denote the constant path at fx_i $(i = 0, 1)$. Let $u = \lambda(\bar{u}, x_0)$, $x_1' = u(1)$. Then

$$T_{\bar{u}}A_0 = [\langle e_{fx_1'}\rangle_K] \in \nabla(f_{b_1}, g_{b_1}: x_1', r_1)$$

where r_1 denotes a path in the fibre homotopic to $-fu+gu$ in E'.

$\Rightarrow$ Let $T_{\bar{u}}A_0 = A_1$. Then

$$[\langle e_{fx_1}\rangle_K] \in \nabla_K(f_{b_1}, g_{b_1}: x_1, e_{fx_1}) \quad \text{and} \quad [\langle e_{fx_1'}\rangle_K] \in \nabla_K(f_{b_1}, g_{b_1}: x_1', r_1)$$

represent the same element in $\nabla_K(f_{b_1}, g_{b_1})$, so for each path w in E_{b_1} from x_1 to x_1' we have

$$v[\langle e_{fx_1}\rangle_K] = [\langle -fw+gw-r_1\rangle_K] = [\langle e_{fx_1'}\rangle_K] \in \nabla_K(f_{b_1}, g_{b_1}: x_1', r_1).$$

Hence for a loop d in E_{b_1} based at x_1'

$$\langle -fw+gw-r_1\rangle_K = \langle fd+r_1-gd-r_1\rangle_K.$$

Now (4.4) implies that the above paths are homotopic in E', so $-fw+gw-r_1 \sim fd+r_1-gd-r_1 \sim fd-fu+gu-gd-r_1$. Therefore $f(u-d-w) \sim g(u-d-w)$ and we put $u_1 = u-d-w$.

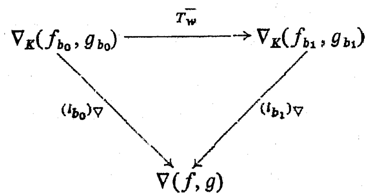
$\Leftarrow$ Suppose that u_1 exists. Then $-u_1+u$ is a path from x_1 to x_1' such that $p(-u_1+u)$ is contractible. Hence there exists a path v in the fibre E_{b_1} joining these points which is homotopic to $-u_1+u$. Then

$$v: \nabla_K(f_{b_1}, g_{b_1}: x_1, e_{fx_1}) \to \nabla_K(f_{b_1}, g_{b_1}: x_1', r_1),$$
$$v[\langle e_{fx_1}\rangle_K] = [\langle -fv+gv-r_1\rangle_K].$$

On the other hand, in E' $-fv+gv-r_1 \sim -fu+fu_1-gu_1+gu-gu+fu \sim e_{fx_1}$, so $v[\langle e_{fx_1}\rangle_K] = [\langle e_{fx'_1}\rangle_K]$. ■

(4.11) LEMMA. *Let $b_0, b_1 \in \Phi(\bar{f}, \bar{g})$ and let $A_i \in \nabla_K(f_{b_i}, g_{b_i})$ $(i = 0, 1)$. Then $(i_{b_0})_\nabla A_0 = (i_{b_1})_\nabla A_1$ iff there exists a path $\bar{w}$ from b_0 to b_1 such that $\bar{f}\bar{w} \sim \bar{g}\bar{w}$ in B' and $T_{\bar{w}}A_0 = A_1$.*

Proof. $\Leftarrow$ follows from the commutativity of the diagram (see (4.7))



Let now $[\langle a_i\rangle_K] \in \nabla_K(f_{b_i}, g_{b_i}: x_i, r_i)$ represent A_i $(i = 0, 1)$ and suppose that $(i_{b_0})_\nabla A_0 = (i_{b_1})_\nabla A_1$. If now v' is a path in E from x_0 to x_1 then

$$[\langle a_1\rangle] = [\langle -fv'+a_0+r_0+gv'-r_1\rangle] \in \nabla(f_{b_1}, g_{b_1}: x_1, r_1),$$

so for a loop d based at x_1

$$\begin{aligned}\langle a_1\rangle &= \langle fd\rangle + \langle -fv'+a_0+r_0+gv'-r_1\rangle + \langle r_1+gd-r_1\rangle \\ &= \langle f(d-v')+a_0+r_0-g(d-v')-r_1\rangle.\end{aligned}$$

We put $v = v'-d$, $\bar{w} = pv$. Then the above equality implies $\bar{f}\bar{w} \sim \bar{g}\bar{w}$.

It remains to show that $T_{\bar w}A_0 = A_1$. Let $w = \lambda(\bar w, x_0)$. Then

$$T_{\bar w}\colon \nabla_K(f_{b_0}, g_{b_0}\colon x_0, r_0) \to \nabla_K(f_{b_1}, g_{b_1}\colon x_1', r_1'),$$
$$T_{\bar w}[\langle a_0\rangle_K] = [\langle a_1'\rangle_K],$$

where $x_1' = w(1)$ and $r_1' \sim -fw+r_0+gw$, $a_1' \sim -fw+a_0+fw$ in E'. Since $p(-w+v)$ is contractible in B, there exists a path l in E_{b_1} from x_1' to x_1 homotopic to $-w+v$. Then $T_{\bar w}A_0 = [\langle -fl+a_1'+r_1'+gl-r_1\rangle_K] \in \nabla_K(f_{b_1}, g_{b_1}\colon x_1, r_1)$. But in E'

$$-fl+a_1'+r_1'+gl-r_1 \sim -f(-w+v)-fw+a_0+fw-fw+r_0+gw+g(-w+v)-r_1$$
$$\sim -fv+a_0+r_0+gv-r_1 \sim a_1.$$

Thus $T_{\bar w}A_0 = [\langle a_1\rangle_K] \in \nabla_K(f_{b_1}, g_{b_1}\colon x_1, r_1)$, proving that $T_w A_0 = A_1$. ∎

(4.12) DEFINITION. Let $f, g\colon X \to Y$ be continuous maps and let $x \in \Phi(f, g)$. Then we put

$$C(f_\#, g_\#)_x = \{\alpha \in \Pi_1(X, x)\colon f_\#\alpha = g_\#\alpha \in \Pi_1(Y, fx)\}.$$

This is a subgroup of $\Pi_1(X, x)$.

Lemma (4.9) says that for a fixed $b \in \Phi(\bar f, \bar g)$ the group $C(\bar f_\#, \bar g_\#)_b$ acts on $\nabla_K(f_b, g_b)$ from the right.

It follows from (4.11) that if $A \in \nabla_K(f_b, g_b)$ then its orbit (under the above action) equals $(i_b)_\nabla^{-1}((i_b)_\nabla A)$.

(4.13) THEOREM. *Suppose we are given a continuous family of commutative diagrams*

$$\begin{array}{ccc} E & \xrightarrow{(f_t, g_t)} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{(\bar f_t, \bar g_t)} & B' \end{array} \qquad (t \in [0, 1]).$$

Set $F(x, t) = f_t(x)$, $G(x, t) = g_t(x)$, $\bar F(x, t) = \bar f_t(x)$, $\bar G(x, t) = \bar g_t(x)$. *Let* $b_i \in \Phi(\bar f_i, \bar g_i)$ $(i = 0, 1)$ *and let $\bar u$ be a path from b_0 to b_1 such that* $\Delta(\bar F, \bar u) \sim \Delta(\bar G, \bar u)$.

