choose $A_1 = \emptyset$. Else we find A_α^1 $(\alpha < \omega_1)$ such that $A_\alpha^1 \cap A_\beta^1 = \emptyset$ if $\alpha \neq \beta$, $A_\alpha^1 \subset G_1 \cap M$ and $A_\alpha^1 \notin \mathfrak{J}$ whenever $\alpha \leqslant \omega_1$. Obviously, if $G_m \cap M \notin \mathfrak{J}$ $(m \geqslant 1)$ then there exists at most one $\alpha < \omega_1$ such that $G_m \cap (M \setminus A_\alpha^1) \in \mathfrak{J}$. Therefore we may find $\alpha_1 < \omega_1$ satisfying $G_m \cap (M \setminus A_{\alpha_1}^1) \notin \mathfrak{J}$ whenever $G_m \cap M \notin \mathfrak{J}$. Now we can put $A_1 = A_{\alpha_1}^1$ and construct A_2 in a similar way, replacing M by $M \setminus A_1$.

Further, for each $n \geqslant 1$ there exist disjoint sets $A_n^1, A_n^2 \subset A_n$ such that $A_n^1, A_n^2 \notin \mathfrak{J}$ if $A_n \notin \mathfrak{J}$. Then it is easy to verify that the sets $M_i = \bigcup_{n \geqslant 1} A_n^i$ $(i = 1, 2)$ have all required properties.

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Weak covering properties and the class MOBI

by

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Abstract. We construct a non-σ-discrete zero-dimensional scattered space which is an open and compact image of a σ-discrete metacompact Moore space. This gives an example showing that weak θ-refinability is not preserved by open and compact mappings between regular spaces and shows that there exist non-weakly θ-refinable spaces in MOBI_3. Moreover, our example is a non-σ-discrete space which is both right and left separated.

In 1966 A. V. Arhangel'skiĭ [A2] defined the class MOBI and asked many questions concerning this class.

The investigation of the class MOBI led to a slight modification of the original definition from [A2].

For a class $\mathscr{K}$ of topological spaces, let $\mathrm{MOBI}_i(\mathscr{K})$ be the minimal class of T_i-spaces containing all metric spaces from $\mathscr{K}$ and invariant under open and compact mappings (see [Ch1]).

It is easy to observe that a T_i-space is in $\mathrm{MOBI}_i(\mathscr{K})$ if and only if it can be obtained as an image of a metric space from $\mathscr{K}$ under a mapping which is a composition of a finite number of open and compact mappings with T_i-domains [B1].

If the class $\mathscr{K}$ contains the class of all metric spaces, we write MOBI_i instead of $\mathrm{MOBI}_i(\mathscr{K})$.

It is well known that open and compact images of metric spaces are metacompact developable T_1-spaces [H], [A1]. It is also known that all the elements of MOBI_2 have a point-countable base and all the elements of MOBI_3 have, moreover, a base of countable order [WW1].

In [Ch4] the first example of a non-weakly θ-refinable space S^* in MOBI_2 was given. The space S^* is the first absolute example of a non-weakly θ-refinable space with a point-countable base and, since it does not have base of countable order, it cannot be in MOBI_3.

In the present paper we shall modify the construction of spaces in MOBI_2 rom [Ch4] and the construction of open and compact mappings from [B2] in order o obtain a non-weakly θ-refinable space Y in MOBI_3.

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The first section contains definitions and preliminary results, in the second section we construct the space Y and show that it has the desired properties and the third section gives some additional informations about the space Y.

1. Definitions and preliminary results. All mappings are assumed to be continuous and onto. Open and compact mappings are the open mappings with compact fibers. The term discrete (σ-discrete) will mean relatively (σ)-discrete.

Most of the pathological spaces in MOBI_i ($i = 2, 3$) constructed thus far are, in fact, in the smaller class MOBI_i(σ-discrete). It turns out (see [Ch4]) that the requirement that the counterexample be in the smaller class, instead of being an additional obstacle in the construction, actually helps by restricting the field of search. Since it is more difficult to find a non-weakly θ-refinable space in MOBI_3 than in MOBI_2, we shall start by adding some more restrictions. We shall look for our counterexample in the class MOBI_3 (scattered). From [N, 3.13, 17, 20], it follows that a metric space is scattered if and only if it is complete and σ-discrete.

It will be convenient to use a non-standard definition of scattered spaces (see [WW2] and [Ch2]).

DEFINITION 1.1. A well-ordered collection $\mathscr{P} = \langle P_\eta \colon \eta \in \nu \rangle$ of subsets of a space Z will be called a *primitive partition of* Z *of type* ν if $\{P_\eta \colon \eta \in \nu\}$ is a partition of Z and $\bigcup \{P_\eta \colon \eta \in \xi\}$ is open in Z for $\xi \in \nu$.

DEFINITION 1.2. A space Z is called *scattered* if it has a primitive partition into discrete subsets; the *height* of Z, denoted by $sht(Z)$, is the smallest type of such a partition.

Observe that the space of countable ordinals is a scattered space of height ω_1 ($\langle \{\alpha\} \colon \alpha \in \omega_1 \rangle$ is a primitive partition of ω_1 of type ω_1 and $sht(\omega_1) \geqslant \omega_1$ follows from the fact that discrete subsets of ω_1 are not stationary, hence ω_1 is not σ-discrete).

