On sets of points of semicontinuity in fine topologies generated by an ideal

by

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Abstract. Let τ be a fine topology generated by an ideal on a metric space X and lef f be a real function defined on X. We study connections between the set of points of metric continuity, the set of points of fine continuity and the set of points of fine semicontinuity.

Let (X, τ) be a topological space. For a function $f\colon X \to R$ and a point $x \in X$ we define as usual

$$\limsup_{y \to x} f(y) = \inf\{t \in R,\ x \in \operatorname{Int}(\{z \in X, f(z) < t\} \cup \{x\})\} \quad \text{and}$$

$$\liminf_{y \to x} f(y) = \sup\{t \in R,\ x \in \operatorname{Int}(\{z \in X, f(z) > t\} \cup \{x\})\} ;$$

here $\operatorname{Int}(A)$ denotes the *interior* of the set A. Further, we put

$$C(f) = \{x,\ \limsup_{y \to x} f(y) \leqslant f(x) \leqslant \liminf_{y \to x} f(y)\},$$

$$S^{+}(f) = \{x,\ \limsup_{y \to x} f(y) \leqslant f(x)\},$$

$$T^{+}(f) = \{x,\ \limsup_{y \to x} f(y) < f(x) \text{ and } x \text{ is not isolated}\},$$

$$S^{-}(f) = \{x,\ \liminf_{y \to x} f(y)\},$$

$$T^{-}(f) = \{x,\ \liminf_{y \to x} f(y) > f(x) \text{ and } x \text{ is not isolated}\} \quad \text{and}$$

$$\mathrm{CH}(f) = (C(f), S^{+}(f), T^{+}(f), S^{-}(f), T^{-}(f)).$$

Clearly, $C(f)$ resp. $S^{+}(f)$, $S^{-}(f)$ is the set of points of continuity resp. upper, lower semicontinuity of the function f.

In the present paper we deal with the question of which quintuples of subsets of X are of the type $\mathrm{CH}(f)$ for some $f\colon X \to R$. For general topological spaces this problem is far from being solved. Here we give a complete characterization of such quintuples for τ belonging to a fairly large subclass of the class of all fine topologies generated by ideals. Therefore we recall the notion of the fine topology generated by an ideal. Let (X, ϱ) be a metric space. In the whole paper we denote by $\mathfrak{Z}$ an ideal of subsets of X satisfying the following (local-global) condition:

(LG): If $A \subset X$ is such that for each $x \in A$ there exists a neighbourhood U_x satisfying $U_x \cap A \in \mathfrak{Z}$ then A belongs to $\mathfrak{Z}$.

It is easy to show that in this case the family

$$\tau(\varrho, \mathfrak{Z}) = \{G \setminus A,\ A \in \mathfrak{Z} \text{ and } G \text{ is open}\}$$

forms a topology on X and that

$$\tau(\varrho, \mathfrak{Z}) = \{A \subset X, \text{ if } x \in A \text{ then } U \setminus A \in \mathfrak{Z} \text{ for some neighbourhood } U \text{ of } x\}.$$

Fundamental properties of these topologies may be found in [2], [4], [5].
Our problem was studied for the first time in [1], where the triples $(C(f), S^+(f), T^+(f))$ were considered. In [6] resp. [7] a complete characterization of $\mathrm{CH}(f)$ for the cases $\tau = \tau(R, \{\emptyset\})$ and $\tau = \tau(R, \{\text{sets of first category}\})$ was given using, however, a result of Sierpiński [10] which fails even for R^2. The papers [8], [9] contain contributions to the more general case where (X, ϱ) is a Polish space and $\mathfrak{Z}$ a σ-ideal containing each finite and nonopen nonempty set. Our result answers several problems stated in [8], [9] and includes all foregoing results as special cases. After some technical preliminaries and definitions in Section I we state our main result in Theorem 6. However, Corollary 10 contains the result most interesting from the "purely fine topological" point of view. Finally, in Section III we show to which concrete fine topologies generated by an ideal our abstract result may be applied.

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Notations. In the sequel we will work on a space X equipped with a topology $\hat{\varrho}$ (induced by a metric ϱ) and with the fine topology $\tau(\hat{\varrho}, \mathfrak{Z})$. Topological notions referring to the fine topology $\tau = \tau(\hat{\varrho}, \mathfrak{Z})$ will be qualified by the prefix τ to distinguish them from those pertaining to the initial topology $\hat{\varrho}$, for example $\mathrm{Int}_\tau M$ resp. $\overline{M}^\tau$ denotes the interior resp. closure of M in the topology τ while ∂G will be the $\hat{\varrho}$-boundary of G.

By $\mathrm{Der}(M)$ we denote the set of all cluster points of M. A set $M \subset X$ is called *discrete* iff $M \cap \mathrm{Der}(M) = \emptyset$ and *isolated* iff $\mathrm{Der}(M) = \emptyset$. Countable unions of discrete resp. isolated sets are called *σ-discrete* resp. *σ-isolated*. If $M \subset X$ and $\varepsilon > 0$ then $U(M, \varepsilon) = \{y \in X,\ \mathrm{dist}(M, y) < \varepsilon\}$ and for $x \in X$ we put $U(x, \varepsilon) = U(\{x\}, \varepsilon)$. Further, we define for given $M \subset X$ and $\varepsilon > 0$

$$\mathfrak{D}_\varepsilon(M) = \{\tilde{M} \subset M;\ \varrho(x, y) \geqslant \varepsilon \text{ for all } x, y \in \tilde{M},\ x \neq y\}.$$

Then by the Hausdorff maximal principle $\mathfrak{D}_\varepsilon(M)$ contains a member maximal with respect to set inclusion. To simplify the notations we use abbreviations like $\{f > a\} = \{x, f(x) > a\}$.

By χ_M we denote the characteristic function of the set M. We define the *support of the ideal* $\mathfrak{Z}$ by $\mathrm{supp}(\mathfrak{Z}) = \{x \in X;\ U(x, \varepsilon) \notin \mathfrak{Z} \text{ if } \varepsilon > 0\}$. Then $\mathrm{supp}(\mathfrak{Z})$ is closed and $[X \setminus \mathrm{supp}(\mathfrak{Z})] \in \mathfrak{Z}$. Further, for the rest of this paper let us fix a sequence a_n, $n \geqslant 1$, of positive numbers satisfying $\sum_{n=1}^{\infty} a_n = 1$.

For a given function f: $X \to R$ we define the *upper* and *lower τ-regularization* by

$$f_\tau^+(x) = \tau\text{-}\limsup_{y \to x} f(y) \quad \text{and} \quad f_\tau^-(x) = \tau\text{-}\liminf_{y \to x} f(y).$$

Observe that f_τ^+ resp. f_τ^- is upper resp. lower semicontinuous even in the weaker topology $\hat{\varrho}$, which is, from our point of view, the most interesting property of the topology τ.

