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UNIVERSITY OF HELSINKI
DEPARTMENT OF MATHEMATICS
Hallituskatu 15
00100 Helsinki 10
Finland

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A proof of Saffe's conjecture

by

Ludomir Newelski (Wrocław)

Abstract. We prove that if T is weakly minimal, $p_0 \in S(\emptyset)$ is non-isolated and has infinite multiplicity, then T has $2^{\aleph_0}$ countable models, thus proving Saffe's conjecture. Together with [B2] this completes the proof of Vaught's conjecture for weakly minimal theories.

§ 0. Introduction. This paper may be regarded as a continuation of the proof of Vaught's conjecture for weakly minimal theories, which was initiated in [B2], and carried on in [B3]. We use a standard set- and model-theoretic terminology. First we shall review shortly what was proved in [B2], [B3], and sketch some proofs to make the paper more self-contained. The reader should know the basic ideas from [B1] and [B2] however, as well as be familiar with stable groups (see [Po]). Vaught's conjecture states that every $1^{\rm st}$-order theory has either countably or $2^{\aleph_0}$ many countable models. Up to now there has been made only a relatively small progress towards proving this conjecture (see [Ls]). Shelah proved Vaught's conjecture for ω-stable theories [SHM]. Thus the natural aim of attack became the case of weakly minimal T. In [B2], Buechler proved that if T is weakly minimal and satisfies

(S) For every finite A, if $p \in S(A)$ is non-isolated then it has finite multiplicity,

then Vaught's conjecture holds for T. Earlier this was also known to Jürgen Saffe. Saffe conjectured that if T is weakly minimal and does not satisfy (S) then T has $2^{\aleph_0}$ countable models. Buechler [B2, Lemma 2.4 Proposition 3.1] reduced proving Saffe's conjecture to proving it for T weakly minimal and unidimensional. This paper is devoted to the proof of Saffe's conjecture for weakly minimal 1-dimensional T. So throughout we assume that T is weakly minimal, 1-dimensional, not ω-stable, does not satisfy (S) and (wlog) is small (i.e. $S_n(\emptyset)$ is countable).

CB denotes Cantor–Bendixson rank defined on $S(A)$ (cf. [B2]), $\mathrm{CB}(a/A)$ abbreviates $\mathrm{CB}(\mathrm{tp}(a/A))$. Recall that by [B1] every non-algebraic weakly minimal strong 1-type over $\emptyset$ is locally modular. For the advantages that local modularity gives, see [B1], [B2], [H]. Also, every such type is non-trivial. This is essentially by [B2, 2.4 and 3.1]. Notice also that if T is weakly minimal, unidimensional, and a non-algebraic $p \in S(\emptyset)$ is trivial, then T is ω-stable.

Given a formula $\varphi(x_1, \ldots, x_n)$, we say that φ is *algebraic in* x_i if, for some $k<\omega$, $(\forall x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ $(\exists^{<k} x_i)\varphi(\bar{x})$ holds. $\mathfrak{C}$ is the monster model. Given a family F of non-empty disjoint sets, we call C a *selector from* F if for every $X \in F$, $|X \cap C| = 1$ and $C \subseteq \bigcup F$.

LEMMA 0.1. *If* $a \in \operatorname{acl}(A \cup \{b\})$ *then* $\mathrm{CB}(a/A) \leqslant \mathrm{CB}(b/A)$.

Proof. An easy induction.

LEMMA 0.2. *If A is finite then there are only finitely many strong non-algebraic (w.m.) 1-types over $\emptyset$ realized in* $\operatorname{acl}(A)$.

Proof. This is Theorem A from [B3]. Following a request of the referee, we give another proof. Let $\mathrm{ST}(A)$ be the topological space of strong w.m. non-algebraic 1-types over A. Suppose that the lemma is false, and wlog $A = \{a\}$. First assume that:

(1) For some $b \in \operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$ there are $b_n \in \operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$ for $n<\omega$ such that $\operatorname{tp}(b/\emptyset) \neq \operatorname{tp}(b_n/\emptyset)$ and $\operatorname{stp}(b/\emptyset) = \lim_n \operatorname{stp}(b_n/\emptyset)$.