Note that there is a one-to-one correspondence between primitive partitions of a space Z and increasing open covers of Z. Since open mappings preserve increasing open covers, we get

LEMMA 1.3. *Suppose* $g \colon Z_1 \to Z_2$.

(a) *If* g *is an open mapping and* Z_1 *is scattered, then* Z_2 *is also scattered and* $sht(Z_2) \leqslant sht(Z_1)$.

(b) *If* g *is a mapping with discrete fibers and* Z_2 *is scattered, then* Z_1 *is also scattered and* $sht(Z_1) \leqslant sht(Z_2)$.

(c) *If* g *is a mapping with scattered fibers and* Z_2 *is scattered, then* Z_1 *is also scattered. If, moreover, all the fibers of* g *are of height not greater than* ν, *then* $sht(Z_1) \leqslant sht(Z_2) \cdot \nu$.

In particular, from 1.3(a) it follows that all spaces in MOBI (scattered) are scattered.

We now turn to the weak covering properties.

DEFINITION 1.4 [BL, Lemma 4]. A space Y is *weakly* θ*-refinable* if every open cover of Y has a σ-discrete refinement.

Clearly σ-discrete spaces are (hereditarily) weakly θ-refinable. Since scattered spaces of countable height are σ-discrete, it follows that a non-weakly θ-refinable scattered space has to have height at least ω_1.

In [N, 3.4.I] it is shown implicitly, that a scattered space of height ω_1 is weakly θ-refinable if and only if it is σ-discrete. Since we shall use this fact in our construction, we shall include, for the sake of completeness, a simple proof (of a slight generalization) of this result. Note that the condition characterizing hereditarily weakly θ-refinable spaces, given in 1.5(b), is equivalent to condition (*) used in [Bu] to show that perfect images of quasi-developable spaces are quasi-developable.

PROPOSITION 1.5. (a) *If* Y *is a weakly* θ*-refinable space, then any primitive partition of* Y *of type* ω_1 *has a* σ*-discrete refinement.*

(b) *The space* Y *is hereditarily weakly* θ*-refinable if and only if any primitive partition of* Y *has a* σ*-discrete refinement.*

Proof. We shall only prove the first part. The second part (we shall not use it) can be proved by induction and the inductive step is similar to the proof of part (a) (see the proof of an analogous characterization of perfectly subparacompact spaces in [J2]).

First observe that if $\mathscr{F}$ is a discrete collection and each $F \in \mathscr{F}$ has a cover $\mathscr{E}(F) = \bigcup \{\mathscr{E}_n(F) \colon n \in \omega\}$ such that each $\mathscr{E}_n(F)$ is a discrete collection of subsets of F, then $\mathscr{E}_n = \bigcup \{\mathscr{E}_n(F) \colon F \in \mathscr{F}\}$ is discrete for $n \in \omega$ and $\mathscr{E} = \bigcup \{\mathscr{E}_n \colon n \in \omega\} = \bigcup \{\mathscr{E}(F) \colon F \in \mathscr{F}\}$ is a σ-discrete cover of $\bigcup \mathscr{F}$.

Suppose that $\mathscr{P} = \langle P_\alpha \colon \alpha \in \omega_1 \rangle$ is a primitive partition of Y. For $\beta \in \omega_1$ put $U_\beta = \bigcup \{P_\alpha \colon \alpha \in \beta\}$ and let $\mathscr{F} = \bigcup \{\mathscr{F}_m \colon m \in \omega\}$ be a refinement of $\{U_\beta \colon \beta \in \omega_1\}$ such that each $\mathscr{F}_m$ is discrete. Fix $m \in \omega$ and consider an $F \in \mathscr{F}_m$. Since $\mathscr{P}$ restricted to F has countable type, it follows that $\mathscr{P}$ has a σ-discrete (in fact, countable) refinement covering F. Using the observation above, we can obtain a σ-discrete collection $\mathscr{E}(m)$ refining $\mathscr{P}$ and covering $\bigcup \mathscr{F}_m$. Clearly, $\bigcup \{\mathscr{E}(m) \colon m \in \omega\}$ is a σ-discrete cover of Y refining $\mathscr{P}$.

COROLLARY 1.6 [N, 3.4]. (a) *A scattered space of height* ω_1 *is weakly* θ*-refinable if and only if it is* σ*-discrete.*

(b) *A scattered space is hereditarily weakly* θ*-refinable if and only if it is a* σ*-discrete.*

In our construction of a non-weakly θ-refinable space in MOBI_3 we will make a strong use of the notion of a neighbornet [J1].

Recall that a *neighbornet* for a space Y is a relation $V \subset Y \times Y$ such that for each $y \in Y$, $y \in \operatorname{Int} V(y)$, where $V(y) = \{z \in Y \colon \langle y, z \rangle \in V\}$. We shall only consider neighbornets V such that $V(y)$ is open in Y for $y \in Y$. The neighbornet V is called *co-countable* (*co-finite*) if $V^{-1}(y) = \{z \in Y \colon y \in V(z)\}$ is countable (finite) for each $y \in Y$ [J3].