Since lower and upper limits and also the fine topology τ depend only on the topology $\hat{\varrho}$, we may assume that our metric ϱ is bounded from above by one and define $\mathrm{dist}(x, \emptyset) = 1$ for $x \in X$. And finally, since there exists an order preserving homeomorphism between $[-1, 1]$ and $[-\infty, \infty]$, we may restrict our considerations to functions mapping X into $[-1, 1]$. That enables us to easily construct uniformly convergent function series and to use their nice limit behaviour.

I

1. Lemma. *Let M be a subset of X.*

(a) *There exists a sequence M_n, $n \geqslant 1$, of mutually disjoint Borel subsets of M such that* $\mathrm{Der}(M_n) = \mathrm{Der}(M)$ *for each* $n \geqslant 1$.

(b) *If $F \subset X$ is a closed set, then there exists some Borel set $\tilde{M} \subset M$ satisfying* $F \cap \mathrm{Der}(M) = \mathrm{Der}(\tilde{M})$.

Proof. (a) Since M is an arbitrary subset of X, it is sufficient to show the existence of two disjoint Borel sets $M_1, M_2 \subset M$ fulfilling $\mathrm{Der}(M_1) = \mathrm{Der}(M_2) = \mathrm{Der}(M)$. By induction we construct a sequence S_n, $n \geqslant 1$, such that each S_n is a maximal member of $\mathfrak{D}_{1/n}(M \setminus \bigcup_{k<n} S_k)$. Now assume that there are $n \geqslant 1$ and $x \in \mathrm{Der}(M)$ such that $U\left(x, \frac{2}{n}\right) \cap S_n = \emptyset$. Since $U\left(x, \frac{1}{2n}\right) \cap \bigcup_{k<n} S_k$ contains at most $n-1$ points we can find some $y \in U\left(x, \frac{1}{2n}\right) \cap (M \setminus \bigcup_{k<n} S_k)$. But now

$$S_n \cup \{y\} \in \mathfrak{D}_{1/n}(M \setminus \bigcup_{k<n} S_k),$$

a contradiction. This shows that we can take

$$M_1 = \bigcup_{k=1}^{\infty} S_{2k} \quad \text{and} \quad M_2 = \bigcup_{k=1}^{\infty} S_{2k+1}.$$

(b) Let $M_0 \subset F \cap M$ be a Borel set satisfying $\mathrm{Der}(M_0) = \mathrm{Der}(F \cap M)$. For each $n \geq 1$ let M_n be a maximal member of $\mathfrak{D}_{1/2n}\left(M \cap U\left(F, \frac{1}{n}\right)\right)$ and now define $\tilde{M} = \bigcup_{n=0}^{\infty} M_n$.

2. Corollary. *Let F be a closed subset of X and let $M \subset X$ satisfy $F \cap M = \emptyset$ and $\mathrm{Der}(M) \supset F$. Then there exists a function $f\colon X \to [-1, 1]$ such that*

(i) *f is continuous on $X \setminus F$,*

(ii) *for each $x \in F$ we have* $\liminf_{\substack{y \to x \\ y \in M}} f(y) = -1$, $\limsup_{\substack{y \to x \\ y \in M}} f(y) = 1$.

If in addition, a closed set $\tilde{F}$ satisfying $\tilde{F} \cap M = \emptyset$ is given then we may require that moreover

(iii) *f identically vanishes on $\tilde{F}$.*

Proof. Clearly it is sufficient to deal with the case that all three sets $\tilde{F}$, F and M are given and moreover we need only define f on $X \setminus F$. By part (b) of Lemma 1 there exists some $\tilde{M} \subset M$ such that $\mathrm{Der}(\tilde{M}) = F$ and the first statement of that Lemma ensures the existence of two disjoint sets $M_1, M_2 \subset \tilde{M}$ satisfying $\mathrm{Der}(M_1) = \mathrm{Der}(M_2) = F$. Therefore the function $f = \chi_{M_1} - \chi_{M_2}$ is continuous on $(M_1 \cup M_2 \cup (\tilde{F} \setminus F), \varrho)$, which is a closed subspace of $X \setminus F$. The proof can now be finished by an application of Tietze's extension theorem.

3. Corollary. *Let F_n, $n \geq 1$, be a sequence of closed subsets of X. If there are given sets M_n^+, M_n^- $(n \geq 1)$ belonging to $\mathfrak{Z}$ and satisfying $(\bigcup_{n=1}^{\infty} M_n^+) \cap (\bigcup_{n=1}^{\infty} M_n^-) = \emptyset$ then there is a Borel measurable function $f\colon X \to [-1, 1]$ such that*

(i) *$\tau\text{-}\lim_{y \to x} f(y) = 0$ for each $x \in \mathrm{Der}_\tau X$,*

(ii) $\{f > 0\} \subset \bigcup_{n=1}^{\infty} M_n^+$, $\{f < 0\} \subset \bigcup_{n=1}^{\infty} M_n^-$,

(iii) *for each $x \in \mathrm{Der}(X)$ we have*

$$\limsup_{y \to x} f(y) > 0 \quad \textit{iff} \quad x \in \bigcup_{n=1}^{\infty} \mathrm{Der}(M_n^+) \cap \bigcup_{n=1}^{\infty} F_n,$$

$$\liminf_{y \to x} f(y) < 0 \quad \textit{iff} \quad x \in \bigcup_{n=1}^{\infty} \mathrm{Der}(M_n^-) \cap \bigcup_{n=1}^{\infty} F_n.$$

Proof. Lemma 1 ensures, for all $i, j \geq 1$, the existence of Borel sets $M_{i,j}^+ \subset M_j^+$ and $M_{i,j}^- \subset M_j^-$ satisfying

$$\mathrm{Der}(M_{i,j}^+) = F_i \cap \mathrm{Der}(M_j^+) \quad \text{and} \quad \mathrm{Der}(M_{i,j}^-) = F_i \cap \mathrm{Der}(M_j^-).$$

Now it is sufficient to define $f = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i a_j (\chi_{M_{i,j}^+} - \chi_{M_{i,j}^-})$.