(a) *The formula*

$$T_{\bar u}\colon \nabla_K(f_{0b_0}, g_{0b_0}\colon x_0, r_0) \to \nabla_K(f_{1b_1}, g_{1b_1}\colon x_1, r_1),$$
$$T_{\bar u}[\langle a_0\rangle_K] = [\langle a_1\rangle_K],$$

where (x_0, r_0) is an arbitrary reference pair for (f_{0b_0}, g_{0b_0}), $u = \lambda(\bar u, x_0)$, $x_1 = u(1)$, r_1 is a path in $E'_{\bar f b_1}$ homotopic in E' to $-\Delta(F, u)+r_0+\Delta(G, u)$, a_1 is a path in $E'_{\bar f b_1}$ homotopic in E' to $-\Delta(F, u)+a_0+\Delta(F, u)$, gives a well-defined bijective map

$$T_{\bar u}\colon \nabla_K(f_{0b_0}, g_{0b_0}) \to \nabla_K(f_{1b_1}, g_{1b_1}).$$

(b) *The following diagram is commutative*:

$$\begin{array}{ccc} \nabla_K(f_{0b_0}, g_{0b_0}) & \xrightarrow{T_{\bar u}} & \nabla_K(f_{1b_1}, g_{1b_1}) \\ {\scriptstyle (i_{b_0})_\nabla}\downarrow & & \downarrow{\scriptstyle (i_{b_1})_\nabla} \\ \nabla(f_0, g_0) & \xrightarrow{\mu(\bar F, \bar G)} & \nabla(f_1, g_1) \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccc} E\times I & \xrightarrow{(F, G)} & E' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle p'} \\ B\times I & \xrightarrow{(\bar F, \bar G)} & B' \end{array}$$

and the path $\bar u_1(t) = (\bar u(t), t)$ (here $P(e, t) = (pe, t)$).

5. The index-product formula. In the next two sections we will consider the commutative diagram

$$(5.0)\qquad \begin{array}{ccc} E & \xrightarrow{(f, g)} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{(\bar f, \bar g)} & B' \end{array}$$

where $p\colon E \to B$, $p\colon E' \to B'$ are locally trivial fibre bundles whose total spaces, base spaces and fibres are compact connected closed oriented topological manifolds Moreover, $\dim E = \dim E'$ $\dim B = \dim B'$ (hence the dimensions of all fibres are equal) and the orientations of the base spaces, total spaces and fibres are compatible. We consider the coincidence indices of (f, g), $(\bar f, \bar g)$, (f_b, g_b) with respect to these orientations.

In this section we prove that a Nielsen class $A \in \Phi'(f, g)$ is essential iff $p_\nabla A \in \Phi'(\bar f, \bar g)$ is essential and $(i_b)_\nabla^{-1}A$ is the sum of essential classes in $\nabla_K(f_b, g_b)$.

(5.1) LEMMA. *The transformation $T_{\bar u}$ (from the diagram* (5.0)*) is an index-preserving bijection.*

Proof. The maps $\tau_{\bar u}\colon E_{b_0} \to E_{b_1}$, $\tau'_{\bar f\bar u}\colon E'_{\bar f b_0} \to E'_{\bar f b_1}$ are homeomorphisms preserving orientations. We apply Theorem (2.1). ∎

(5.2) LEMMA. *Let* $A \in \Phi'(f, g)$, $p_\nabla A \in \Phi'(\bar f, \bar g)$, $b_0, b_1 \in p_\nabla A$ *and let* $A_0 \in \nabla_K(f_{b_0}, g_{b_0})$, $A_1 \in \nabla_K(f_{b_1}, g_{b_1})$ *satisfy* $(i_{b_0})_\nabla A_0 = (i_{b_1})_\nabla A_1 = A$. *Then*

$$\operatorname{ind}(f_{b_0}, g_{b_0}\colon A_0) = \operatorname{ind}(f_{b_1}, g_{b_1}\colon A_1) \quad \textit{and} \quad \#(i_{b_0})_\nabla^{-1}A = \#(i_{b_1})_\nabla^{-1}A$$

($\# X$ here denotes the cardinality of the set X).

Proof. If $(i_{b_0})_\nabla A_0 = (i_{b_1})_\nabla A_1$ then, by (4.11), $A_1 = T_{\bar u}A_0$ for a path $\bar u$ from b_0 to b_1 and $T_{\bar u}$ preserves index. On the other hand, $T_{\bar u}$ is bijective, so by (4.7)

$$T_{\bar u}((i_{b_0})_\nabla^{-1}A) = (i_{b_1})_\nabla^{-1}A. \ ∎$$

(5.3) DEFINITION. Let $A \in \Phi'(f, g)$. We define the numbers

$$j(A) = \operatorname{ind}(f_b, g_b\colon A_0), \quad k(A) = \#(i_b)_\nabla^{-1}A$$

(where $b \in p_\nabla A$, $A_0 \in (i_b)_\nabla^{-1}A$). It follows from Lemma (5.2) that the definition is correct.

(5.4) Remark. If $A \in \Phi'(f, g)$ and $j(A) \neq 0$ then

(a) $k(A)$ is a natural number,

(b) $p_\nabla A = pA$ as sets.

Proof. (a) Since $A\in\Phi'(f,g)$, $pA\neq\emptyset$; choose $b\in pA$. Then $p^{-1}b\cap A\neq\emptyset$, hence $k(A)>0$. On the other hand, the indices of all classes in $(i_{b_0})_{\nabla}^{-1}(A)$ are equal and nonzero, so the compactness of the fibre implies that the number of classes must be finite.

(b) It is obvious that $pA\subseteq p_{\nabla}A$. Let now $b_1\in p_{\nabla}A$. Fix $b_0\in pA$. Then b_0 and b_1 are Nielsen equivalent, so there is a path $\bar u$ from b_0 to b_1 such that $\bar f\bar u\sim\bar g\bar u$.

Then $T_{\bar u}\colon (i_{b_0})_{\nabla}^{-1}A\to(i_{b_1})_{\nabla}^{-1}A$ is a bijective index-preserving map. In this way we get, in E_{b_1}, $k(A)$ essential classes contained in A. Thus $b_1\in pA$. ■

Now we are in a position to formulate the main result of this section.

(5.5) Theorem. *Let $A\in\Phi'(f,g)$. Then*

$$\operatorname{ind}(f,g\colon A)=k(A)\operatorname{ind}(\bar f,\bar g\colon p_{\nabla}(A))=\operatorname{ind}(f_b,g_b\colon A\cap E_b)\operatorname{ind}(\bar f,\bar g\colon p_{\nabla}A)$$

(for any $b\in p_{\nabla}A$).