It is easy to check that first-countable spaces with a co-countable neighbornet have a point-countable base. In the class of first countable T_1-spaces the property of having a co-finite neighbornet is equivalent to being a σ-discrete metacompact and developable space (see [J3]).

The following lemma will be used to show that the space that we are going to construct has co-finite neighbornets locally. This lemma is a part of the proof of the equivalence mentioned above. We include its proof for the sake of completeness.

LEMMA 1.7. *Assume that a space Z has a countable cover $\{F_n: n \in \omega\}$ by closed discrete subspaces. If for each $n \in \omega$, there exists a pairwise disjoint (or, even weaker, point-finite) collection $\{B_n(z): z \in F_n\}$ of open sets such that $B_n(z) \cap F_n = \{z\}$ for $z \in F_n$, then Z has a co-finite neighbornet.*

Proof. For each $z \in Z$, define $V(z) = B_n(z) \setminus \bigcup \{F_m: m < n\}$, where n is the first natural number such that $z \in F_n$. It is easy to see that $\{V(z): z \in Z\}$ defines a co-finite neighbornet in Z.

In [J3] first-countable T_1-spaces with co-finite neighbornets have been characterized as open finite-to-one images of σ-discrete metric spaces (another proof of this characterization is given in [Ch.3, Proposition 1]). From 1.3(a) and 1.3(b) it follows that first-countable scattered T_1-spaces with co-finite neighbornets can be characterized as open finite-to-one images of scattered metric spaces.

The next lemma, together with the observation above, shows that any first-countable zero-dimensional (scattered) space having a co-countable neighbornet and, locally, co-finite neighbornets is in MOBI_3(σ-discrete) (MOBI_3(scattered)). In the second section we shall construct a non-weakly θ-refinable space having these properties.

LEMMA 1.8. *If Y is a first-countable zero-dimensional (and scattered) space having a co-countable neighbornet and, locally, co-finite neighbornets, then there exists an open and compact mapping $f: X \to Y$ of a first-countable zero-dimensional (and scattered) space X having a co-finite neighbornet onto Y.*

Proof. First choose $\{V(y): y \in Y\}$ defining a co-countable neighbornet such that each $V(y)$ is clopen in Y and each $V(y)$ has a co-finite neighbornet given by a clopen collection $\{W_y(z): z \in V(y)\}$. Next, note that, since Y is first-countable and $V^{-1}(z)$ is countable, we can assume (by consecutively replacing $\{W_y(z): y \in V^{-1}(z)\}$ with smaller clopen sets if necessary) that, for each $z \in Y$, if $z \in U$ open in Y, then $\{y \in V^{-1}(z): W_y(z) \not\subset U\}$ is finite. In particular, if $V^{-1}(z)$ is infinite, then $\{W_y(z): y \in V^{-1}(z)\}$ is a base at z.

We are now ready to construct f and X. Our construction is a modification of a construction from [B2] (see [Ch4, 4.4]).

Put $X' = \{\langle y, z\rangle \in Y \times Y: z \in V(y)\} = V \subset Y \times Y$ and consider X' with the topology of the subspace of $D(Y) \times Y$, where $D(Y)$ is Y with the discrete topology (less formally, $X' = \oplus\{V(y): y \in Y\}$).

Let $e: X' \to Y$ be the projection onto the second factor. Clearly $e^{-1}(z) = V^{-1}(z)$.

Put $X = X' \cup \{z \in Y: e^{-1}(z) \text{ is infinite}\}$ and consider X with a topology such that X' is an open subspace of X while neighborhoods of a point $z \in X \setminus X'$ are of the form $B(z, \varphi) = \{z\} \cup \bigcup \{\{y\} \times W_y(z): y \in e^{-1}(z) \setminus \varphi\}$, where φ is a finite subset of $e^{-1}(z)$.

Clearly X is a first-countable Hausdorff space with a co-finite neighbornet and the natural function $f: X \to Y$ is continuous, open, onto and the infinite fibers of f are convergent sequences. If Y is a scattered space, then it follows from 1.3(c) that X is also a scattered space. It remains to prove that the space X is zero-dimensional, more precisely, it is enough to show that the sets of the form $B(z, \varphi)$ are clopen in X.

Consider a set $B(z_1, \varphi)$. It is clear that this set has no boundary points in X'. Pick a $z_2 \neq z_1$ in $X \setminus X'$. Since Y is a Hausdorff space, there exist disjoint open sets U_1, U_2 in Y containing z_1 and z_2 respectively. Let φ_i be a finite subset of $e^{-1}(z_i)$ such that $W_y(z_i) \subset U_i$ for $y \in e^{-1}(z_i) \setminus \varphi_i$, where $i = 1, 2$. It is easy to see that for $\psi = \varphi_2 \cup (e^{-1}(z_2) \cap \varphi_1)$ we have $B(z_1, \varphi) \cap B(z_2, \psi) = \emptyset$.

2. A non-weakly θ-refinable space in MOBI_3. By Lemma 1.8, in order to construct a non-weakly θ-refinable space in MOBI_3, it is sufficient to construct a first-countable zero-dimensional non-weakly θ-refinable space Y having a co-countable neighbornet and, locally, co-finite neighbornets.