4. Lemma. (a) *Each τ-discrete set is a τ-F_σ set and therefore also a σ-τ-isolated set.*

(b) *Let $M \subset X$ be a σ-τ-discrete set. Then there exists some $f\colon X \to [0, 1]$ such that $M = \{f > 0\}$ and $\tau\text{-}\lim_{y \to x} f(y) = 0$ for each $x \in \mathrm{Der}_\tau X$.*

Proof. (a) Clearly it is enough to show that for a τ-discrete set M and $\varepsilon > 0$ the set $M_\varepsilon = \{x \in M, M \cap (U(x, \varepsilon) \setminus \{x\}) \in \mathfrak{Z}\}$ is τ-closed. To do this, assume $x \in \overline{M}_\varepsilon^\tau \setminus M_\varepsilon$. Then we have $U(x, r) \cap M_\varepsilon \notin \mathfrak{Z}$ if $r > 0$; in particular two distinct points $y_1, y_2 \in U\left(x, \frac{\varepsilon}{2}\right) \cap M_\varepsilon$ exist. Since $U\left(x, \frac{\varepsilon}{2}\right) \subset U(y_i, \varepsilon)$ $(i = 1, 2)$ we obtain $M_\varepsilon \cap U(y_i, \varepsilon) \notin \mathfrak{Z}$. Now the definition of M_ε implies that $\{y_1\}, \{y_2\} \notin \mathfrak{Z}$. But this contradicts the fact that

$$\{y_2\} \subset M \cap (U(y_1, \varepsilon) \setminus \{y_1\}) \quad \text{and} \quad y_1 \in M_\varepsilon.$$

(b) By (a) the classes of σ-τ-discrete and of σ-τ-isolated sets coincide. Therefore $M = \bigcup_{n=1}^{\infty} M_n$ where the M_n are τ-isolated sets and we can define

$$f = \sum_{n=1}^{\infty} a_n \cdot \chi_{M_n}.$$

5. Definition. We say that the ideal $\mathfrak{Z}$ on (X, ϱ) has the *property* SPL (*splitting*) if the following is true. Whenever $U \subset \mathrm{Der}_\tau X$ is open in $\mathrm{supp}(\mathfrak{Z})$ and $A_1, A_2 \subset U$ satisfy $A_1 \cap A_2 = \emptyset$, $\mathrm{Int}_\tau A_1 = \mathrm{Int}_\tau A_2 = \emptyset$ then there exists a sequence K_n $(n \geq 0)$ of sets such that

(i) $\bigcup_{n=0}^{\infty} K_n = U \setminus (A_1 \cup A_2)$ and $K_n \cap K_m = \emptyset$ for $n \neq m$,

(ii) for each $x \in U$ and for each $n \geq 0$ we have

$$x \in \mathrm{Der}_\tau(\bigcup_{m=n}^{\infty} K_{2m} \cup A_1) \quad \text{and} \quad x \in \mathrm{Der}_\tau(\bigcup_{m=n}^{\infty} K_{2m+1} \cup A_2).$$

After these preparations we formulate our main theorem.

II

6. Theorem. *Assume that the ideal $\mathfrak{Z}$ on (X, ϱ) has the properties* LG *and* SPL. *For a given sixtuple $(C_\varrho, C, S^+, T^+, S^-, T^-)$ of subsets of X the following are equivalent:*

(a) *We have*

(i) $(X \setminus \mathrm{Der}_\tau X) \subset C$,

(ii) $C = S^+ \cap S^-$,

(iii) *$T^+ \subset S^+ \setminus C$, $T^- \subset S^- \setminus C$ and both T^+ and T^- are σ-τ-discrete sets,*

(iv) *there exist disjoint sets G_c, $G \subset \mathrm{supp}(\mathfrak{Z})$ such that*

$$C = [X \setminus \mathrm{supp}(\mathfrak{Z})] \cup [G_c \setminus (T^+ \cup T^-)],$$

G_c is a G_δ-set, G is open and of first category in $\mathrm{supp}(\mathfrak{Z})$ and fulfils

$$G \supset [\mathrm{Int}_\tau(S^+ \cap \mathrm{supp}(\mathfrak{Z})) \cup \mathrm{Int}_\tau(S^- \cap \mathrm{supp}(\mathfrak{Z}))] \setminus \overline{G_c},$$

(v) $C_\varrho \subset C$ *and* C_ϱ *is of type* G_δ,

(vi) *there are sets* D_n^+, D_n^- $(n \geq 1)$ *belonging to* $\mathfrak{Z}$ *such that*

$$\bigcup_{n=1}^{\infty} D_n^+ \subset T^+, \quad \bigcup_{n=1}^{\infty} D_n^- \subset T^- \quad \text{and}$$

$$C \setminus C_\varrho \subset \bigcup_{n=1}^{\infty} \operatorname{Der}(D_n^+ \cup D_n^-) \cup \operatorname{Der}(X \setminus \operatorname{supp}(\mathfrak{Z})) .$$

(b) *There exists a function* $f\colon X \to [-1, 1]$ *such that*

$$C(f) = C_\varrho \quad \text{and} \quad \mathrm{CH}_\tau(f) = (C, S^+, T^+, S^-, T^-) .$$

First we will derive some lemmas used in the proof of this theorem.

7. Lemma. *Assume that a function* $f\colon X \to [-1, 1]$ *is given. If*

$$(C, S^+, T^+, S^-, T^-) = \mathrm{CH}_\tau(f)$$

then there exist sets G_c, G *satisfying the condition* (a) (iv) *of Theorem* 6.

Proof. We already know that the function $f_\tau^+ - f_\tau^-$ is upper semicontinuous, therefore $G_c = \{f_\tau^+ \leq f_\tau^-\} \cap \operatorname{supp}(\mathfrak{Z})$ is a G_δ-set and it is not hard to show that $C = [X \setminus \operatorname{supp}(\mathfrak{Z})] \cup [G_c \setminus (T^+ \cup T^-)]$. Set $\tilde{S}^+ = S^+ \cap \operatorname{supp}(\mathfrak{Z})$, $\tilde{S}^- = S^- \cap \operatorname{supp}(\mathfrak{Z})$, $\tilde{f} = f|_{\operatorname{supp}(\mathfrak{Z})}$. In the remainder of this proof we will work only in the subspace $(\operatorname{supp}(\mathfrak{Z}), \varrho)$. We have $G_c = \bigcap_{n=1}^{\infty} G_n$ where the G_n are open and we may require $G_n \subset U\left(G_c, \frac{1}{n}\right)$; therefore $\overline{G_c} = \bigcap_{n=1}^{\infty} \overline{G_n}$. Clearly $\tilde{S}^+ = S_\tau^+(f)$; we define $U_n^+ = \operatorname{Int}_\tau \tilde{S}^+ \setminus \overline{G_n}$. Since U_n^+ is τ-open there exist $P_n^+ \in \mathfrak{Z}$ and G_n^+ open such that $U_n^+ = G_n^+ \setminus P_n^+$. From the definition of $\operatorname{supp}(\mathfrak{Z})$ and from the inclusion $G_n^+ \cap G_n \subset P_n^+ \in \mathfrak{Z}$ it follows that $G_n^+ \cap G_n = \emptyset$.