The proof of the theorem will be preceded by two lemmas:

(5.6) Lemma. *Suppose we are given a fibre homotopy $(F,G)=(f_t,g_t)$, i.e. a continuous family of commutative diagrams*

$$\begin{array}{ccc} E & \xrightarrow{(f_t,g_t)} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{(\bar f_t,\bar g_t)} & B' \end{array}\qquad (t\in[0,1])$$

Let $A_0\in\Phi'(f_0,g_0)$ and define $A_1=\mu_{(F,G)}A_0\in\nabla(f_1,g_1)$. Then $\operatorname{ind}(f_0,g_0\colon A_0)=\operatorname{ind}(f_1,g_1\colon A_1)$,

$$\operatorname{ind}(\bar f_0,\bar g_0\colon p_{\nabla}A_0)=\operatorname{ind}(\bar f_1,\bar g_1\colon p_{\nabla}A_1).$$

If moreover $p_{\nabla}(A_1)\in\Phi'(\bar f_1,\bar g_1)$ then

$$j(A_0)=j(A_1),\qquad k(A_0)=k(A_1).$$

Proof. The first two equalities follow from (1.7) and from the commutativity of the diagram

$$\begin{array}{ccc} \nabla(f_0,g_0) & \xrightarrow{\mu(F,G)} & \nabla(f_1,g_1) \\ {\scriptstyle p_{\nabla}}\downarrow & & \downarrow{\scriptstyle p_{\nabla}} \\ \nabla(\bar f_0,\bar g_0) & \xrightarrow{\mu(\bar F,\bar G)} & \nabla(\bar f_1,\bar g_1) \end{array}$$

Now assume that $p_{\nabla}(A_1)\in\Phi'(\bar f_1,\bar g_1)$. Fix $b_0\in p_{\nabla}A_0$, $b_1\in p_{\nabla}A_1$. Then (1.5d) gives a path $\bar u$ from b_0 to b_1 such that $\Delta(\bar F,\bar u)\sim\Delta(\bar G,\bar u)$. Now (4.13) yields a bijective map $T_{\bar u}$ such that $T_{\bar u}(i_{b_0})_{\nabla}^{-1}A_0=(i_{b_1})_{\nabla}^{-1}A_1$, thus $k(A_0)=k(A_1)$. On the other hand, $T_{\bar u}$ preserves index, so $j(A_0)=j(A_1)$. ■

(5.7) Lemma. *Let $b_0\in\Phi(\bar f,\bar g)$ be an isolated coincidence point and let A be closed and open in $\Phi(f_{b_0},g_{b_0})$. Then*

$$\operatorname{ind}(f,g\colon A)=\operatorname{ind}(\bar f,\bar g\colon b_0)\operatorname{ind}(f_{b_0},g_{b_0}\colon pA).$$

Proof. Let m, n denote the dimensions of the base spaces and fibres respectively. Fix trivializations over some neighbourhoods U_0 and U' ($b_0\in U_0\subset B$, $\bar f b_0\in U'\subset B'$):

$$\begin{array}{ccc} p^{-1}U_0 & \xrightarrow{h} & U_0\times N \\ & {\scriptstyle p}\searrow\ \ \swarrow{\scriptstyle \pi_1} & \\ & U_0 & \end{array}\qquad\qquad \begin{array}{ccc} p'^{-1}U' & \xrightarrow{h'} & U'\times N' \\ & {\scriptstyle p}\searrow\ \ \swarrow{\scriptstyle \pi_1'} & \\ & U' & \end{array}$$

We may assume that U_0 is homeomorphic to R^m and that $\Phi(\bar f,\bar g)\cap U_0=\{b_0\}$. We choose an open ball $U\subset V_0$ centred at b_0. We may choose U so small that $\bar f(\operatorname{cl}U)\cup\bar g(\operatorname{cl}U)\subset U'$ (cl denotes closure). We get the commutative diagram

$$\begin{array}{ccc} p^{-1}(\operatorname{cl}U) & \xrightarrow{(f,g)} & p^{-1}U' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \operatorname{cl}U\times N & \xrightarrow{(h'fh^{-1},\,h'gh^{-1})} & U'\times N' \end{array}$$

where $h'fh^{-1}$, $h'gh^{-1}$ are of the form

$$h'fh^{-1}(b,x)=(\bar fb,\varphi(b,x)),$$
$$h'gh^{-1}(b,x)=(\bar gb,\psi(b,x)).$$

Let $b_1\in\partial U$; then the formula

$$d((1-t)b_0+tb_1,s)=\begin{cases} b_0, & t\leqslant s/2,\\ \dfrac{2-2t}{2-s}b_0+\dfrac{2t-s}{2-s}b_1, & t\geqslant s/2,\end{cases}$$

defines a deformation of the closed ball $\operatorname{cl}U$. It gives rise to homotopies

$$(F,G)\colon(\operatorname{cl}U\times N)\times I\to U'\times N'$$

given by $F(b,x,t)=(\bar fb,\varphi(d(b,t),x))$, $G(b,x,t)=(\bar gb,\psi(d(b,t),x))$. Then $\Phi(F,G)\subset b_0\times N\times I$, so $\varphi(F,G)$ is compact. Moreover,

$$F(b,x,0)=(\bar fb,\varphi(b,x)),\quad G(b,x,0)=(\bar gb,\psi(b,x))\quad\text{for } b\in\operatorname{cl}U,$$
$$F(b,x,1)=(\bar fb,\varphi(b_0,x)),\quad G(b,x,1)=(\bar gb,\psi(b_0,x))\quad\text{for } \|b-b_0\|\leqslant 1/2.$$

The above homotopies allow us to replace U by the ball of radius $1/2$ and to assume that φ and ψ do not depend on b. So it remains to study the coincidence index of

$$(\bar f\times f_{b_0},\bar g\times g_{b_0})\colon U\times N\to B'\times N'.$$

The diagram (5.8) is commutative, so that the index equals

$$\operatorname{ind}(\bar f,\bar g\colon U)\operatorname{ind}(f_{b_0},g_{b_0}\colon A).$$

Proof of Theorem (5.5). We consider two cases:

(a) $\Phi(\bar f,\bar g)$ finite. Let $p_{\nabla}A=\{b_1,\dots,b_s\}$. Then (5.7) implies

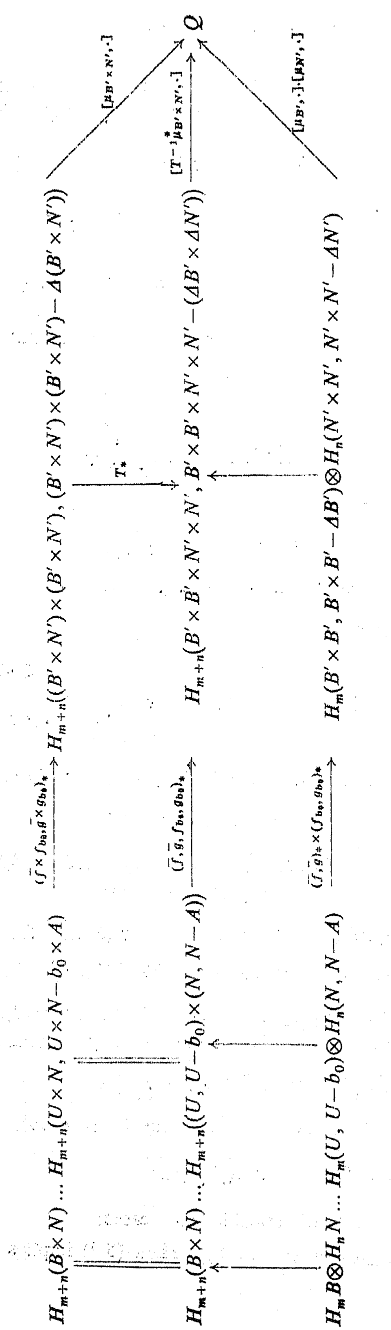


Diagram (5.8). Here $T(b_0', n_0', b_1', n_1') = (b_0', b_1', n_0', n_1')$ and the lower vertical lines denote Künneth homomorphisms

$$\operatorname{ind}(f, g\colon A) = \sum_{i=1}^{s} \operatorname{ind}(f, g\colon A \cap E_{b_i}) = \sum_{i=1}^{s} \operatorname{ind}(f_{b_i}, g_{b_i}\colon A \cap E_{b_i}) \operatorname{ind}(\bar f, \bar g\colon b_i)$$
$$= \sum_{i=1}^{s} k(A) j(A) \operatorname{ind}(\bar f, \bar g\colon b_i) = k(A) j(A) \operatorname{ind}(\bar f, \bar g\colon p_\nabla A).$$

(b) $\Phi(\bar f, \bar g)$ arbitrary. Then by (3.1), $(\bar f, \bar g)$ is homotopic to a map $(\bar f_1, \bar g_1)$ such that $\Phi(\bar f_1, \bar g_1)$ is finite. Let $(\bar F, \bar G)$ denote this homotopy. It lifts to a homotopy $(F, G)\colon (f, g) \sim (f_1, g_1)$ where (f_1, g_1) covers $(\bar f_1, \bar g_1)$. Set $A_1 = \mu_{(F,G)} A \in \nabla(f_1, g_1)$.