Then by the exchange principle, $b \in \operatorname{acl}(b_n)$ and $b_n \in \operatorname{acl}(b)$ for $n<\omega$. By Lemma 0.1 we get $\mathrm{CB}(b/\emptyset) = \mathrm{CB}(b_n/\emptyset)$. On the other hand, by the definition of CB-rank we have $\mathrm{CB}(b/\emptyset) > \limsup_n (\mathrm{CB}(b_n/\emptyset)$, a contradiction.

Now we shall show how to reduce the general case to (1). $\mathrm{ST}(\emptyset)$ is compact, hence there are $b_n \in \operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$ for $n<\omega$, and there is a $q \in \mathrm{ST}(\emptyset)$ such that $q = \lim_n \operatorname{stp}(b_n/\emptyset)$. Let $b \in q(\mathfrak{C}) \setminus \operatorname{acl}(a)$. Let us work in $T(a, b)$. Let $a' \overset{s}{\equiv} a$, $a' \downarrow ab$. By [B1] there is $b' \overset{s}{\equiv} b$ with $b' \in \operatorname{acl}(a, a', b) \setminus \operatorname{acl}(a, b)$, and for $n<\omega$ there are $b'_n \in \operatorname{acl}(a') \setminus \operatorname{acl}(a, b)$ with $b'_n \overset{s}{\equiv} b_n$. In particular, $\operatorname{stp}(b'/\{a, b\}) = \lim_n \operatorname{stp}(b'_n/\{a, b\})$ and also $\{\operatorname{tp}(b'_n/\{a, b\})\colon n<\omega\}$ is infinite, so we get (1) for $T := T(a, b)$ and $A := \{a'\}$.

LEMMA 0.3 [B2]. *If A is finite and $\operatorname{stp}(a/A)$ is modular then $\operatorname{Mlt}(a/A)$ is finite.*

We shall rely heavily on Hrushovski's result asserting existence (in $\mathfrak{C}^{eq}$) of a definable w.m. abelian group G, such that on the connected component G^0 of G the forking dependence relation is particularly well describable. If $S \leqslant G^0$ is finite and $r\colon G^0 \to G^0/S$ is a homomorphism, we say that r is *definable* if S is definable and there is a formula $\varphi(x, y)$ such that for $a, b \in G^0$, $\models \varphi(a, b)$ iff $ra = b+S$. The definition of r works (by compactness) also on some definable $G_1 \leqslant G$ such that $G^0 \leqslant G_1$. We take the following presentation of Hrushovski's result from [Lo].

THEOREM 0.4 [H]. (1) *If $S \leqslant G^0 \cap \operatorname{acl}(\emptyset)$ and $r\colon G^0 \to G^0/S$ is a definable group-homomorphism then r is almost 0-definable.*

(2) *If G^0 is locally modular, $A \cup \{b\} \subseteq G^0$ and $b \in \operatorname{acl}(A)$ then there are $a_1, \ldots, a_n \in A$, $S \leqslant G^0$ almost 0-definable and definable group-homomorphisms $r_i\colon G^0 \to G^0/S$ such that $(b+S) - \Sigma r_i a_i \subseteq \operatorname{acl}(\emptyset)$.*

We can regard every homomorphism r definable in the above way as acting on G^0/S_0, where $S_0 := \operatorname{acl}(\emptyset) \cap G^0$. Then we can define D as the set of all such homomorphisms of G^0/S_0. D forms a division ring, and G^0/S_0 is a vector space

over D. D does not depend on the choice of G but only on T, as T is unidimensional. Thus we can define $F(T) := D$ (see [H]).

T is 1-dimensional, hence if $p \in S(\emptyset)$ is non-isolated and $\operatorname{Mlt}(p) \geqslant \omega$ then also in $T \restriction G$, for some finite $A \subseteq G$ there is $q \in S(A)$ which is non-isolated and $\operatorname{Mlt}(q) \geqslant \omega$. So to prove Saffe's conjecture it suffices to prove it when $\mathfrak{C} = G$ is an abelian group (with some further structure possible). Hence from now on until § 3 we assume that $T = \operatorname{Th}(G, +, \ldots)$, $p_0 \in S(\emptyset)$ is non-isolated, $\operatorname{Mlt}(p_0) \geqslant \omega$, T is small weakly minimal and not ω-stable.