Now assume that G_n^+ is of second category. Then by a well-known property of functions belonging to the first Baire class, see [3, § 27. X], there exists a point $x \in G_n^+$ such that $\tilde{f}_\tau^+$ is continuous at x. Since $\tilde{f} \geq \tilde{f}_\tau^+$ on U_n^+ and $G_n^+ \setminus U_n^+ \in \mathfrak{Z}$ we easily obtain

$$\tau\text{-}\liminf_{y \to x} \tilde{f}(y) \geq \liminf_{\substack{y \to x \\ y \in U_n}} \tilde{f}_\tau^+(y) \geq \tilde{f}_\tau^+(x) = \tau\text{-}\limsup_{y \to x} \tilde{f}(y) .$$

Since $x \notin G_n$ the point x is not τ-isolated and therefore $\tau\text{-}\lim\limits_{\substack{y \to x \\ y \in G_n}} \tilde{f}(y) = \tau\text{-}\lim\limits_{\substack{y \to x \\ y \in X}} f(y)$ and consequently $x \in G_c \subset G_n$, a contradiction.

The set G_n^+ is therefore of first category. Analogously we find an open set G_n^- of first category such that $G_n \cap G_n^- = \emptyset$ and $\operatorname{Int}_\tau \tilde{S}^- \setminus \overline{G_n} \subset G_n^-$. The required set G is now defined to be $\bigcup_{n=1}^{\infty} (G_n^+ \cup G_n^-)$.

8. Lemma. *Assume that a function* $f\colon X \to [-1, 1]$ *is given. If we define* $C = C_\tau(f)$, $C_\varrho = C(f)$, $T^+ = T_\tau^+(f)$, $T^- = T_\tau^-(f)$ *then there exist sets* D_n^+, D_n^- $(n \geq 1)$ *satisfying the condition* (a) (vi) *of Theorem* 6.

Proof. Let q_n, $n \geq 1$, be an enumeration of all rationals belonging to $(-1, 1)$. We define

$$D_n^+ = \{x \in T^+,\ \{x\} \in \mathfrak{Z} \text{ and } f(x) > q_n > \tilde{f}_\tau^+(x)\} ,$$

$$D_n^- = \{x \in T^-,\ \{x\} \in \mathfrak{Z} \text{ and } f(x) < q_n < \tilde{f}_\tau^-(x)\} .$$

Since for each $x \in D_n^+$ there is some $\delta > 0$ such that the inclusion $D_n^+ \cap U(x, \delta) \subset \{y \in U(x, \delta), f(y) > q_n\} \in \mathfrak{Z}$ is true we conclude that $D_n^+ \in \mathfrak{Z}$ for each $n \geq 1$. Analogously, $D_n^- \in \mathfrak{Z}$ holds for each $n \geq 1$.

Now let $x \in C \setminus C_\varrho$, for example $f(x) > q_n > \liminf\limits_{y \to x} f(y)$ for some suitable $n \geq 1$ (the case $f(x) < \limsup\limits_{y \to x} f(y)$ is quite analogous). Then $f_\tau^-(x) > q_n$ and since f_τ^- is lower semicontinuous there is some $\varepsilon > 0$ such that we have both $U(x, \varepsilon) \subset \{f_\tau^- > q_n\}$ and $\{y \in U(x, \varepsilon) \setminus \{x\}, f(y) < q_n\} \in \mathfrak{Z}$. But for each such ε there exists some $y_\varepsilon \in U(x, \varepsilon) \setminus \{x\}$ satisfying $f(y_\varepsilon) < q_n$. Therefore $\{y_\varepsilon\} \in \mathfrak{Z}$ and now either $y_\varepsilon \in \operatorname{Der}_\tau X$ and then $y_\varepsilon \in D_n$, or $y_\varepsilon \notin \operatorname{Der}_\tau X$ and in this case $y_\varepsilon \notin \operatorname{supp}(\mathfrak{Z})$. Since ε can be chosen arbitrarily small we have $x \in \operatorname{Der}\big(D_n^- \cup (X \setminus \operatorname{supp}(\mathfrak{Z}))\big) = \operatorname{Der}(D_n^-) \cup \operatorname{Der}(X \setminus \operatorname{supp}(\mathfrak{Z}))$, which finishes the proof.

9. Lemma. *Assume that the ideal* $\mathfrak{Z}$ *on* (X, ϱ) *has the property* SPL. *Let* (C, S^+, T^+, S^-, T^-) *be a quintuple of subsets of* X *satisfying the statements* (a)(i) ... (a)(iv) *of Theorem* 6. *If moreover* $T^+ = T^- = \emptyset$ *then there exists a function* $f\colon X \to [-1, 1]$ *such that* $\mathrm{CH}(f) = \mathrm{CH}_\tau(f) = (C, S^+, T^+, S^-, T^-)$.

Proof. First we assume additionally that the following is true:

(*) The set C is open.

After constructing the required function in this case, we will return to the general case. By assumption 6 (a) (iv) there are closed sets F_n $(n \geq 1)$ such that we have $G \subset \bigcup_{n=1}^{\infty} F_n \subset \operatorname{supp}(\mathfrak{Z})$ and each F_n is nowhere dense in $\operatorname{supp}(\mathfrak{Z})$. For $n \geq 1$ we denote by $\tilde{g}_n$ the function corresponding to the sets F_n, $\operatorname{supp}(\mathfrak{Z}) \setminus F_n$ in the sense of the first conclusion of Corollary 2; moreover, we may define $\tilde{g}_n$ on F_n as follows: for $x \in F_n$ set

$$\tilde{g}_n(x) = \begin{cases} 1 & \text{if } x \in S^+ \cap G , \\ -1 & \text{if } x \in S^- \cap G , \\ 0 & \text{else} . \end{cases}$$

Then the function $g_n(x) = \tilde{g}_n(x) \cdot \operatorname{dist}\big(x, (\operatorname{supp}(\mathfrak{Z}) \setminus G) \cup \bigcup_{j<n} F_j\big)$ satisfies

$$\mathrm{CH}(g_n) = \mathrm{CH}_\tau(g_n) = \big([X \setminus (F_n \cap G)] \cup \bigcup_{j<n} F_j,\ C(g_n) \cup S^+, \emptyset, C(g_n) \cup S^-, \emptyset\big)$$

since the continuity of $\tilde{g}_n$ on $\operatorname{supp}(\mathfrak{Z}) \setminus F_n$ implies that for each $x \in F_n$

$$\tau\text{-}\liminf_{\substack{y \to x \\ y \in \operatorname{supp}(\mathfrak{Z}) \setminus F_n}} \tilde{g}_n(y) = -1 \quad \text{and} \quad \tau\text{-}\limsup_{\substack{y \to x \\ y \in \operatorname{supp}(\mathfrak{Z}) \setminus F_n}} \tilde{g}_n(y) = 1 .$$