The space Y will be a modification of the space ω_1 of countable ordinals. Observe that the space ω_1 is first-countable, zero-dimensional, non-weakly θ-refinable [BL] (in fact, ω_1 is a non-σ-discrete scattered space of height ω_1, see 1.6.(a)) and has, locally, co-finite neighbornets (in fact, is locally countable, see 1.7). In order to construct the space Y having these properties and, in addition, a co-countable neighbornet we shall apply to ω_1 a modification similar to the one used in [Ch4, 3].

Let Y be the set of all one-to-one functions from countable successor ordinals into ω_1. For a function $s \in Y$, let $|s|$ denote the ordinal which is the domain of s and let $e(s) = s(\{|s|-1\})$ be the last term of s.

Fix a decreasing sequence $\langle D_k: k \in \omega\rangle$ of uncountable subsets of ω_1 having empty intersection and such that $D_0 = \omega_1$.

For $s \in Y$, $k \geqslant 0$ and a clopen neighborhood U of $e(s)$ in ω_1 define

$$B(s, k, U) = \{s\} \cup \{t \in Y \setminus \{s\}: t \supset s,\ t(\{|s|\}) \in D_k \text{ and } e(t) \in U\}.$$

Observe that $t \in B(s, k, U)$, $l \geqslant k$ and $e(t) \in V \subset U$ imply that $B(t, l, V) \subset B(s, k, U)$. This shows that the sets $B(s, k, U)$ form a base for a (first-countable) topology on Y.

We shall show that Y has the desired properties.

To simplify the notation, put $B(s) = B(s, 0, \omega_1)$ for $s \in Y$.

2.1. *The function $e: Y \to \omega_1$ is continuous onto and*

$$e(B(s, k, U)) = U \setminus s(\{|s|-1)\} \subset U;$$

moreover, the restriction of e to the set $Y_\omega = \{s \in Y: s \text{ is finite}\}$ is an open mapping.

2.2. *For each $\alpha \in \omega_1$, the fiber $e^{-1}(\{\alpha\})$ is closed, discrete and has a pairwise disjoint open expansion in Y.*

Proof. Consider $\{B(s)\colon s \in e^{-1}(\{\alpha\})\}$. If $s, s' \in e^{-1}(\{\alpha\})$ and $t \in B(s) \cap B(s')$, then $s \subset t$ and $s' \subset t$. Since $e(s) = \alpha = e(s')$, the fact that t is one-to-one implies $|s| = |s'|$ and, consequently, $s = s'$.

2.3. *The space Y is a scattered space of height ω_1.*

Proof. From 1.3(b), 2.1 and 2.2, it follows that Y is scattered and $sht(Y) \leqslant sht(\omega_1)$. On the other hand, from 1.3(a) and 2.1, we obtain, $sht(\omega_1) \leqslant sht(Y_\omega) \leqslant sht(Y)$. Thus, $sht(Y) = sht(\omega_1) = \omega_1$.

2.4. *The space Y is a Hausdorff space.*

Proof. This follows easily from 2.1 and 2.2.

2.5. *The space Y is a zero-dimensional space.*

Proof. We shall show that each basic open set $B = B(s, k, U)$ (with U clopen in ω_1) is a clopen subset of Y. Suppose that $s' \notin B$. By 2.1, if $e(s') \in U$, then $e^{-1}(\omega_1 \setminus U)$ is a neighborhood of s' disjoint from B. Assume now that $e(s') \in U$ and $B(s') \cap B \neq \emptyset$, which implies either $s' \supset s$ or $s \supset s'$. If $s \supset s'$, then $s' \notin B$ and $e(s') \in U$ imply $s'(\{|s|\}) \notin D_k$ and this contradicts $B(s') \cap B \neq \emptyset$. Thus we are left with the case $s' \supset s$. Choose an l such that $s(\{|s'|\}) \notin D_l$. Clearly, $B(s', l, \omega_1) \cap B = \emptyset$ and the proof of 2.5 is finished.

2.6. *The space Y has a co-countable neighbornet.*

Proof. It is easy to see that $\{B(s)\colon s \in Y\}$ defines a co-countable neighbornet in Y.

2.7. *The space Y has, locally, co-finite neighbornets.*

Proof. This follows from 1.7, 2.1 and 2.2.

2.8. *The space Y is not weakly θ-refinable.*

Proof. Suppose that Y is weakly θ-refinable. From 2.3 and 1.6(a) it follows that $Y = \bigcup\{A_n\colon n \in \omega\}$, where each A_n is a discrete subset of Y. Before showing that this gives a contradiction, we shall introduce some notation and indicate connections between Y and the non-weakly θ-refinable space S^* with a point-countable base constructed in [Ch4]. Our proof can be viewed as a modification of the proof that S^* is not weakly θ-refinable.

For a function $s \in Y$ let $p(s)$ be the restriction of s to $|s|-1$. Thus s can be identified with the pair $\langle \alpha, p\rangle$, where $\alpha = e(s)$ and $p = p(s)$. Let $Y^* = \{\langle e(s), p(s)\rangle\colon s \in Y\}$ and define $p^\frown\alpha = s \in Y$ for the pair $\langle \alpha, p\rangle = \langle e(s), p(s)\rangle \in Y^*$.