If $p_\nabla A_1 \in \Phi'(\bar f_1, \bar g_1)$ then by Lemma (5.6) and part (a) $\operatorname{ind}(f, g\colon A) = \operatorname{ind}(f_1, g_1\colon A_1) = k(A_1) j(A_1) \operatorname{ind}(\bar f_1, \bar g_1\colon p_\nabla A_1) = k(A) j(A) \operatorname{ind}(\bar f, \bar g\colon p_\nabla A)$.

If $p_\nabla A_1 \notin \Phi'(\bar f_1, \bar g_1)$ then $A_1 \notin \Phi'(f_1, g_1)$, so by (5.6)

$$\operatorname{ind}(f, g\colon A) = \operatorname{ind}(f_1, g_1\colon A_1) = 0,$$
$$\operatorname{ind}(\bar f, \bar g\colon p_\nabla A) = \operatorname{ind}(\bar f_1, \bar g_1\colon p_\nabla A_1) = 0.$$

Thus the both sides of the desired equality are zero. ■

Now we are going to find conditions under which the product formula for Lefschetz numbers holds. Notice first that this formula makes no sense if $\Phi(\bar f, \bar g) = \emptyset$. But in this case or more generally if $N(\bar f, \bar g) = 0$ then by (5.5) also $N(f, g) = 0$ and hence $L(f, g) = 0 = L(\bar f, \bar g)$. Therefore, we will assume that $N(f, g) \neq 0$. Notice that the formula makes no sense either if $L(f_b, g_b)$ is not the same for all points $b \in \Phi'(\bar f, \bar g)$.

(5.9) Lemma. *Consider the fibre homotopy from Lemma* (5.6). *Let* $b_0 \in A_0 \in \Phi'(\bar f_0, \bar g_0)$ *and let* $b_1 \in A_1 = \mu(A_0) \in \Phi'(\bar f_1, \bar g_1)$. *Then*

$$L(f_{0b_0}, g_{0b_0}) = L(f_{1b_1}, g_{1b_1}).$$

In particular, if $(f_0, g_0) = (f_1, g_1) = (f, g)$ *then the Lefschetz numbers over the Nielsen equivalent points are the same.*

Proof. There is a path $\bar u$ from b_0 to b_1 such that $\bar F \bar u \sim \bar G \bar u$ ((1.5d)). Then the induced transformation $T_{\bar u}$ carries A_0 to A_1 and preserves index. ■

(5.10) Theorem. *Suppose that* $N(\bar f, \bar g) \neq 0$ *and that* $L(f_b, g_b)$ *is the same for all* $b \in \Phi(\bar f, \bar g)$. *Then*

$$L(f, g) = L(\bar f, \bar g) L(f_b, g_b).$$

Proof. (f, g) is homotopic to (f_1, g_1) such that $\Phi(\bar f_1, \bar g_1)$ is finite. Let A denote the sum of all essential Nielsen classes of $(\bar f_1, \bar g_1)$ and let $A_1, \ldots, A_k$ be its unessential classes. Then Lemma (5.9) implies that for any $b \in \Phi(\bar f, \bar g)$ and $b_1 \in A$ the Lefschetz numbers $L(f_b, g_b)$ and $L(f_{1b_1}, g_{1b_1})$ are equal. So

$$L(f, g) = L(f_1, g_1) = \sum_{b \in \Phi(\bar f_1, \bar g_1)} \operatorname{ind}(f_1, g_1\colon E_b) = \sum_{b \in \Phi(\bar f_1, \bar g_1)} L(f_{1b}, g_{1b}) \operatorname{ind}(\bar f_1, \bar g_1\colon b)$$
$$= \sum_{b \in A} L(f_{1b}, g_{1b}) \operatorname{ind}(\bar f_1, \bar g_1\colon b) + \sum_{i=1}^{k} \sum_{b \in A_i} L(f_{1b}, g_{1b}) \operatorname{ind}(\bar f_1, \bar g_1\colon b)$$

$$= L(f_{b_0}, g_{b_0}) \sum_{b\in A} \operatorname{ind}(\bar f_1, \bar g_1 \colon b) + \sum_{i=1}^{k} L(f_{1b_i}, g_{1b_i}) \sum_{b\in A_i} \operatorname{ind}(\bar f_1, \bar g_1 \colon b)$$

$$= L(f_b, g_b) L(\bar f_1, \bar g_1) = L(f_b, g_b) L(\bar f, \bar g). \ \blacksquare$$

(5.11) Definition. A fibre bundle is called *orientable* if for any loop $\bar u$ based at $b \in B$ the induced map $\tau_{\bar u}$ is homotopic to the identity map of the fibre E_b.

(5.12) Corollary. *If either (E', p', B') is orientable or $\bar f \sim \bar g$ then for any $b_0, b_1 \in \Phi(\bar f, \bar g)$*

$$L(f_{b_0}, g_{b_0}) = L(f_{b_1}, g_{b_1}),$$

$$N_K(f_{b_0}, g_{b_0}) = N_K(f_{b_1}, g_{b_1}), \quad N(f_{b_0}, g_{b_0}) = N(f_{b_1}, g_{b_1}).$$

Proof. Notice that in both cases for any $b_0 \in \Phi(\bar f, \bar g)$ and a path $\bar u$ joining them the maps $\tau'_{\bar f\bar u}, \tau'_{\bar g\bar u} \colon E'_{\bar f b_0} \to E'_{\bar f b_1}$ are homotopic. Then the diagram

$$\begin{array}{ccc} E_{b_0} & \xrightarrow{(f_{b_0}, g_{b_0})} & E'_{\bar f b_0} \\ \downarrow{\scriptstyle \tau_{\bar u}} & & \downarrow{\scriptstyle \tau'_{\bar f \bar u}} \\ E_{b_1} & \xrightarrow{(f_{b_1}, g_{b_1})} & E'_{\bar f b_1} \end{array}$$

is homotopy commutative. We apply Theorem (2.1) to it.

6. The Nielsen number product formula. Let $\bar A_1, \ldots, \bar A_s$ denote all essential classes of $(\bar f, \bar g)$. It follows from (5.5) that if $A \in \nabla(f, g)$ is essential then so is $p_\nabla A \in \nabla(\bar f, \bar g)$, hence $p_\nabla A = A_i$ for some $i = i, \ldots, s$. Define

$$C_i = \#\{A \in \Phi'(f, g) \colon \operatorname{ind}(f, g \colon A) \neq 0, p_\nabla A = \bar A_i\}.$$

Then $N(f, g) = C_1 + \ldots + C_s$ and $N(\bar f, \bar g) = s$.