Since $C(g_n)\cup C(g_m) = X$ if $n\neq m$ we deduce that the function $g = \sum_{n=1}^{\infty} a_n g_n$ maps X into $[-1,1]$ and fulfils

$$\mathrm{CH}(g) = \mathrm{CH}_\tau(g) = \Big(\bigcap_{n=1}^{\infty} C(g_n),\ \bigcup_{n=1}^{\infty} \big(S^+(g_n)\setminus C(g_n)\big)\cup C(g),\ \emptyset,$$
$$\bigcup_{n=1}^{\infty} \big(S^-(g_n)\setminus C(g_n)\big)\cup C(g),\ \emptyset\Big) = \big(X\setminus G,\ S^+\cup(X\setminus G),\ \emptyset,\ S^-\cup(X\setminus G),\ \emptyset\big).$$

Further, also the set $F = \overline{(G\cup G_c)}\setminus(G\cup G_c)$ is closed and nowhere dense in $\mathrm{supp}(3)$; therefore F and $\mathrm{supp}(3)\setminus F$ satisfy the assumptions of Corollary 2. We denote by h the corresponding function; again we may define h additionally on F, namely for $x\in F$

$$h(x) = \begin{cases} 1 & \text{if } x\in S^+, \\ -1 & \text{if } x\in S^-, \\ 0 & \text{else}. \end{cases}$$

Clearly now $\mathrm{CH}(h) = \mathrm{CH}_\tau(h) = (X\setminus F,\ S^+\cup(X\setminus F),\ \emptyset,\ S^-\cup(X\setminus F),\ \emptyset)$.

Finally, since the sets $U = \mathrm{supp}(3)\setminus\overline{(G\cup G_c)}$, $A_1 = S^+\cap U$ and $A_2 = S^-\cap U$ have all properties required in Definition 5 we may find a sequence K_n $(n\geq 0)$ of sets fulfilling statements 5 (i) and 5 (ii). We now define the function $\bar{k}\colon X\to[-1,1]$ by

$$\bar{k}(x) = \begin{cases} 1 & \text{if } x\in A_1, \\ \dfrac{n}{n+1} & \text{if } x\in K_{2n}, \\ 0 & \text{if } x\notin U, \\ \dfrac{1-n}{n} & \text{if } x\in K_{2n-1}, \\ -1 & \text{if } x\in A_2, \end{cases}$$

and then we put $k(x) = \bar{k}(x)\cdot\mathrm{dist}(x, \overline{G\cup G_c})$. It is quite easy to show that $\mathrm{CH}(k) = \mathrm{CH}_\tau(k) = (X\setminus U,\ A_1\cup(X\setminus U),\ \emptyset,\ A_2\cup(X\setminus U),\ \emptyset)$.

From the fact that at each point $x\in X$ at most one of the functions g, h, k is discontinuous it follows immediately that the map $f = \frac{1}{3}(g+h+k)$ has the required properties, i.e. $f\colon X\to[-1,1]$ and

$$\mathrm{CH}(f) = \mathrm{CH}_\tau(f) = \big(X\setminus(G\cup F\cup U),\ S^+\cup C(f),\ \emptyset,\ S^-\cup C(f),\ \emptyset\big)$$
$$= (C,\ S^+,\ T^+,\ S^-,\ T^-).$$

We may now easily deal with the general case without assumption $(*)$. We may find open sets $X = G_0\supset G_1\supset\ldots$ such that $C = (X\setminus\mathrm{supp}(3))\cup G_c = \bigcap_{n=1}^{\infty} G_n$. It is quite easy to verify that for each $n\geq 1$ the quintuple $(G_n,\ S^+\cup G_n,\ T^+,\ S^-\cup G_n,\ T^-)$

satisfies the assumption of Lemma 9 and the condition $(*)$. Therefore we conclude the existence of a function $\tilde{f}_n\colon X\to[-1,1]$ such that

$$\mathrm{CH}(\tilde{f}_n) = \mathrm{CH}_\tau(\tilde{f}_n) = (G_n,\ S^+\cup G_n,\ T^+,\ S^-\cup G_n,\ T^-).$$

Now define $f_n(x) = \tilde{f}_n(x)\cdot\mathrm{dist}(x, X\setminus G_{n-1})$. Then

$$\mathrm{CH}(f_n) = \mathrm{CH}_\tau(f_n) = \big(G_n\cup(X\setminus G_{n-1}),\ S^+\cup C(f_n),\ T^+,\ S^-\cup C(f_n),\ T^-\big)$$

and since again $C(f_n)\cup C(f_m) = X$ for $n\neq m$ it follows immediately that the function $f = \sum_{n=1}^{\infty} a_n\cdot f_n$ has all required properties.

Now we have finished the necessary preparations and return to the

Proof of Theorem 6. We have to prove two implications. First assume that the statement 6 (a) holds. By Lemma 9 there exists a function $\tilde{f}\colon X\to[-1,1]$ such that

$$\mathrm{CH}(\tilde{f}) = \mathrm{CH}_\tau(\tilde{f}) = \big((X\setminus\mathrm{supp}(3))\cup G_c,\ S^+\cup C(\tilde{f}),\ \emptyset,\ S^-\cup C(\tilde{f}),\ \emptyset\big).$$

Further, we can find closed sets $F_n\subset F_{n+1}$ $(n\geq 1)$ such that $X\setminus C_\varrho = \bigcup_{n=1}^{\infty} F_n$ and Lemma 4 (b) ensures, for each $n\geq 1$, the existence of functions $g_n^+, g_n^-\colon X\to[0,1]$ satisfying $\{g_n^+>0\} = F_n\cap T^+$, $\{g_n^->0\} = F_n\cap T^-$ and

$$\tau\text{-}\lim_{y\to x} g_n^+(y) = \tau\text{-}\lim_{y\to x} g_n^-(y) = 0$$

for each $x\in\mathrm{Der}_\tau X$. Now define, for $x\in X$, $g(x) = \sum_{n\geq 1} a_n(g_n^+(x)-g_n^-(x))$. Then $\{g>0\} = T^+$, $\{g<0\} = T^-$, $\lim_{y\to x} g(y) = 0$ for $x\in\mathrm{Der}(X)\cap C_\varrho$, and $x\in\mathrm{Der}_\tau X$ implies $\tau\text{-}\lim_{y\to x} g(y) = 0$.

Denote by h the function corresponding to the sets F_n $(n\geq 1)$ and D_n^+, D_n^- $(n\geq 1)$ in the sense of Corollary 3.