The following lemma was already known to Buechler and Hrushovski.

LEMMA 0.5. *$F(T)$ is locally finite, and so $\operatorname{char}(F(T)) \neq 0$ and $F(T)$ is commutative.*

Sketch of the proof. First, for every finite $F_0 \subseteq F(T)$, by compactness and by Lemma 0.2, we prove that for every definable $G_0 \leqslant G$ there is $x \in G_0 \setminus G^0$ such that for each $n<\omega$ if $a_1, \ldots, a_n \in F_0$, then $a_1 \ldots a_n(x+S_0)$ is well defined. By Lemma 0.2, the closure C of $\{x+S_0\}$ under multiplication by elements of F_0, and by natural numbers, is finite, and we can assume that $C \subseteq G_1 \leqslant G$ for some G_1 such that for every $a \in F_0$, multiplication by a is distributive w.r. to $+$ on G_1. The set $\bigcup \{r(\mathfrak{C})\colon r \in \mathrm{ST}(\emptyset)$ and $r(C) \neq \emptyset\}$ generates a non-algebraic type-definable subgroup $G_2 \leqslant G_1$ such that for every $y \in G_2$, the closure of $\{y+S_0\}$ under multiplication by elements of F_0 and by natural numbers is finite. We have $G^0 \leqslant G_2$. By Lemma 0.2 again we see that F_0 generates a finite division subring of $F(T)$, so we are done.

The next lemma shows that if $p \in S(A)$ has infinite multiplicity, then in some neighbourhood of p (in $S(A)$) there are no algebraic types. This strengthens Lemma 0.2.

LEMMA 0.6. *Assume that A is finite and $p \in S(A)$ has infinite multiplicity. Then there is an $E^0 \in \mathrm{FE}(\emptyset)$ such that for $a \in p(\mathfrak{C})$ we have $E^0(\mathfrak{C}, a) \cap \operatorname{acl}(A) = \emptyset$. In particular, for some $\varphi_0 \in p$ we have $\varphi_0(\mathfrak{C}) \cap \operatorname{acl}(A) = \emptyset$.*

Proof. First we shall prove the lemma for $A = \emptyset$. Let a realize p and suppose that

(1) for every $E \in \mathrm{FE}(\emptyset)$ we have $E(\mathfrak{C}, a) \cap \operatorname{acl}(\emptyset) \neq \emptyset$.

We have $\operatorname{Mlt}(p) = 2^{\aleph_0}$, and the set $P = \{q \in \mathrm{ST}(\emptyset)\colon p \subseteq q\}$ is closed, has no isolated points, and for every open $U \subseteq \mathrm{ST}(\emptyset)$, if $U \cap P \neq \emptyset$ then $|U \cap P| = 2^{\aleph_0}$. Also, by Lemma 0.3, every $q \in P$ is non-modular. Let $q_0 = \operatorname{stp}(a/\emptyset)$. The set of types from $\mathrm{ST}(\emptyset)$ realized in $\operatorname{acl}(q_0(\mathfrak{C}))$ is countable. Hence we can choose a $q_1 \in P$ such that q_1 is not realized in $\operatorname{acl}(q_0(\mathfrak{C}))$. Let b realize q_1. q_0 and q_1 are non-orthogonal, hence by [B1] there is a formula $\varphi(x, y, z, t)$ algebraic in x, y, z, t and $c \overset{s}{\equiv} a$, $d \overset{s}{\equiv} b$ such that $\varphi(a, b, c, d)$ holds and any 3 elements from $\{a, b, c, d\}$ are independent.

Fix an $E \in \mathrm{FE}(\emptyset)$. By compactness, if d' realizes a type $r_E \in \mathrm{ST}(\emptyset)$ sufficiently close to q, then for some $c' \in E(\mathfrak{C}, a)$ we have $\varphi(a, b, c', d')$. Choose such a d' realizing a type $r_E \in P$ which satisfies moreover

(2) r_E is not realized in $\mathrm{acl}(q_0(\mathfrak{C}) \cup q_1(\mathfrak{C}))$.