Let $b \in \Phi(\bar f, \bar g)$ and let $A \in \operatorname{im}(i_b)_\nabla$. Then $(i_b)_\nabla^{-1} A$ is an orbit of the action of the group $C(\bar f_\#, \bar g_\#)_b$ on $\nabla_K(f_b, g_b)$ and every orbit is of the above form ((4.11)). All elements of one orbit have the same index. We call an orbit *essential* iff it contains essential classes.

(6.1) Lemma. *Let $b \in \bar A_i$ for some $i = 1, \ldots, s$. Then the number of essential orbits in $\nabla_K(f_b, g_b)$ equals C_i.*

Proof. Each element $A \in \operatorname{im}(i_b)_\nabla$ determines an orbit in $\nabla_K(f_b, g_b)$. This is a bijection under which essential classes correspond to essential orbits (5.5). On the other hand, by (4.11), $\operatorname{im}(i_b)_\nabla = p_\nabla^{-1} \bar A_i$ and the number of essential classes in $p_\nabla^{-1} \bar A_i$ equals C_i. $\blacksquare$

The next two corollaries follow from (4.10).

(6.2) Corollary. *Let $b \in \Phi(\bar f, \bar g)$, $A \in \Phi'_K(f_b, g_b)$, $\alpha \in C(\bar f_\#, \bar g_\#)_b$. Then the following conditions are equivalent*:

(a) $T_\alpha A = A$.

(b) $\alpha \in p_*(C(f_\#, g_\#)_x)$ *for any* $x \in A$.

(c) $\alpha \in p_*(C(f_\#, g_\#)_x)$ *for some* $x \in A$.

(6.3) Corollary. *Let $b \in \Phi(\bar f, \bar g)$, $A \in \Phi'_K(f_b, g_b)$. Then the length of the orbit containing A is equal to the index of the subgroup $p_*(C(f_\#, g_\#)_x)$ in $C(\bar f_\#, \bar g_\#)_b$ (written $[C(\bar f_\#, \bar g_\#)_b \colon p_* C(f_\#, g_\#)_x]$) where $x \in A$, $b = px$.*

Now we are ready to discuss the product formula

$$N(f, g) = N(\bar f, \bar g) N(f_b, g_b).$$

This formula makes sense only if $N(f_b, g_b)$ is independent of $b \in \Phi(\bar f, \bar g)$. By Corollary (5.12) this is satisfied if the fibre bundle (E', p', B') is orientable. But even without this assumption we have the following:

(6.4) Theorem. (a) *If $N(\bar f, \bar g) = 0$ then $N(f, g) = 0$.*

(b) *If $N(f_b, g_b) = 0$ for all b lying in an essential class then $N(f, g) = 0$.*

(c) *If $N(\bar f_b, \bar g_b) = 1$ for all b lying in an essential class then $N(f, g) = N(\bar f, \bar g)$.*

Proof. If the assumption of (a) or (b) is satisfied then (5.5) implies that every class of (f, g) is unessential. The assumption of (c) implies that $C_i = 1$ for all $i = 1, \ldots, s$. $\blacksquare$

(6.5) Theorem. *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{(f, g)} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{(\bar f, \bar g)} & B' \end{array}$$

where E, E', B, B' and all fibres are manifolds of respectively equal dimensions. Suppose that the bundle (E', p', B') is orientable and $N(f, g) \neq 0$.

Then the formula $N(f, g) = N(\bar f, \bar g) N(f_b, g_b)$ holds if and only if the following two conditions hold:

(a) $N_K(f_b, g_b) = N(f_b, g_b)$.

(b) *For any $x \in \Phi(f, g)$ in an essential class $p_*(C(f_\#, g_\#)_x) = C(\bar f_\#, \bar g_\#)_b$ $(b = px)$.*

Proof. By (6.3), (b) is equivalent to the condition that for each $b \in \Phi(\bar f, \bar g)$ in an essential class the length of each essential orbit in $\nabla_K(f_b, g_b)$ equals 1. So by (6.1), $C_i = N_K(f_b, g_b)$, $i = 1, \ldots, s$. Thus (b) is equivalent to the formula

$$N(f, g) = N(\bar f, \bar g) N_K(f_b, g_b).$$

The conclusion of the theorem now follows since $N(f, g) \neq 0$. $\blacksquare$

Now let us consider the case $N(f, g) = 0$. Recall that for a space X and a point $x_0 \in X$ the Jiang subgroup $J(X, x_0) \subset \Pi_1(X, x_0)$ is given by

$$J(X, x_0) = \{\alpha \in \Pi_1(X, x_0) \colon \text{there is a cyclic homotopy } H \colon \mathrm{id}_X \sim \mathrm{id}_X \text{ such that } \langle H(x_0, \cdot)\rangle = \alpha\}.$$

(6.6) LEMMA. *Let $(f, g)\colon M \to N$ be maps between manifolds of the same dimension, and let $H \subset \Pi_1 N$ be a normal subgroup such that $H \subset J(N)$. Suppose that $A_0, A_1 \in \nabla(f, g)$ satisfy $\mathrm{id}_\nabla A_0 = \mathrm{id}_\nabla A_1$ where $\mathrm{id}_\nabla\colon \nabla(f,g) \to \nabla_H(f,g)$ is induced by $(\mathrm{id}_M, \mathrm{id}_N)$. Then $\mathrm{ind}(f,g\colon A_0) = \mathrm{ind}(f,g\colon A_1)$.*

Proof. Fix a reference pair (x_0, r). Let $\langle a_0\rangle, \langle a_1\rangle \in \Pi_1(N, fx_0)$ satisfy $A_0 = [\langle a_0\rangle]$, $A_1 = [\langle a_1\rangle]$. Since $\mathrm{id}_\nabla A_0 = \mathrm{id}_\nabla A_1$ we have $[\langle a_0\rangle_H] = [\langle a_1\rangle_H]$, so we may assume that $\langle a_0\rangle_H = \langle a_1\rangle_H$. Hence $\langle a_1\rangle = \langle -u+a_0\rangle$ for a $\langle u\rangle \in H(fx_0) \subset J(N, fx_0)$. Select a cyclic homotopy $F\colon \mathrm{id}_N \sim \mathrm{id}_N$ satisfying $F(fx_0, \cdot) \sim u$. Then $F'(x, t) = F(fx, t)$ is a cyclic homotopy $F'\colon f \sim f$ satisfying $F'(x_0, \cdot) \sim u$. Thus letting G be the constant homotopy at g we get

$$\mu_{(F,G)} A_0 = \mu_{(F,G)}[\langle a_0\rangle] = [\langle -u+a_0\rangle] = [\langle a_1\rangle] = A_1 .$$

Since μ is index-preserving ((1.7)), the indices of A_0 and A_1 are equal.