Next observe that Corollary 2 guarantees, for each $n\geq 1$, the existence of a function $\bar{k}_n\colon X\to[-1,1]$ such that $\bar{k}_n|_{\mathrm{supp}(3)}\equiv 0$, $\bar{k}_n$ is continuous on $X\setminus\tilde{F}_n$ and for arbitrary $x\in\tilde{F}_n$ we have

$$\liminf_{\substack{y\to x\\ y\in C_\varrho\setminus\mathrm{supp}(3)}} \bar{k}_n(y) = -1 \quad\text{and}\quad \limsup_{\substack{y\to x\\ y\in C_\varrho\setminus\mathrm{supp}(3)}} \bar{k}_n(y) = 1$$

where $\tilde{F}_n = F_n\cap\mathrm{Der}(C_\varrho\setminus\mathrm{supp}(3))$. Define $k_n(x) = \bar{k}_n(x)\,\mathrm{dist}(x, \bigcup_{j<n} F_j)$ and $k(x) = \sum_{n=1}^{\infty} a_n k_n(x)$. As before we derive that k is τ-continuous on X and that

$$\liminf_{\substack{y\to x\\ y\in C_\varrho\setminus\mathrm{supp}(3)}} k(y) < \limsup_{\substack{y\to x\\ y\in A_\varrho\setminus\mathrm{supp}(3)}} k(y) \quad\text{if } x\in[\mathrm{Der}(C_\varrho\setminus\mathrm{supp}(3))]\setminus C_\varrho,$$

else k is continuous at x.

Finally, by Lemma 1 there exist two disjoint subsets M^+, M^- of

$$X\setminus(\operatorname{supp}(\mathfrak{Z})\cup C_\varrho)$$

satisfying

$$\operatorname{Der}(M^+)=\operatorname{Der}(M^-)=\operatorname{Der}[X\setminus(\operatorname{supp}(\mathfrak{Z})\cup C_\varrho)].$$

Denote by l the function corresponding in the sense of Corollary 3 to the sets F_n $(n\geqslant 1)$ and $M_n^+=M^+$, $M_n^-=M^-$ $(n\geqslant 1)$.

We put $f=\frac{1}{5}(\tilde{f}+g+h+k+l)$. Then $f\colon X\to[-1,1]$ and since $\tau\text{-}\lim_{y\to x}(g+k+l)=0$ whenever $x\in\operatorname{Der}_\tau X$, and k is τ-continuous on X, we obtain

$$\begin{aligned}\operatorname{CH}_\tau(f)&=(C(\tilde{f})\setminus(T^+\cup T^-),\ C_\tau(f)\cup S^+,\ T^+,\ C_\tau(f)\cup S^-,\ T^-)\\&=(C,S^+,T^+,S^-,T^-).\end{aligned}$$

Further, we note that $g+h+k+l$ is continuous at each point $x\in C_\varrho$. It remains to show that $g+h+k+l$ is discontinuous at x whenever $x\in C\setminus C_\varrho$. If

$$x\in\operatorname{Der}(C_\varrho\setminus\operatorname{supp}(\mathfrak{Z}))$$

this statement follows from the fact that $g,h,l\equiv 0$ on $C_\varrho\setminus\operatorname{supp}(\mathfrak{Z})$. Else we have $x\in\bigcup_{n=1}^{\infty}\operatorname{Der}(D_n^+\cup D_n^-)\cup\operatorname{Der}[X\setminus(\operatorname{supp}(\mathfrak{Z})\cup C_\varrho)]$ and k is continuous at x. Now it is sufficient to observe that $g\cdot h\geqslant 0$, $l\equiv 0$ on $T^+\cup T^-\supset\{g\neq 0\}\cup\{h\neq 0\}$ and $g,h\equiv 0$ on $[X\setminus\operatorname{supp}(\mathfrak{Z})]\supset\{l\neq 0\}$, see Corollary 3 (iii). So the first implication is proved.

Now assume that a function $f\colon X\to[-1,1]$ is given and write

$$(C,S^+,T^+,S^-,T^-)=\operatorname{CH}_\tau(f)$$

and $C_\varrho=C(f)$. Then the statements 6 (a) (i), (ii) and (v) are trivial or well known. The conditions 6 (a) (iv) and (vi) are fulfilled because of Lemma 7 and Lemma 8. Finally, the statement 6 (a) (iii) follows directly from the argument used in the proof of Lemma 8, see also [1].

10. Corollary. *Assume that the ideal $\mathfrak{Z}$ on (X,ϱ) has the properties* LG *and* SPL. *For a given quintuple (C,S^+,T^+,S^-,T^-) of subsets of X the following are equivalent*:

(a) *The conditions* 6 (a) (i), ..., (a) (iv) *hold.*
(b) *There exists a function $f\colon X\to[-1,1]$ such that*

$$\operatorname{CH}_\tau(f)=(C,S^+,T^+,S^-\ T^-).$$

Proof. By Theorem 6 statement (b) implies (a). Now assume that a quintuple (C,S^+,T^+,S^-,T^-) satisfying the condition 10 (a) is given. By Lemma 4 we may write $T^+=\bigcup_{n=1}^{\infty}T_n^+$, $T^-=\bigcup_{n=1}^{\infty}T_n^-$ where T_n^+, T_n^- are τ-*isolated* sets. The property LG implies that for each $n\geqslant 1$ the sets

$$D_n^+=\{x\in T_n^+,\{x\}\in\mathfrak{Z}\}\quad\text{and}\quad D_n^-=\{x\in T_n^-,\{x\}\in\mathfrak{Z}\}$$

belong to $\mathfrak{Z}$. Moreover, the sets $M_n^+=T_n^+\setminus D_n^+$ and $M_n^-=T_n^-\setminus D_n^-$ are *isolated* and therefore *closed*. Consequently, the set

$$C_\varrho=[(X\setminus\operatorname{supp}(\mathfrak{Z}))\cup G_c]\setminus\bigcup_{n=1}^{\infty}(\bar{D}_n^+\cup M_n^+\cup\bar{D}_n^-\cup M_n^-)$$

satisfies conditions 6 (a) (v) (vi) and Theorem 6 may be applied to finish the proof.

However, the definition of the property SPL could easily give the impression that its formulation was determined mainly by the requirements of our proof. This impression is refuted by the following lemma, at least for a *fairly large* class of metric spaces.

11. Lemma. *Assume that for the ideal $\mathfrak{Z}$ on (X,ϱ) the metric subspace $(\operatorname{supp}(\mathfrak{Z}),\varrho)$ is a Baire space. If the statement* 10 (a) *implies* 10 (b) *then $\mathfrak{Z}$ has the property* SPL.