For $p^\frown\alpha \in Y$ put

$$C(p^\frown\alpha) = \{q^\frown\alpha \in Y\colon q \supset p \text{ and } \mathrm{rg}(q\setminus p) \cap \alpha+1 = \emptyset\},$$

where $\mathrm{rg}(q\setminus p)$ denotes the range of the function $q\setminus p$. Less formally, $q^\frown\alpha$ is in $C(p^\frown\alpha)$ if q extends p through the ordinals greater than α.

Note that if $\alpha \in \omega_1$ is fixed, then $\mathscr{C}_\alpha = \{C(p^\frown\alpha)\colon p^\frown\alpha \in Y\}$ is a base of a T_0-topology in $e^{-1}(\{\alpha\})$. Since the intersection of any decreasing sequence of sets from $\mathscr{C}_\alpha$ is also in $\mathscr{C}_\alpha$, it follows that $e^{-1}(\{\alpha\})$ with this topology satisfies the Baire Theorem. In particular, since each open set $C(p^\frown\alpha)$ is covered by

$$\{C(p^\frown\alpha) \cap A_n\colon n \in \omega\},$$

the Baire Theorem implies that for a certain $n \in \omega$ and an open subset $C(q^\frown\alpha)$ of $C(p^\frown\alpha)$, A_n has to intersect any open subset $C(r^\frown\alpha)$ of $C(q^\frown\alpha)$. This means that

(*) for each $p^\frown\alpha \in Y$, there exists an $n \in \omega$ and a $q^\frown\alpha \in C(p^\frown\alpha)$ such that if $r^\frown\alpha \in C(q^\frown\alpha)$, then $C(r^\frown\alpha) \cap A_n \neq \emptyset$.

Using (*), we can construct, by induction on $\eta \in \omega_1$, transfinite sequences $\langle \alpha_\eta\colon \eta \in \omega_1\rangle$ of countable ordinals, $\langle n_\eta\colon \eta \in \omega_1\rangle$ of natural numbers, $\langle q_\eta\colon \eta \in \omega_1\rangle$ of functions such that $q_\eta{}^\frown\alpha_\eta \in Y$ and a transfinite sequence $\langle B_\eta\colon \eta \in \omega_1\rangle$ of neighborhoods of $q_\eta{}^\frown\alpha_\eta$ in Y such that for all $\eta \in \omega_1$ the following conditions are satisfied:

(0) $p_\eta = \bigcup\{q_\xi\colon \xi \in \eta\}$ is a function,

(1) $\alpha_\eta = \min\{\alpha\colon \mathrm{rg}(p_\eta) \subset \alpha\}$, $q_\eta{}^\frown\alpha_\eta \in C(p_\eta{}^\frown\alpha_\eta) \setminus \{p_\eta{}^\frown\alpha_\eta\}$,

(2) if $r^\frown\alpha_\eta \in C(q_\eta{}^\frown\alpha_\eta)$ then $C(r^\frown\alpha_\eta) \cap A_{n_\eta} \neq \emptyset$,

(3) $q_\eta{}^\frown\alpha_\eta \in A_{n_\eta}$,

(4) $B_\eta = B(q_\eta{}^\frown\alpha_\eta, k_\eta, (\gamma_\eta, \alpha_\eta])$ and $B_\eta \cap A_{n_\eta} = \{q_\eta{}^\frown\alpha_\eta\}$.

To see that such a construction is possible, note that (0) gives $p_0 = \emptyset$. Assume that p_η is given by (0) after all the sequences have been defined up to the level $\eta \in \omega_1$. Condition (1) then gives α_η and forces $\mathrm{rg}(q_\eta) \setminus (\alpha_\eta+1) \neq \emptyset$ (this assures that $\alpha_{\eta+1} > \alpha_\eta$ and, in fact, that the set $\{\alpha_\eta\colon \eta \in \omega_1\}$ will be a closed unbounded subset of ω_1). Now, we can use (*) to extend, if necessary, q_η (without destroying the second part of (1)) and find an n_η so that (2) is satisfied. Furthermore, we can use (2) with $r = q_\eta$ in order to extend, if necessary, q_η (without destroying (1), (2)) so that (3) is satisfied and we can find a basic neighborhood B_η of $q_\eta{}^\frown\alpha_\eta$ in Y as in (4) by using the fact that A_{n_η} is discrete in Y.

This completes the inductive construction. As we have observed, (1) implies that the set $C = \{\alpha_\eta\colon \eta \in \omega_1\}$ is a closed and unbounded subset of ω_1. Since $\gamma_\eta < \alpha_\eta$ (see (4)), we can find a $\gamma \in \omega_1$, $n \geqslant 0$ and a stationary subset $S \subset C$ such that $\gamma_\eta = \gamma$ and $n_\eta = n$ for $\alpha_\eta \in S$. Pick $\alpha_\xi < \alpha_\eta$ in S and $\delta \in D_{k_\eta} \setminus \alpha_{\eta+1}$. Conditions (0) and (1) imply that for $r = q_\eta{}^\frown\alpha_\eta{}^\frown\delta$, $r^\frown\alpha_\xi \in C(q_\xi{}^\frown\alpha_\xi)$ and from (2), it follows that $C(r^\frown\alpha_\xi) \cap A_n \neq \emptyset$. Since, on the other hand, $C(r^\frown\alpha_\xi) \subset B_\eta$, we obtain a contradiction with (4).