(6.7) COROLLARY. *Let a commutative diagram (5.0) be given. Let $b \in \Phi(\bar f, \bar g)$, $A \in \nabla_K(f_b, g_b)$ and let $\mathrm{id}_\nabla\colon \nabla(f_b, g_b) \to \nabla_K(f_b, g_b)$ be the map induced by $(\mathrm{id}_{E_b}, \mathrm{id}_{E'_{\bar f b}})$* (see (2.1)). *Then all the classes from $(\mathrm{id}_\nabla)^{-1}A$ have the same index.*

Proof. It follows from [11] that $K \subset J(E'_{\bar f b})$. Now apply (6.6). ■

(6.8) THEOREM. *If (E', p', B') is orientable and $N(f, g) = 0$ then the product formula also holds.*

Proof. We have to prove that either $N(\bar f, \bar g) = 0$ or $N(f_b, g_b) = 0$. By (3.1) and (1.7) we may assume that $\Phi(\bar f, \bar g)$ is finite. Suppose that $N(\bar f, \bar g) \neq 0$ and take $b \in \Phi(\bar f, \bar g)$ from an essential class. Let $A \in \Phi'(f_b, g_b)$ and let $A_1 = (\mathrm{id}_\nabla)A$. Then the assumption $N(f,g) = 0$ and Theorem (5.5) imply that $\mathrm{ind}(f_b, g_b\colon A_1) = 0$, so by (6.7), $\mathrm{ind}(f_b, g_b\colon A) = 0$. Hence A is unessential. Thus $N(f_b, g_b) = 0$. ■

If (E', p', B') is orientable then $C_1, \ldots, C_s \leqslant N_K(f_b, g_b) \leqslant N(f_b, g_b)$, so $N(f,g) \leqslant N(\bar f, \bar g) N_K(f_b, g_b) \leqslant N(\bar f, \bar g) N(f_b, g_b)$.

For $N(f, g) \neq 0$ we define rational numbers (Pak numbers [24])

$$P_K(f,g) = N(\bar f, \bar g) N_K(f_b, g_b)/N(f,g),$$
$$P(f,g) = N(\bar f, \bar g) N(f_b, g_b)/N(f,g).$$

(6.9) THEOREM. *Let (E', p', B') be orientable, $N(f,g) \neq 0$ and let $\Pi_1 E'$ be abelian. Then*

$$P_K(f,g) = [C(\bar f_\#, \bar g_\#)_b : p_\#(C(f_\#, g_\#)_x)] \quad \text{for any } x \in \Phi(f,g),\ b = px.$$

Moreover, $P_K(f,g)$ divides $N_K(f_b, g_b)$.

Proof. First of all we prove that $[C(\bar f_\#, \bar g_\#)_b : p_\#(C(f_\#, g_\#)_x)]$ is independent of $x \in \Phi(f,g)$. Let $x_0, x_1 \in \Phi(f,g)$ and let u be a path from x_0 to x_1. We will show that the isomorphism $h_{\langle u\rangle}\colon \Pi_1(E, x_0) \to \Pi_1(E, x_1)$ given by $h_{\langle u\rangle}\langle a\rangle = \langle -u+a+u\rangle$ maps $C(f_\#, g_\#)_{x_0}$ into $C(f_\#, g_\#)_{x_1}$.

Let $\langle a\rangle \in C(f_\#, g_\#)_{x_0}$. Then since $\Pi_1 E'$ is abelian $f(-u+a+u)$ is homotopic to $g(-u+a+u)$, which proves that $h_{\langle u\rangle}\langle a\rangle \in C(f_\#, g_\#)_{x_1}$.

Similarly we prove that $h_{\langle \bar u\rangle}\colon C(\bar f_\#, \bar g_\#)_{b_0} \to C(\bar f_\#, \bar g_\#)_{b_1}$ is an isomorphism (where $\bar u = pu$, $b_0 = px_0$, $b_1 = px_1$). Thus we get the commutative diagram which

$$\begin{array}{ccc} C(f_\#, g_\#)_{x_0} & \xrightarrow{p_\#} & C(\bar f_\#, \bar g_\#)_{b_0} \\ \downarrow{\scriptstyle h_{\langle u\rangle}} & & \downarrow{\scriptstyle h_{\langle \bar u\rangle}} \\ C(f_\#, g_\#)_{x_1} & \xrightarrow{p_\#} & C(\bar f_\#, \bar g_\#)_{b_1} \end{array}$$

implies the equality

$$[C(\bar f_\#, \bar g_\#)_{b_0} : p_\#(C(f_\#, g_\#)_{x_0})] = [C(\bar f_\#, \bar g_\#)_{b_1} : p_\#(C(f_\#, g_\#)_{x_1})] .$$

Then for any $b \in \Phi(\bar f, \bar g)$ in an essential class and for $x \in \Phi(f_b, g_b)$ the length of each essential orbit in $\nabla_K(f_b, g_b)$ equals $[C(\bar f_\#, \bar g_\#)_b : p_\#(C(f_\#, g_\#)_x)]$ (see the remark after (4.12)). Denote it by m. Then $C_1 = \ldots = C_s = C$. Hence $N_K(f_b, g_b) = mC$ and m divides $N_K(f_b, g_b)$. Then $mN(f,g) = mCN(\bar f, \bar g) = N_K(f_b, g_b) N(\bar f, \bar g)$. Dividing by $N(f,g)$ we obtain $P_K(f,g) = m$.

(6.10) THEOREM. *Let (E', p', E') be orientable, $N(f,g) \neq 0$ and let $\Pi_1 E'$, $\Pi_1 E'_{b'}$ be abelian. Then $P(f,g)$ is equal to the order of the kernel of the homomorphism*

$$(i_b)'_\nabla\colon \Pi_1(E'_{\bar f b}, fx)/\mathrm{im}(f_{b\#} - g_{b\#}) \to \Pi_1(E', fx)/\mathrm{im}(f_\# - g_\#)$$

induced by the inclusion $E'_{\bar f b} \hookrightarrow E'$.

Proof. Take $x \in \Phi(f,g)$ and the constant path r_0 at $fx = gx$ as references. Consider the diagram

$$\begin{array}{ccc} \nabla(f_b, g_b\colon x, r_0) & \xrightarrow{(i_b)'_\nabla} & \nabla(f, g\colon x, r_0) \\ {\scriptstyle \mathrm{id}_\nabla}\searrow & & \nearrow{\scriptstyle (i_b)_\nabla} \\ & \nabla_K(f_b, g_b\colon x, r_0) & \end{array}$$

Since $\Pi_1 E'$ and $\Pi_1 E'_b$ are abelian, the quotients

$$\nabla(f, g\colon x, r_0) = \Pi_1(E', fx)/\mathrm{im}(f_\# - g_\#),$$
$$\nabla(f_b, g_b\colon x, r_0) = \Pi_1(E'_{b'}, fx)/\mathrm{im}(f_{b\#} - g_{b\#}),$$
$$\nabla_K(f_b, g_b\colon x, r_0) = (\Pi_1(E'_{b'}, fx)/K)/\mathrm{im}((f_{b\#})_K - (g_{b\#})_K)$$

are also abelian groups, id_∇, $(i_b)_\nabla$, $(i_b)'_\nabla$ are homomorphisms and $(i_b)'_\nabla$ is the homomorphism specified in the theorem.

Since id_∇ is an epimorphism $\#(\mathrm{id}_\nabla)^{-1}A$ is the same for all $A \in \nabla_K(f_b, g_b\colon x, r_0)$ and by (6.7) all classes in $(\mathrm{id}_\nabla)^{-1}A$ have the same index. Thus $N(f_b, g_b) = N_K(f_b, g_b)(\#\ker(\mathrm{id}_\nabla))$. Then $\#\ker(i_b)'_\nabla N(f,g) = (\#\ker(\mathrm{id}_\nabla))(\#\ker(i_b)_\nabla)$

$N(f, g) = (\# \ker(\mathrm{id}_\nabla))\ P_K(f, g)\ N(\bar f, \bar g)$ (by Theorem (6.9)) $= (\# \ker(\mathrm{id}_\nabla))\ N_K(f_b, g_b)N(\bar f, \bar g) = N(f_b, g_b)N(\bar f, \bar g)$.