Proof. Let $A_1,A_2\subset U$ be arbitrary subsets of X such that $\emptyset\neq U\subset\operatorname{Der}_\tau X$, U is *open* in $\operatorname{supp}(\mathfrak{Z})$ and $A_1\cap A_2=\operatorname{Int}_\tau A_1=\operatorname{Int}_\tau A_2=\emptyset$. Define $C=(X\setminus U)$, $S^+=C\cup A_1$, $S^-=C\cup A_2$ and $T^+=T^-=\emptyset$. It is easy to show that (C,S^+,T^+,S^-,T^-) satisfies the statement 10 (a); therefore we can choose a function $f\colon X\to[-1,1]$ such that $\operatorname{CH}_\tau(f)=(C,S^+,T^+,S^-,T^-)$. We consider again the functions f_τ^+ and f_τ^-. Obviously $f_\tau^+(x)>f_\tau^-(x)$ if $x\in U$, $f_\tau^+(x)=f(x)$ if $x\in A_1$, $f_\tau^+(x)>f(x)>f_\tau^-(x)$ if $x\in U\setminus(A_1\cup A_2)$ and $f(x)=f_\tau^-(x)$ if $x\in A_2$. This implies that

$$U=\bigcup_{n=1}^{\infty}\left\{x\in U,f_\tau^+(x)-f_\tau^-(x)\geqslant\frac{1}{n}\right\}.$$

Denote by M_n $(n\geqslant 1)$ the interior of

$$\left\{x\in U,f_\tau^+(x)-f_\tau^-(x)\geqslant\frac{1}{n}\right\}$$

w.r.t. the subspace $(\operatorname{supp}(\mathfrak{Z}),\varrho)$. A well-known category argument ensures that $U\subset\overline{\bigcup_{n\geqslant 1}M_n}$. Finally, we define for each $n\geqslant 0$ the sets

$$K_{2n}=\left\{x\in U,1-\frac{1}{2(n+1)}<\frac{f(x)-f_\tau^-(x)}{f_\tau^+(x)-f_\tau^-(x)}\leqslant 1-\frac{1}{2(n+2)}\right\},$$

$$K_{2n+1}=\left\{x\in U,\frac{1}{2(n+2)}<\frac{f(x)-f_\tau^-(x)}{f_\tau^+(x)-f_\tau^-(x)}\leqslant\frac{1}{2(n+1)}\right\}.$$

Clearly, $U\setminus(A_1\cup A_2)=\bigcup_{n=0}^{\infty}K_n$ and the sets K_n are mutually disjoint. Now assume that for some $x\in U$ and some $n\geqslant 0$ we have for example $x\notin\operatorname{Der}_\tau(\bigcup_{k=n}^{\infty}K_{2k}\cup A_1)$. Then there exist $\varepsilon>0$ and $m>0$ satisfying $(\bigcup_{k=n}^{\infty}K_{2k}\cup A_1)\cap(U(x,\varepsilon)\setminus\{x\})\in\mathfrak{Z}$ and

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$y \in (U(x,\varepsilon)\setminus\{x\}) \cap M_m$. Therefore the set $[(U(x,\varepsilon)\setminus\{x\}) \cap M_m]\setminus(\bigcup_{k=n}^{\infty} K_{2k} \cup A_1)$ fulfils $y \in \mathrm{Int}_\tau S$. Obviously, we have for each $z \in S$ the estimates

$$f_\tau^+(z) - f_\tau^-(z) \geqslant \frac{1}{m}, \qquad (f(z)-f_\tau^-(z))/(f_\tau^+(z)-f_\tau^-(z)) \geqslant 1 - \frac{1}{2(n+2)}$$

and consequently

$$f_\tau^+(z) - f(z) \geqslant c = \frac{1}{2(n+2)m}.$$

This implies

$$f_\tau^+(y) \geqslant \tau\text{-}\limsup_{\substack{z\to y\\ z\in S}} f_\tau^+(z) \geqslant \tau\text{-}\limsup_{\substack{z\to y\\ z\in S}} f(z) + c = f_\tau^+(y) + c,$$

a contradiction. Therefore the sets K_n, $n \geqslant 0$, have indeed all required properties.

12. Remark. Our main results are formulated under quite general assumptions. If we add further conditions concerning $\mathfrak{Z}$ then the statements (a) (iv), (a) (vi) can be given a shorter formulation. This may be left to the reader; here we only mention the interesting case $\mathrm{supp}(\mathfrak{Z}) = X$ and $\mathfrak{Z}$ is a σ-ideal. Then one may quite directly derive from Corollary 10 the following (slight) generalization of Proposition 3 in [9]: Denote by $\mathfrak{N}$ the σ-ideal of all sets of first category. If $\mathfrak{N} \subset \mathfrak{Z}$ resp. $\mathfrak{Z} \subset \mathfrak{N}$ then (a)(iv) can be replaced by

(a) (iv)′: *$C = G_c\setminus(T^+ \cup T^-)$, G_c is a G_δ and $\mathrm{Int}_\tau(S^+\setminus C)$, $\mathrm{Int}_\tau(S^-\setminus C)$ are both empty resp. first category sets.*

III

After establishing our quite abstract Theorem 6 we have to deal with the question which ideals $\mathfrak{Z}$ on metric spaces (X, ϱ) do have the properties LG and SPL.

13. Definition. We say that the ideal $\mathfrak{Z}$ on (X, ϱ) has the *property* SPL* if the following holds. Whenever $M \subset X$ then there are $M_1, M_2 \subset M$ such that $M_1 \cap M_2 = \emptyset$ and $\mathrm{Der}_\tau M = \mathrm{Der}_\tau M_1 = \mathrm{Der}_\tau M_2$.

Obviously, if $\mathfrak{Z}$ has the property SPL* then for any $M \subset X$ there exists a sequence M_n $(n \geqslant 0)$ of mutually disjoint subsets of M satisfying $\mathrm{Der}_\tau M_n = \mathrm{Der}_\tau M$ for each $n \geqslant 0$. Consequently, SPL* implies SPL.

From Lemma 1 we already know that the ideal $\mathfrak{Z} = \{\emptyset\}$ has the property SPL* on an arbitrary metric space; in this case we have $\tau(\varrho, \mathfrak{Z}) = \hat{\varrho}$ and LG is trivially fulfilled. Because of the fundamental significance of this case we formulate explicitly the "metric" variant of Corollary 10. Note that the case $\mathfrak{A} = \exp(X)$ is of particular interest.

14. Corollary. *Let (X, ϱ) be an arbitrary metric space and $\mathfrak{A}$ a σ-algebra on X containing all open subsets of X. For a given quintuple (C, S^+, T^+, S^-, T^-) of subsets of X the following statements are equivalent:*

(a) (i)* *C is a G_δ-set containing all isolated points,*

(ii)* *$C = S^+ \cap S^-$ and $S^+, S^- \in \mathfrak{A}$,*

(iii) *$T^+ \subset S^+\setminus C$, $T^- \subset S^-\setminus C$ and both T^+, T^- are σ-discrete sets,*

(iv)* *$\mathrm{Int}(S^+)\setminus\bar{C}$, $\mathrm{Int}(S^-)\setminus\bar{C}$ are first category sets.*

(b) *There exists an $\mathfrak{A}$-measurable function $f\colon X \to [-1, 1]$ such that $\mathrm{CH}(f) = (C, S^+, T^+, S^-, T^-)$.*

Proof. Assume (b) holds. From Lemma 4 we conclude that T^+, T^- are F_σ (6 (a) (iii)) and consequently C is a G_δ, see 6 (a) (iv). The use of condition (a) (iv)* instead of 6 (a) (iv) is correct by Remark 12. The $\mathfrak{A}$-measurability of $S^+(f)$ and $S^-(f)$ follows from the fact that both f^+, f^- are Borel measurable.