In particular any 3 elements from $\{a, b, c', d'\}$ are independent, and c' does not realize q_0. Thus for some $E' \in \mathrm{FE}(\emptyset)$, $\neg E'(a, c')$ holds. Again by compactness there is $E^* \in \mathrm{FE}(\emptyset)$ refining E such that if $E^*(d'', d')$ holds then for some c'' we have

(3) $\varphi(a, b, c'', d'') \& E(a, c'') \& \neg E'(a, c'')$ holds.

By (1) we can choose $d'' \in E^*(\mathfrak{C}, d') \cap \mathrm{acl}(\emptyset)$, and let $p_E = \mathrm{stp}(c''/\emptyset)$. If p_E is algebraic then we get $b \in \mathrm{acl}(a)$, a contradiction. So $p_E \in \mathrm{ST}(\emptyset)$ and by (3), $p_E \neq q_0$.

Let $A' = \{a, b\}$. We see that for every $E \in \mathrm{FE}(\emptyset)$, p_E is realized in $\mathrm{acl}(A')$. By (3), the set $\{p_E\colon E \in \mathrm{FE}(\emptyset)\}$ is infinite. So we get a contradiction with Lemma 0.2.

Now the general case (with $A \neq \emptyset$) follows, because by Lemma 0.2 there are only finitely many types from $\mathrm{ST}(\emptyset)$ realized in $\mathrm{acl}(A)$. The φ_0 from the last clause of the lemma can be defined as the union of E^0-classes meeting $p(\mathfrak{C})$.

Remark. The above proof would be easier to carry out if we used the notion of acl*-dependence introduced in [N] (see also § 3 below).

§ 1. A duality theorem.

LEMMA 1.1. *For every infinite 0-definable group $G_0 \leqslant G$ and for every finite $F_0 \subseteq F(T)$ there is a 0-definable infinite $G_1 \leqslant G_0$ such that G_1/S_0 is closed under multiplication by elements of F_0.*

Proof. Follows by Lemma 0.5.

COROLLARY 1.2. *There is a sequence $G_{n+1} \leqslant G_n$, $n < \omega$, of 0-definable subgroups of G such that $\bigcap_n G_n = G^0$ (the connected component of G), and for every $a \in F(T)$ there is $n(a) < \omega$ such that for every $n > n(a)$, G_n/S_0 is closed under a and a^{-1}.*

Using G_n's from Corollary 1.2, we define equivalence relations E_n, $n < \omega$, by xE_ny iff $x - y \in G_n$, and by Corollary 1.2, E_n's generate the topology on $\mathrm{ST}(\emptyset)$.

For a finite set A we say that the set $\{p_n\colon n < \omega\} \subseteq S(A)$ is *almost orthogonal* if every selector C from $\{p_n(\mathfrak{C})\colon n < \omega\}$ is independent over A (i.e. for $c \in C$, $c \downarrow C \setminus \{c\}$ (over A)).

For a finite set A and an element a, we define a function

$$f(a/A) = f(\mathrm{tp}(a/A)) \in {}^{\omega}\omega \text{ by}$$

$$f(a/A)(i) = \text{the number of } E_i\text{-classes consistent with } \{E_{i-1}(x, a)\} \cup \mathrm{tp}(a/A)(x)\,.$$

We define $\mathbf{1} \in {}^{\omega}\omega$ by $\mathbf{1}(i) = 1$, and for $f, g \in {}^{\omega}\omega$, $f = {}^*g$, $f \leqslant {}^*g$ mean that $f(n) = g(n)$, $f(n) \leqslant g(n)$ respectively, eventually.

If $A \subseteq B$ are finite, $p \in S(A)$, $q \in S(B)$, $q \vdash p$, then we say that q is *isolated in p* iff for some $\varphi(x) \in L(B)$, $q(x) \equiv p(x) \cup \{\varphi(x)\}$. If no such φ exists we call q *non-isolated* in p.