Dividing by $N(\bar f, \bar g)$ we obtain $\# \ker(i_b)'_\nabla = P(f, g)$. ■

(6.11) COROLLARY. *Under the above assumptions*:

$P(f, g) = (\# \ker \mathrm{id}_\nabla) P_K(f, g)$,

$P_K(f, g)$ *divides* $N_K(f_b, g_b)$,

$P(f, g)$ *divides* $N(f_b, g_b)$

and

$N_K(f_b, g_b)/P_K(f, g) = N(f_b, g_b)/P(f, g) = N(f, g)/N(\bar f, \bar g)$.

7. The Anosov theorem. In this section we apply the above theory to prove the following

(7.1) THEOREM. *For any two self-maps of a nilmanifold M the formula $N(f, g) = |L(f, g)|$ holds.*

This generalizes [0] and [9]. We will follow the method used in [9].

(7.2) LEMMA. *Let M, N be compact connected smooth closed oriented n-dimensional manifolds and let $f, g\colon M \to N$ be C^1 maps such that for some $x_0 \in \Phi(f, g)$ the difference of tangent maps $(Df)_{x_0} - (Dg)_{x_0}$ is an isomorphism. Then x_0 is an isolated coincidence point and $\mathrm{ind}(f, g\colon x) = +1(-1)$ if $(Df)_{x_0} - (Dg)_{x_0}$ preserves (reverses) the local orientation.*

Proof. We choose euclidean neighbourhoods $U \subset M$, $V \subset N$ such that $x_0 \in U$, $fx_0 = gx_0 \in V$, $fU \cup gU \subset V$. We identify U and V with R^n. Then we have the commutative diagram

$$\begin{array}{ccccc}
H_n(U, U-x_0) & \xrightarrow{(f,g)_*} & H_n(N\times N, N\times N-\Delta N) & \xrightarrow{[\mu_N, \cdot]} & Q\\
 & \searrow^{(f,g)_*} & \uparrow \alpha_* & & \nearrow\\
 & & H_n(V\times V, V\times V-\Delta N) & & \\
 & & \uparrow \beta_* & & \\
 & {\scriptstyle (f-g)_*}\searrow & H_n((V, V-fx_0)\times V) & & \nearrow {\scriptstyle [s^*, \cdot]}\\
 & & \uparrow \gamma_* & & \\
 & & H_n(V, V-fx_0) & &
\end{array}$$

where $\alpha(z, z') = (z, z')$, $\beta(z, z') = (z+z'-fx_0, z')$, $\gamma(z) = (z, gx_0)$, $(f-g)x = fx_0+fx-gx$ and s^* is a generator of $H^n(V, V-fx_0)$ corresponding to the chosen orientation. Let now $\bar z_M \in H_n(U, U-x_0)$, $\bar z_N \in H_n(V, V-fx_0)$ be generators corresponding to the chosen orientations of M and N. Then

$$(f-g)_*\bar z_M = \begin{cases} +\bar z_N & \text{if } (Df)_x - (Dg)_{x_0} \text{ preserves orientation,}\\ -\bar z_N & \text{otherwise.}\end{cases}$$

Finally, $\mathrm{ind}(f, g\colon x_0) = [\mu_N, (f, g)_*\bar z_M] = [s^*, (f-g)_*\bar z_M] = [s^*, \pm\bar z_N] = +1(-1)$.

(7.3) LEMMA. *Let T^n denote the n-dimensional torus and let $f, g\colon T^n \to T^n$ be continuous maps. Let A, B denote the $n\times n$ matrices representing the endomorphisms $f_\#, g_\#\colon \Pi_1 T^n \to \Pi_1 T^n$. Then*

$$N(f, g) = |L(f, g)| = |\det(A-B)|\,.$$

Proof. We represent T^n as the quotient space R^n/Z^n and denote by $\pi\colon R^n \to T^n$ the natural projection. We write $\pi(x) = \bar x$. We may assume that f and g are induced by linear maps represented by A and B. We consider two cases:

(a) $\det(A-B) = 0$. Then $\mathrm{im}(A-B) \not\supseteq R^n$. For a $v \notin \mathrm{im}(A-B)+Z^n$ we put $A'(x) = A(x)-v$. Then the map f' induced by A' is homotopic to f and $\Phi(f', g) = \emptyset$. Thus $N(f, g) = L(f, g) = 0$.

(b) $\det(A-B) \neq 0$. Consider $\bar x_0 \in \Phi(f, g)$. Take $x_0 \in \pi^{-1}\bar x_0$ and set $r = Ax_0 - Bx_0 \in Z^n$. Let $B'(x) = B(x)+r$. Then the diagram

$$\begin{array}{ccc}
R^n & \xrightarrow{(A, B')} & R^n\\
\pi\downarrow & & \downarrow\pi\\
T^n & \xrightarrow{(f, g)} & T^n
\end{array}$$

commutes and $x_0 \in \Phi(A, B')$. Since $D_{x_0}A = D_{\bar x_0}f = A$ and $D_{x_0}B' = D_{x_0}g = B$, it follows that $D_{\bar x}f - D_{\bar x}g$ is an isomorphism. Therefore (7.2), $\bar x_0$ is an isolated coincidence point and $\mathrm{ind}(f, g\colon \bar x_0) = \mathrm{sgn}(A-B)$. Thus $\#\Phi(f, g) = |\mathrm{ind}(f, g)| = |L(f, g)|$.

It remains to show that no two distinct points $\bar x_0, \bar x_1 \in \Phi(f, g)$ are Nielsen equivalent. Suppose the contrary: then $\bar x_0, \bar x_1 \in \pi\Phi(f, g')$ for some B' ($B'(x) = B(x)+r$, $r \in Z^n$). Then $A(x_i) = B'(x_i)$ for some $x_i \in \pi^{-1}\bar x_i (i = 0, 1)$. This implies $A(x_1-x_0) = B(x_1-x_0)$, so $x_1-x_0 \in \ker(A-B) = 0$ since $A-B$ is an isomorphism. Thus $x_0 = x_1$. On the other hand, $\#\Phi(f, g)$ equals the order of the quotient group $Z^n/\mathrm{im}(A-B)$ and the latter is $|\det(A-B)|$ ([2]). ■

(7.4) LEMMA. *If in the diagram (5.0), $B = B' = T^n$ and (E', p', B') is orientable then the product formula $N(f, g) = N(\bar f, \bar g)\ N(f_b, g_b)$ holds for any (f, g).*

Proof. If $N(\bar f, \bar g) = 0$ then the formula follows from (6.4a). Suppose that $N(f, g) \neq 0$. We will show that the conditions (a) and (b) of Theorem (6.5) are satisfied. Since $\Pi_2(T^n) = 0$ the homotopy exact sequence of (E', p', B') implies $K = 0$, hence (a) holds. Now we show (b) by proving that $C(\bar f_\#, \bar g_\#)_b$ is the zero subgroup of $\Pi_1 B$. Suppose that $\bar f_\# \alpha = \bar g_\# \alpha$ for some $\alpha \in \Pi_1 B$. Let $n\times n$ matrices A, B and an element $x \in Z^n$ represent $\bar f_\#$, $\bar g_\#$ and α respectively. Then $Ax = Bx$, hence $(A-B)x = 0$ and the assumption $0 \neq N(f, g) = |\det(A-B)|$ implies $x = 0$. ■

Recall that a nilmanifold is a quotient space of a nilpotent simply-connected Lie group by its discrete subgroup. We consider only compact nilmanifolds. We base on the following.