Conversely, by conditions (a) (i)*, (a) (iii) the sets C, T^+ and T^- always belong to $\mathfrak{A}$. Further, one can immediately verify that in the "metric" case all functions or sets occurring in the proofs of Theorem 6 and Corollary 10 could be chosen to be $\mathfrak{A}$-measurable.

The second type of ideals having all required properties are, under some set-theoretic assumptions, σ-ideals on separable metric spaces. The property LG is then a consequence of the Lindelöf property of X. In [9] it is shown that a σ-ideal $\mathfrak{Z}$ on a separable space (X, ϱ) has the property SPL* provided that:

— $\mathfrak{Z}$ contains all sets of cardinality less than 2^ω (the cardinality of the continuum),

— for each $A \in \mathfrak{Z}$ there exists also a Borel superset $B \in \mathfrak{Z}$.

The first condition is of course closely related to some set-theoretic assumptions (Martin's axiom, CH). We present here a slightly different result.

15. Lemma. *Assume that the following condition ($\mathfrak{C}$) is satisfied.*

($\mathfrak{C}$): *There exists no weakly unaccessible cardinal less than or equal to 2^ω. Then each σ-ideal on an arbitrary separable metric space has the property* SPL*.

Proof. We will only outline the proof. For a given σ-ideal $\mathfrak{Z}$ on (X, ϱ) set $D(\mathfrak{Z}) = \{x \in X, \{x\} \notin \mathfrak{Z}\}$. Then $\mathfrak{Z}$ is a σ-ideal on $X\setminus D(\mathfrak{Z})$. Since $M = (M \cap D(\mathfrak{Z})) \cup (M\setminus D(\mathfrak{Z}))$ and $\mathrm{Der}_\tau \tilde{M} = \mathrm{Der}(\tilde{M})$ for each $\tilde{M} \subset D(\mathfrak{Z})$ we may by Lemma 1 (a) restrict our attention to the case $M \subset X\setminus D(\mathfrak{Z})$.

In [11], [12] the following was shown: Let $\tilde{M}$ be a set such that $\mathrm{card}(\tilde{M})$ is less than the first weakly unaccessible cardinal and let $\mathfrak{M}$ be a family of subsets of $\tilde{M}$ such that for each sequence $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{M}$ the set $\tilde{M}\setminus\bigcup_{n=1}^{\infty} A_n$ is uncountable. Then there exists an uncountable system of pairwise disjoint subsets of $\tilde{M}$ not belonging to $\mathfrak{M}$. Consequently, condition ($\mathfrak{C}$) ensures that for each set $\tilde{M} \subset X\setminus D(\mathfrak{Z})$ such that $\tilde{M} \notin \mathfrak{Z}$ mutually disjoint subsets M_α, $\alpha < \omega_1$, of $\tilde{M}$ satisfying $M_\alpha \notin \mathfrak{Z}$ for each $\alpha < \omega_1$ may be found; here ω_1 denotes the first uncountable cardinal.

Let $\mathfrak{B} = \{G_n\}_{n=1}^{\infty}$ be an open base of (X, ϱ). Then we may by induction construct a sequence A_n $(n \geqslant 1)$ of mutually disjoint subsets of M such that $A_n \subset G_n$ for each $n \geqslant 1$ and $A_n \notin \mathfrak{Z}$ provided that $M \cap G_n \notin \mathfrak{Z}$. Indeed, if $G_1 \cap M \in \mathfrak{Z}$ then

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choose $A_1 = \emptyset$. Else we find A_α^1 $(\alpha < \omega_1)$ such that $A_\alpha^1 \cap A_\beta^1 = \emptyset$ if $\alpha \neq \beta$, $A_\alpha^1 \subset G_1 \cap M$ and $A_\alpha^1 \notin \mathfrak{Z}$ whenever $\alpha \leqslant \omega_1$. Obviously, if $G_m \cap M \notin \mathfrak{Z}$ $(m \geqslant 1)$ then there exists at most one $\alpha < \omega_1$ such that $G_m \cap (M \setminus A_\alpha^1) \in \mathfrak{Z}$. Therefore we may find $\alpha_1 < \omega_1$ satisfying $G_m \cap (M \setminus A_{\alpha_1}^1) \notin \mathfrak{Z}$ whenever $G_m \cap M \notin \mathfrak{Z}$. Now we can put $A_1 = A_{\alpha_1}^1$ and construct A_2 in a similar way, replacing M by $M \setminus A_1$.

Further, for each $n \geqslant 1$ there exist disjoint sets $A_n^1, A_n^2 \subset A_n$ such that $A_n^1, A_n^2 \notin \mathfrak{Z}$ if $A_n \notin \mathfrak{Z}$. Then it is easy to verify that the sets $M_i = \bigcup_{n \geqslant 1} A_n^i$ $(i = 1, 2)$ have all required properties.

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Weak covering properties and the class MOBI

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Abstract. We construct a non-σ-discrete zero-dimensional scattered space which is an open and compact image of a σ-discrete metacompact Moore space. This gives an example showing that weak θ-refinability is not preserved by open and compact mappings between regular spaces and shows that there exist non-weakly θ-refinable spaces in MOBI_3. Moreover, our example is a non-σ-discrete space which is both right and left separated.

In 1966 A. V. Arhangel'skiĭ [A2] defined the class MOBI and asked many questions concerning this class.

The investigation of the class MOBI led to a slight modification of the original definition from [A2].

For a class $\mathscr{K}$ of topological spaces, let $\mathrm{MOBI}_i(\mathscr{K})$ be the minimal class of T_i-spaces containing all metric spaces from $\mathscr{K}$ and invariant under open and compact mappings (see [Ch1]).

It is easy to observe that a T_i-space is in $\mathrm{MOBI}_i(\mathscr{K})$ if and only if it can be obtained as an image of a metric space from $\mathscr{K}$ under a mapping which is a composition of a finite number of open and compact mappings with T_i-domains [B1].

If the class $\mathscr{K}$ contains the class of all metric spaces, we write MOBI_i instead of $\mathrm{MOBI}_i(\mathscr{K})$.

It is well known that open and compact images of metric spaces are metacompact developable T_1-spaces [H], [A1]. It is also known that all the elements of MOBI_2 have a point-countable base and all the elements of MOBI_3 have, moreover, a base of countable order [WW1].

In [Ch4] the first example of a non-weakly θ-refinable space S^* in MOBI_2 was given. The space S^* is the first absolute example of a non-weakly θ-refinable space with a point-countable base and, since it does not have base of countable order, it cannot be in MOBI_3.

In the present paper we shall modify the construction of spaces in MOBI_2 rom [Ch4] and the construction of open and compact mappings from [B2] in order o obtain a non-weakly θ-refinable space Y in MOBI_3.

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