3. Additional properties of the example. In further investigations of Y, we shall need some new notation. We shall identify a function $b\colon \eta \to \omega_1$ with the sequence $\langle b(\xi)\colon \xi \in \eta\rangle$.

For two sequences $b_0, b_1\colon \eta \to \omega_1$, let $b_0 * b_1\colon \eta\cdot 2 \to \omega_1$ be a sequence whose "even" terms are the consecutive terms of b_0 and "odd" terms are the consecutive terms of b_1. More precisely, $b_0 * b_1(\tau(\xi, i)) = b_i(\xi)$, where $i = 0, 1$ and τ is the identification of $\eta\times 2$ with $\eta\cdot 2$ inducing the lexicographic order on $\eta\times 2$.

Let Λ denote the set of limit ordinals in ω_1 and let $\Sigma = \omega_1\setminus\Lambda$.

3.1. *The space Y is neither normal nor perfect. Moreover, there exists a countable discrete in Y collection $\{E_n: n\in\omega\}$ of closed subsets of Y contained in $e^{-1}(\Lambda)$ such that if $E_n\subset U_n$ is open in Y for $n\in\omega$, then $\bigcap\{U_n: n\in\omega\}\cap e^{-1}(\Sigma)\neq\emptyset$.*

Proof. Since the sequence $\langle D_k: k\in\omega\rangle$ is decreasing and its intersection is empty, there exists an $l\in\omega$ such that $I = \Sigma\setminus D_l$ is uncountable.

Let C_k be the set of limit points of D_k in ω_1 and put $C = \bigcap\{C_k: k\in\omega\}$. Clearly, C is a closed unbounded subset of ω_1 and, for each $k\in\omega$, $D_k' = D_k\cap D_l\setminus C$ is an uncountable subset of D_k disjoint from $I\subset\Sigma$ and $C\subset\Lambda$.

Fix a partition $\{S_n: n\in\omega\}$ of C such that each S_n is a stationary subset of ω_1 (see [K, 2.6.12]). For $n\in\omega$ put

$$E_n = \{s\in e^{-1}(S_n):\ C\cap \mathrm{rg}(s)\setminus e(s)\subset S_n\}.$$

Suppose that s is in $e^{-1}(C)$, $e(s) = \alpha\in S_m$ and the neighborhood $B(s, 0, [0, \alpha])$ of s in Y contains a $t\in E_n$. Since $s\subset t$ and $e(t)\in[0,\alpha]$, we get

$$\alpha\in C\cap\mathrm{rg}(s)\setminus e(s)\subset \mathrm{rg}(t)\setminus e(t).$$

Thus, from $t\in E_n$ and $\alpha\in S_m$, it follows that $n = m$ and $s\in E_n$. This shows that the collection $\{E_n: n\in\omega\}$ is closed and discrete in $e^{-1}(C)$ and, consequently, in Y.

Assume now that $E_n\subset U_n$ and U_n is open in Y for $n\in\omega$. For each $n\in\omega$ and all $s\in E_n$ fix open sets $B_s = B(s, k(s), (\gamma(s), e(s)])$ contained in U_n. We shall construct a $t\in e^{-1}(I)$ and a sequence $\langle s_n: n\in\omega\rangle$ such that $s_n\in E_n$ and $t\in\bigcap\{B_{s_n}: n\in\omega\}\subset\bigcap\{U_n: n\in\omega\}$.

Let $C = \{\alpha_\eta: \eta\in\omega_1\}$ be an increasing enumeration of C and, for each $\eta\in\omega_1$, choose the $n_\eta\in\omega$ such that $\alpha_\eta\in S_{n_\eta}$.

By induction on $\eta\in\omega_1$, we can define sequences $\langle\delta_\eta: \eta\in\omega_1\rangle$ of countable ordinals and $\langle s_\eta: \eta\in\omega_1\rangle$ of elements of Y such that, for $\eta\in\omega_1$,

(1) $$s_\eta = \langle\alpha_\xi: \xi\in\eta\rangle * \langle\delta_\xi: \xi\in\eta\rangle^\frown\alpha_\eta\in E_{n_\eta},$$

(2) $$\delta_\eta\in D'_{k(s_\eta)}\setminus\mathrm{rg}(s_\eta).$$

For each $n\in\omega$, let $\gamma_n\in\omega_1$ be an ordinal such that $T_n = \{\alpha_\eta\in S_n: \gamma(s_\eta) = \gamma_n\}$ is stationary.

Choose an ordinal $\beta\in I\setminus\sup\{\gamma_n: n\in\omega\}$ and, for each $n\in\omega$, find η_n such that $\beta<\alpha_{\eta_n}\in T_n$. Put $s_n = s_{\eta_n}\in E_n$ and let $\eta = \sup\{\eta_n: n\in\omega\}$. Since $s_n\subset s_\eta$ and $\gamma_n<\beta<e(s_n) = \alpha_{\eta_n}\in T_n$, (2) implies that $t = s_\eta^\frown\beta\in\bigcap\{B_{s_n}: n\in\omega\}$.