LEMMA 1.3. *Assume that $A \subseteq B$ are finite and $a, b \in \mathfrak{C}$.*

(1) *Let $f_{\max} \in {}^{\omega}\omega$ be defined by $f_{\max}(i) = [G_{i-1}\colon G_i]$ (and $G_{-1} = G_0$). Then $\mathbf{1} \leqslant f(a/A) \leqslant f_{\max}$.*

(2) *$f(a/A) = {}^*f_{\max}$ iff $\mathrm{tp}(a/A)$ is isolated and non-algebraic.*

(3) *$f(a/A) = {}^*\mathbf{1}$ iff $\mathrm{Mlt}(a/A)$ is finite.*

(4) *(monotonicity) $f(a/B) \leqslant {}^*f(a/A)$.*

(5) *If $\mathrm{Mlt}(a/A)$ is infinite then $f(a/A) = {}^*f(a/B)$ iff $\mathrm{tp}(a/B)$ is non-algebraic and isolated in $\mathrm{tp}(a/A)$.*

(6) *If $B \subseteq \mathrm{acl}(A)$, then $f(a/A) = {}^*f(a/B)$.*

(7) *(symmetry) $f(a/A \cup \{b\}) = {}^*f(a/A)$ iff $f(b/A \cup \{a\}) = {}^*f(b/A)$.*

Proof. (1), (2)$\leftarrow$, (3), (4), (5)$\leftarrow$ and (6) are easy.

(2)$\rightarrow$. Assume $f(a/A) = {}^*f_{\max}$. Choose $n_0 < \omega$ such that for $n \geqslant n_0$, $f(a/A)(n) = f_{\max}(n)$. By (3), $\mathrm{Mlt}(a/A)$ is infinite. Let $E = E_{n_0}$. By Lemma 0.2 there are only finitely many types $q \in \mathrm{ST}(\emptyset)$ such that for c realizing q, $(c + G^0) \cap \mathrm{acl}(\emptyset) \neq \emptyset$. Also, whenever $(c + G^0) \cap \mathrm{acl}(\emptyset) = \emptyset$ then $\mathrm{stp}(c/\emptyset)$ is generated by $\{E_n(x, c)\colon n < \omega\}$. So by Lemma 0.6, wlog

(a) $E(\mathfrak{C}, a) \cap \mathrm{acl}(A) = \emptyset$.

We shall prove

(b) If $b \in E(\mathfrak{C}, a)$ then $b \equiv a$ (over A).

By (a) we know that $\mathrm{tp}(b/A)$ is non-algebraic and $\{E_n(x, b)\colon n < \omega\}$ generates $\mathrm{stp}(b/A)$. Suppose that for some $\varphi \in L(A)$, $\varphi(a) \& \neg\varphi(b)$ holds. For some $n > n_0$ we get $E_n(\mathfrak{C}, b) \subseteq \neg\varphi(\mathfrak{C})$. By the choice of n_0 we see that for some $a' \equiv a$ (over A), we have $E_n(a', b)$. So $\neg\varphi(a)$ holds, a contradiction.

To finish, notice that $\mathrm{tp}(a/A)$ is isolated by the union of E-classes consistent with $\mathrm{tp}(a/A)$.

(5)$\rightarrow$ is similar to (2)$\rightarrow$. Let n_0 be so large that for $n \geqslant n_0$, $f(a/A)(n) = f(a/B)(n)$, and let $E = E_{n_0}$. By Lemma 0.6 wlog

(a′) $E(\mathfrak{C}, a) \cap \mathrm{acl}(B) = \emptyset$.

We prove

(b′) If $b \in E(\mathfrak{C}, a)$ and $b \equiv a$ (over A) then $b \equiv a$ (over B).

Suppose not and choose $\varphi(x) \in L(B)$ such that $\varphi(a) \& \neg\varphi(b)$ holds. By (a′), $\neg\varphi$ is non-algebraic and $\{E_n(x, b)\colon n < \omega\}$ generates $\mathrm{stp}(b/B)$. So for some $n > n_0$, $E_n(\mathfrak{C}, b) \subseteq \neg\varphi(\mathfrak{C})$. By the choice of n_0, for some $a' \equiv a$ (over B) we have $E_n(a', b)$. In particular, $E_n(\mathfrak{C}, a') \subseteq \neg\varphi(\mathfrak{C})$ hence also $E_n(\mathfrak{C}, a) \subseteq \neg\varphi(\mathfrak{C})$ and $\neg\varphi(a)$ holds, a contradiction. Now, $\mathrm{tp}(a/B)$ is isolated in $\mathrm{tp}(a/A)$ by the union of E-classes consistent with $\mathrm{tp}(a/B)$.