(7.5) THEOREM [9]. *Let M be a nilmanifold. If M is not a torus then M is a total space of a T-principal fibre bundle over another nilmanifold B (T is a torus). Moreover, for any continuous map f': $M \to M$ there exists a map f: $M \to M$ homotopic to f such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\ \bar f\ } & B \end{array}$$

commutes.

(7.6) LEMMA. *Let M be a nilmanifold and let (f, g): $M \to M$ satisfy $L(f, g) \neq 0$. Then $C(f_{\#}, g_{\#}) = 0$.*

Proof. The proof is by induction on the dimension. The case of M = torus is contained in the proof of Lemma (7.4). Now assume that the lemma holds for a nilmanifold B and let (M, p, B, T) be a principal torus bundle. We prove the lemma for M.

Let (f, g): $M \to M$ satisfy $L(f, g) \neq 0$. We may assume that f, g are fibre maps. Then we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Pi_1 T & \xrightarrow{\ i_{\#}\ } & \Pi_1 M & \xrightarrow{\ p_{\#}\ } & \Pi_1 B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (f_{0\#}, g_{0\#})} & & \downarrow{\scriptstyle (f_{\#}, g_{\#})} & & \downarrow{\scriptstyle (\bar f_{\#}, \bar g_{\#})} & & \\ 0 & \longrightarrow & \Pi_1 T & \xrightarrow{\ i_{\#}\ } & \Pi_1 M & \xrightarrow{\ p_{\#}\ } & \Pi_1 B & \longrightarrow & 0 \end{array}$$

Since the principal bundle is orientable and $L(f, g) \neq 0$ the equality $L(f, g) = L(f_0, g_0)L(\bar f, \bar g)$ implies $L(f_0, g_0) \neq 0$ and $L(\bar f, \bar g) \neq 0$, hence by the inductive assumption $C(\bar f_{\#}, \bar g_{\#}) = 0$ and $C(f_{0\#}, g_{0\#}) = 0$. Suppose that $f_{\#} u = g_{\#} u$. Then $\bar f_{\#} p_{\#} u = p_{\#} f_{\#} u = p_{\#} g_{\#} u = \bar g_{\#} p_{\#} u$, so $p_{\#} u = 0$. Thus $u = i_{\#} v$ for some $v \in \Pi_1 T$. But then $i_{\#} f_{0\#} v = f_{\#} i_{\#} v = f_{\#} u = g_{\#} u = g_{\#} i_{\#} v = i_{\#} g_{0\#} v$. But $i_{\#}$ is injective, so $f_{0\#} v = g_{0\#} v$ and hence $v \in C(f_{0\#}, g_{0\#}) = 0$. Thus $u = i_{\#} v = 0$. ■

Proof of Theorem (7.1). The proof is by induction (as for (7.6)). The case of M = torus is Lemma (7.3). Suppose now that the theorem holds for a nilmanifold B and let (M, p, B, T) be a principal torus bundle from (7.5). Let (f', g'): $M \to M$ be continuous. Then (7.5) gives rise to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\ (f_b, g_b)\ } & T \\ \cap\downarrow & & \downarrow\cap \\ M & \xrightarrow{\ (f, g)\ } & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\ (\bar f, \bar g)\ } & B \end{array}$$

with (f, g) homotopic to (f', g').

If $L(\bar f, \bar g) = 0$ then by the inductive assumption $N(\bar f, \bar g) = 0$ and $(\bar f, \bar g)$ is homotopic to a coincidence free pair (if $\dim B \geqslant 3$ we apply the Wecken type theorem [5], for $B = T^1$, T^2 see the proof of (7.3a)). This homotopy may be lifted giving rise to a deformation of (f, g) to a coincidence free pair. Thus $N(f, g) = 0$, so $L(f, g) = 0$.

Now assume that $L(f, g) \neq 0$. Then the assumptions of Theorem (6.5) are satisfied: $\Pi_2 M = 0$ implies (a) and Lemma (7.6) implies $C(\bar f_{\#}, \bar g_{\#}) = 0$, so (b) is also satisfied. Thus the product formula $N(f, g) = N(\bar f, \bar g)N(f_b, g_b)$ holds, so

$$|L(f, g)| = |L(\bar f, \bar g)L(f_b, g_b)| = N(\bar f, \bar g)N(f_b, g_b) = N(f, g).$$

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On some famous examples in dimension theory

by

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Abstract. It was proved in [M–R] that for each $n \geqslant 2$ there exists an n-dimensional compactum X such that $\mathscr{E}(X, R)^{2n}$ is dense in $\mathscr{C}(X, R^{2n})$. In this note we prove that the classical examples of V. G. Boltyanskii [B] and Y. Kodama [K2], and their natural higher dimensional counterparts, have the same property.

In [M–R], D. McCullough and L. R. Rubin proved the following result: For each $n \geqslant 2$ there exists an n-dimensional compactum X such that the space $\mathscr{E}(X, \boldsymbol{R}^{2n})$ of imbeddings from X into $\boldsymbol{R}^{2n}$ is dense in the space $\mathscr{C}(X, \boldsymbol{R}^{2n})$ of mappings to $\boldsymbol{R}^{2n}$.

The aim of this note is to show that some famous examples first studied by V. G. Boltyanskii [B] (see also [K1], [K2], [Wi]) and their straightforward n-dimensional counterparts, $n \geqslant 2$, also have this property (see Th. 5.2 and Th. 5.3).

It was shown in [Kr] that this property implies $\dim X \times X < 2n$. This gives an elementary proof of the fundamental property of the examples in which only elementary algebraic topology is needed (all previous used the Künneth formula).

Lately S. Spież [S] has proved that the latter property implies the property from the McCullough–Rubin theorem for $n \geqslant 3$. (1)

All spaces in this note are assumed to be metric with a metric denoted by d.

1. A lemma on imbeddings. Let $\mathscr{U}$ be a cover of a space X. A mapping $f\colon X \to Y$ is said to be a *$\mathscr{U}$-mapping* provided for every $y \in f(X)$ there is a $U \in \mathscr{U}$ such that $f^{-1}(y) \subset U$. If $\mathscr{V}$ is a collection of subsets of Y then we denote

$$f^{-1}(\mathscr{V}) = \{f^{-1}(V)\colon V \in \mathscr{V}\}.$$

The following lemma is a slight generalization of Proposition 1.7 from [M–R].

1.1. *Let* $\boldsymbol{X} = \{X_1 \overset{\varrho_{12}}{\leftarrow} X_2 \overset{\varrho_{23}}{\leftarrow} \ldots\}$ *be an inverse sequence of compacta satisfying the condition*

(∗) *for each $i \geqslant 1$, for every mapping $f\colon X_i \to R^n$, for every open cover $\mathscr{U}$ of X_i and every positive number $\delta > 0$ there exist an index $j \geqslant i$ and a $\varrho_{ij}^{-1}(\mathscr{U})$-mapping $g\colon X_j \to \boldsymbol{R}^n$ such that $d(f\varrho_{ij}, g) < \delta$.*

Then $\mathscr{E}(\varprojlim \boldsymbol{X}, \boldsymbol{R}^n)$ is a dense G_δ in $\mathscr{C}(\varprojlim \boldsymbol{X}, \boldsymbol{R}^n)$.

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(1) Added in proof. Extended to $n = 2$ as well.