The second part of 3.1 shows that the space Y is very far from being normal or countably metacompact (hence perfect). In fact, by partitioning ω into infinitely many infinite subsets and grouping the corresponding sets E_n, one obtains a discrete in Y collection $\mathscr{F}$ of closed subsets of $e^{-1}(\Lambda)$ such that any expansion of $\mathscr{F}$ by G_δ-subsets of Y has a common point in $e^{-1}(\Sigma)$.

Another consequence of the second part of 3.1 is given by

3.2. *The space Y does not have a G_δ-diagonal and is not a p-space (see [G]); in particular, Y is not Čech complete.*

Proof. Suppose that, for each $n\in\omega$ and $s\in Y$, $B_n(s)$ is a basic open neighborhood of s. Put $U_n = \bigcup\{B_n(s): s\in E_n\}$. From 3.1, it follows that there exist a sequence $\langle s_n: n\in\omega\rangle$ of elements of $e^{-1}(\Lambda)$ and a $t = p^\frown\beta\in e^{-1}(\Sigma)$ such that $t\in\bigcap\{B_n(s_n): n\in\omega\}$. Choose a $\delta\in D_k\setminus\mathrm{rg}(t)$, where $k = \max\{l: \beta\in D_l\}$.

It is easy to check that the large closed and discrete set $C(p^\frown\delta^\frown\beta)$ is contained in $\bigcap\{B_n(s_n): n\in\omega\}$ (we use $p^\frown\delta$ instead of p in order to assure that $C(p^\frown\delta^\frown\beta)\subset B_n(s_n)$ for n such that $s_n = p$).

In order to investigate the local properties of Y, we define the notion of height of a point in a scattered space. This is usually done by using the derived sets of the space to construct a standard primitive partition of the space into discrete sets (these sets are called levels in [N, 2]). Our definition of scattered spaces leads to a different approach.

For a point z of a scattered space Z, the *height* of z in Z denoted by $sht(z, Z)$ (or $sht(z)$ if it does not lead to a misunderstanding) will be the ordinal $\min\{sht(V): V$ is a neighborhood of z in $Z\}$, where $sht(V)$ is given by 1.2.

Observe that, according to the above definition, the height of a point is always a non-limit ordinal and $\{z\in Z: sht(z) = 1\}$ is the set of the isolated points of Z.

Note that every point $z\in Z$ has a neighborhood V such that $sht(z) = sht(V) > sht(V)\setminus\{z\}$. This, in particular, implies that the sets $L_\eta = \{z\in Z: sht(z) = \eta+1\}$, where $\eta\in sht(Z)$, form a primitive partition of Z into discrete sets (see [N, 2]).

We shall show that a point $s\in Y$ has a metrizable neighborhood if and only if its height is not greater that 3.

First, we shall give a description of the levels of Y.

3.3. *For a point $s = p^\frown\alpha\in Y$, $sht(s, Y) = sht(\alpha, \omega_1\setminus\mathrm{rg}(p))$.*

Proof. For a clopen neighborhood U of α in ω_1 and a neighborhood $B = B(s, k, U)$ of s in Y the restriction of e to B maps B onto $e(B) = U\setminus\mathrm{rg}(p)$ (see 2.1). Moreover, the restriction of e to $B_\omega = \{t\in B: t\setminus s$ is finite$\}$ is an open mapping from B_ω onto $e(B)$. Thus 1.3(a) and (b) imply (as in the proof of 2.3) that $sht(B) = sht(U\setminus\mathrm{rg}(p))$. Since the sets B (the sets $U\setminus\mathrm{rg}(p)$) as above form a base of neighborhoods of s in Y (of α in $\omega_1\setminus\mathrm{rg}(p)$), we obtain $sht(s, Y) = sht(\alpha, \omega_1\setminus\mathrm{rg}(p))$.

We are now ready to prove

3.4. *If $s\in Y$ and $sht(s)\leqslant 3$, then s has a metrizable neighborhood in Y.*

Proof. Let $s = p^\frown\lambda$ and put $Z = \omega_1\setminus\mathrm{rg}(p)$. Since $sht(\lambda, Z)\leqslant 3$, we can find a $\gamma<\lambda$ such that $sht((\gamma,\lambda)\cap Z)\leqslant 2$.

In order to show that the clopen neighborhood $B = B(s, 0, (\gamma, \lambda])$ of s has a base which is σ-discrete in Y, it suffices to show that for each $\alpha \in e(B)$, $e^{-1}(\{\alpha\}) \cap B$ has an open discrete in Y expansion. If $\alpha = \lambda$, then $e^{-1}(\{\alpha\}) \cap B = \{s\}$, so we can assume that $\alpha \neq \lambda$ and since $\alpha \in e(B) \setminus \{\lambda\} = (\gamma, \lambda) \cap Z$, $sht(\gamma, Z) \leqslant 2$.

If $sht(\alpha, Z) = 1$, then $e^{-1}(\{\alpha\}) \cap B$ is closed, discrete in Y and consists of isolated points of Y. If $sht(\alpha, Z) = 2$, then there exists a neighborhood $(\gamma', \alpha] \cap Z$ of α in Z such that $sht((\gamma', \alpha] \cap Z) = 1$. Put $B' = B \cap e^{-1}((\gamma', \alpha])$. Since $B' \setminus e^{-1}(\{\alpha\})$ consists of isolated points of Y, the restriction to B' of the pairwise disjoint open expansion of $e^{-1}(\{\alpha\}) \cap B'$ given by 2.2 is, in fact, discrete in B' and hence in Y.