(7) is easy by (1)–(6).

The next lemma is the only essential place where we use the assumption that $\mathfrak{C}$ is a group. Moreover, it is the crucial point of the whole proof.

LEMMA 1.4. *If A is finite and $a \in \mathrm{acl}(A \cup \{b\})$ then $f(a/A) \leqslant {}^*f(b/A)$.*

Proof. Wlog $A = \emptyset$ and $\mathrm{Mlt}(a/\emptyset)$ is infinite (by Lemma 1.3(3)). Thus $\mathrm{stp}(a/\emptyset)$ is not modular, as well as $\mathrm{stp}(b/\emptyset)$, and we have also $b \in \mathrm{acl}(a)$. Take $\varphi(x, y) \in L(\emptyset)$ algebraic in x and in y such that $\varphi(a, b)$ holds, $\varphi(x, b) \vdash \mathrm{tp}(a/b)(x)$ and

$\varphi(a, y) \vdash \operatorname{tp}(b/a)(y)$. Take a' and b' such that $a' \overset{s}{\equiv} a$, $b' \overset{s}{\equiv} b$, $a' \downarrow ab$ and $a'b' \equiv ab$. Then

$$b'-b,\ a'-a \in G^0 \setminus S_0 \quad \text{and} \quad b'-b \in \operatorname{acl}(a, a'-a).$$

If $b'-b \notin \operatorname{acl}(a'-a)$ then $a \in \operatorname{acl}(b'-b, a'-a)$. In [B1] or in [N, Proposition 2.7] it is proved that if p, q are w.m. locally modular, $p^a \not\perp q^\omega$ and q is modular, then p is modular. Thus in our case we would get that $\operatorname{stp}(a/\emptyset)$ is modular, a contradiction.

So $b'-b \in \operatorname{acl}(a'-a)$ which means that for some $\lambda \in F(T)$, $\lambda(a'-a+S_0) = b'-b+S_0$, i.e. $b' \in b+\lambda(a'-a+S_0)$. By compactness we see that for some n_0, whenever $a_1 \in E_{n_0}(\mathfrak{C}, a) \setminus \operatorname{acl}(a, b)$, then for some $b_1 \in b+\lambda(a_1-a+S_0)$, $\varphi(a_1, b_1)$ holds.

By Corollary 1.2 and the choice of E_n's, take n_1 so large that for $n \geqslant n_0+n_1$, G_n is closed under λ and λ^{-1}. In particular, for every $m > n \geqslant n_0+n_1$, λ induces a permutation of G_n/G_m. Let $n > n_0+n_1$. We shall show $f(a/\emptyset)(n) \leqslant f(b/\emptyset)(n)$.

Let $k = f(a/\emptyset)(n)$. That means that we can choose $a_0, \ldots, a_{k-1} \notin \operatorname{acl}(a, b)$ realizing $\operatorname{tp}(a/\emptyset)$ such that $E_{n-1}(a_i, a)$ holds (i.e. $a_i - a \in G_{n-1}$), and for $i \neq j < k$, $\neg E_n(a_i, a_j)$ holds, i.e. $(a_i-a)-(a_j-a) = a_i - a_j \notin G_n$. Choose $b_0, \ldots, b_{k-1}$ so that $\varphi(a_i, b_i)$ holds and $b_i \in b+\lambda(a_i-a+S_0)$. It follows that for $i < k$, $E_{n-1}(b_i, b)$ holds and $b_i - b \in \lambda(a_i-a+S_0)$. $\{a_i - a\colon i < k\}$ are in distinct cosets of G_n in G_{n-1}, hence for $i \neq j < k$, $b_i - b_j = (b_i-b)-(b_j-b) \notin G_n$. So $\neg E_n(b_i, b_j)$. Also, $b_i \equiv b$ by the choice of φ. We see that $f(b/\emptyset)(n) \geqslant k$.

THEOREM 1.5 (the duality theorem). *Assume that T is small, weakly minimal, unidimensional, not ω-stable, $p_0 \in S(\emptyset)$ is non-isolated and $\operatorname{Mlt}(p_0)$ is infinite. Then either* (A) *or* (B) *holds, where*

(A) *If A is finite, $q \in S(A)$ is non-isolated and $\operatorname{Mlt}(q)$ is infinite then there are $b_1, \ldots, b_m \in q(\mathfrak{C})$ (for some m) such that $\operatorname{tp}(b_i/A \cup \{b_1, \ldots, b_{i-1}\})$ is non-algebraic and isolated in q and there is $r \in S(A \cup \{b_1, \ldots, b_m\})$ with $r \vdash q$ and such that $\operatorname{Mlt}(r)$ is infinite and r is non-isolated in q.*

(B) *There is a finite set A and an almost orthogonal set of non-isolated types $\{q_n\colon n < \omega\} \subseteq S(A)$.*

Proof. Suppose (A) does not hold for some A_0 and q, and we shall prove (B). The proof is split into two cases depending on whether the assumption of case (a) below holds or not.

Case (a). Assume that for some finite B there are infinitely many non-isolated types $r \in S(B)$ with finite multiplicity.

In this case put $A = A_0 \cup B$, and by Lemma 0.2 we can define inductively $q_n \in S(A)$ so that $\operatorname{Mlt}(q_n)$ is finite, q_n is non-isolated and whenever $a_i \in q_i(\mathfrak{C})$ for $i < n$ then $q_n(\mathfrak{C}) \cap \operatorname{acl}(A \cup \{a_i\colon i < n\}) = \emptyset$, so we are done.

Case (b). Suppose that the assumption of Case (a) does not hold.

First notice the trivial fact that

(1) If $A \subseteq B$ are finite and $p \in S(A)$ has infinite multiplicity, then p has only finitely many extensions over B iff all extensions of p over B are non-algebraic and isolated in p.

Take any $a \in q(\mathfrak{C})$ and let $A = A_0 \cup \{a\}$. We define $q_n \in S(A)$ by induction on n so that the following hold:

(i) $q_n \vdash q$, q_n is isolated in q and non-algebraic,

(ii) if $a_i \in q_i(\mathfrak{C})$ for $i < n$ then q_n splits over $A \cup \{a_i\colon i < n\}$ into finitely many types, i.e. there are $r_0, \ldots, r_k \in S(A \cup \{a_i\colon i < n\})$ such that $q_n(C) = \bigcup_{i \leqslant k} r_i(\mathfrak{C})$.

By (1) we see that (i) and (ii) for q_n, $n < \omega$, imply (B). Suppose we have found q_i for $i < n$ and we want to find q_n satisfying (i) and (ii). By the inductive hypothesis, the set

$$P = \{\operatorname{tp}(\bar{a}/A)\colon \bar{a} = \langle a_0, \ldots, a_{n-1}\rangle \text{ and } a_i \in q_i(\mathfrak{C})\}$$

is finite. Choose $\bar{a}_0, \ldots, \bar{a}_m$ so that $P = \{\operatorname{tp}(\bar{a}_i/A)\colon i \leqslant m\}$. As we assume that (A) does not hold for A_0 and q, and the assumption of case (a) does not hold either, we see that for every $i \leqslant m$, for all but finitely many non-algebraic $r \in S(A)$ such that $r \vdash q$ and r is isolated in q, we have that r splits over $A \cup \bar{a}_i$ into finitely many types. A contains a realization of q and T is small, hence by Lemma 0.2 there are infinitely many non-algebraic $r \in S(A)$ isolated in q. Consequently we see that there is a non-algebraic $q_n \in S(A)$ isolated in q such that for every $i \leqslant m$, q_n splits over $A \cup \bar{a}_i$ into finitely many types. It is easy to see that $q_0, \ldots, q_n$ satisfy (ii).

FACT 1.6. *If* 1.5(B) *holds then T has $2^{\aleph_0}$ countable models.*

Proof. See [B3]. The point is that by the omitting types theorem, for every $X \subseteq \omega$ there is a model M_X of $T(A)$ such that $q_i(M_X) \neq \emptyset$ iff $i \in X$.