3.5. *If a point of Y has height greater than 3, then it has no metrizable neighborhood. Moreover, no neighborhood of this point is even σ-paralindelöf.*

Proof. In order to simplify the notation, we shall only prove that if $\alpha \in \omega_1$ and $sht(\alpha, \omega_1) > 2$, then no open expansion of $e^{-1}(\{\alpha\})$ in Y is σ-locally countable. After the proof of 3.4, it should be clear that the same reasoning gives 3.5.

Fix $\alpha \in \omega_1$ of height greater than 2 and an increasing to α sequence $b = \langle \beta_n: n \in \omega \rangle$ of limit ordinals. Suppose that $\mathscr{B} = \{B_p: p^\frown\alpha \in Y\}$, where $B_p = B(p^\frown\alpha, k_p, (\gamma_p, \alpha])$ is an open expansion of $e^{-1}(\{\alpha\})$ in Y. We shall find an $s \in Y$ such that every neighborhood of s intersects continuum many elements of $\mathscr{B}$ and it will be easy to see how to modify our reasoning if $\mathscr{B}$ is partitioned into countably many subcollections.

For each $n \in \omega$ fix a one-to-one function $\Delta_n: \omega \to \omega_1 \setminus (\alpha+1)$ such that $\Delta_n(l) \in D_l$ for $l \in \omega$ and $\mathrm{rg}(\Delta_n) \cap \mathrm{rg}(\Delta_m) = \emptyset$ if $m \neq n$.

Define $D = \Pi\{\mathrm{rg}(\Delta_n): n \in \omega\}$ and put $p_d = b * d$: $\omega \cdot 2 = \omega \to \omega_1$ for $d \in D$.

For each $d \in D$, $p_d{}^\frown\alpha \in Y$ and we can use the Baire property in $C(p_d{}^\frown\alpha)$ (see the proof of 2.8) to find a $\gamma(d) < \alpha$ and a $q(d) \supset p_d$ such that $q(d)^\frown\alpha \in C(p_d{}^\frown\alpha)$ and

(*) for each $r^\frown\alpha \in C(q(d)^\frown\alpha)$ there exists a $p^\frown\alpha \in C(r^\frown\alpha)$ such that $\gamma_p = \gamma(d)$.

Consider the set D with the Baire metric. The Baire Theorem for D implies that there exists a $\gamma < \alpha$ such that $D(\gamma) = \{d \in D: \gamma(d) = \gamma\}$ is dense in an open subset of D.

Choose an $n \in \omega$ such that $\beta_n > \gamma$ and $D(\gamma)$ is dense in a ball $G = \{d \in D: d \supset \langle \delta_m: m < n \rangle\}$.

Let $s \in Y$ be the extension of $\langle \beta_m: m < n \rangle * \langle \delta_m: m < n \rangle$ by β_n and suppose that $B = B(s, l, V)$ is a neighborhood of s in Y. Since $\beta_n = e(s) \in V$ is a limit ordinal and $\beta_n > \gamma$, we can find an ordinal $\beta \in V$ such that $\gamma < \beta < \beta_n$. Furthermore, put $\delta_n = \Delta_n(l) \in D_l$ and consider $H = \{d \in D: d \supset \langle \delta_m: m \leqslant n \rangle\}$. Since $H \subset G$ and H is open in D, there exists a $d \in H \cap D(\gamma)$. Our construction assures that $\gamma(d) = \gamma$ and, since $q(d) \supset p_d \supset s$, any $t \in Y$ such that $t \supset q(d)$ and $e(t) = \beta$ is an element of B. We shall show that continuum many different elements of B contain points t satisfying the above two conditions.

For the $d \in D(\gamma)$ and any $r^\frown\alpha \in C(q(d)^\frown\alpha)$, we can use (*) to find $p^\frown\alpha \in C(r^\frown\alpha)$ such that $\gamma_p = \gamma(d) = \gamma$. It is easy to see that if $\delta \in D_{k_p} \setminus (\mathrm{rg}(p) \cup (\alpha+1))$, then $t = p^\frown\alpha^\frown\delta^\frown\beta \in B_p \cap B$. Since there are continuum many pairwise incomparable functions r such that $r^\frown\alpha \in C(q(d)^\frown\alpha)$, this gives continuum many different elements of $\mathscr{B}$ intersecting the neighborhood B of s.

Since the pairwise disjoint open expansions given by 2.2 cannot be shrunk to open and discrete in Y expansions, we obtain another proof of the fact that Y is not a normal space (see 3.1).

3.6. *If a point of Y has height greater than 3, then it has no normal neighborhood.*

In [R], a non-σ-discrete right and left separated space was constructed with the use of the continuum hypothesis. The space Y is an absolute example having the same properties (see [GJ]).

3.7. *The space Y is both right and left separated.*

Proof. Since Y is scattered, it is right separated (see [GJ] for the definitions).

The natural lexicographic order on Y shows that Y is left separated. In fact, by [J3] and the proof of (c)$\Rightarrow$(a) in Theorem 2.2 of [F], all T_1-spaces with co-countable neighbornets are left separated.

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