In [B3] S. Buechler proved that if $F(T)$ is finite then 1.5(B) holds, thus proving Saffe's conjecture for the case of finite $F(T)$. In the next section we shall prove that 1.5(A) also implies that T has $2^{\aleph_0}$ countable models, thus completing the proof of Saffe's conjecture. Notice that in Theorem 1.5 we did not use the assumption that $\mathfrak{C}$ is a group.

§ 2. 1.5(*A*) implies that *T* has $2^{\aleph_0}$ countable models. In this section we assume that 1.5(A) holds and we shall construct $2^{\aleph_0}$ countable models of T. Recall that $p_0 \in S(\emptyset)$ is non-isolated and $\operatorname{Mlt}(p_0)$ is finite.

LEMMA 2.1. *There are $a_n \in p_0(\mathfrak{C})$ and $f_k \in {}^\omega\omega$ for $n, k < \omega$ and an increasing sequence n_k, $k < \omega$ (with $n_0 = 0$) such that if $A_n = \{a_i\colon i < n\}$ then the following hold:*

(i) $f_{k+1} \leqslant^* f_k$, $\neg f_{k+1} =^* f_k$.

(ii) *For each k there is $q_k \in S(A_{n_k})$ such that if $n \geqslant n_k$ then*

$$f(a_n/A_{n_k}) =^* f_k \quad \text{and} \quad \operatorname{tp}(a_n/A_{n_k}) = q_k.$$

(iii) *If $n_k \leqslant n < n_{k+1}$ then $f(a_n/A_n) =^* f_k$.*

Proof. Straightforward by 1.5(A).

We call a countable model M of T *constrained* if it satisfies

(1) $a_n \in M$ for $n < \omega$ and

(2) M omits every modular w.m. non-algebraic type.

For a constrained model M we define $g_M \in {}^{\omega}2$ as follows.

$g_M(i) = 0$ iff for every finite $A \subseteq p_0(M)$ there is a finite $B \subseteq p_0(M)$ containing A such that for every $a \in p_0(M)$, if $f(a/B) = {}^*f_i$ then for some $m < \omega$ there are $b_0, \ldots, b_m \in p_0(M)$ such that $a \in \operatorname{acl}(B \cup \{b_0, \ldots, b_m\})$, and for each $t \leqslant m$, $f(b_t/B) \leqslant {}^*f_i$ and $\neg f(b_t/B) = {}^*f_i$,

$g_M(i) = 1$ otherwise.

We shall show that for every $g \in {}^{\omega}2$ there is a constrained model M of T such that $g = g_M$. Clearly this will give $2^{\aleph_0}$ countable models of T. So fix $g \in {}^{\omega}2$ and let $A = \{a_n\colon n < \omega\}$. We define by induction sets B_k, $-1 \leqslant k < \omega$, where

$$B_k = \{b_k^m\colon m < \omega\}$$

or $B_k = \varnothing$. Let $k = -1$. We define b_{-1}^m by induction on m so that $\operatorname{acl}(A \cup B_{-1})$ is a model of T and the following hold for $k = -1$ (and we stipulate $q_{-1} = \varnothing$, $n_{-1} = 0$ and $f_{-1} = f_{\max}$).

(i) for every m, $\operatorname{tp}(b_k^m/A_{n_k}) \vdash q_k$ and

(ii) if B is finite and $A_{n_k} \subseteq B \subseteq A \cup \bigcup \{B_i\colon -1 \leqslant i < k\} \cup \{b_k^s\colon s < m\}$ then $f(b_k^m/B) = {}^*f_k$.

It is easy to define B_{-1} because T is small. If we have defined B_t for $t < k$ and $g(k) = 1$ then we define $B_k = \{b_k^m\colon m < \omega\}$ so that (i) and (ii) hold. If $g(k) = 0$ then we put $B_k = \varnothing$. Let

$$C = A \cup \bigcup \{B_k\colon -1 \leqslant k < \omega\} \quad \text{and} \quad M = \operatorname{acl}(C)\,.$$

LEMMA 2.2. (1) *M is a model of T.*

(2) *C is independent over $\varnothing$.*

(3) *If $b \in B_k$ then for every finite B such that $A_{n_k} \subseteq B \subseteq C \setminus \{b\}$ we have $f(b/B) = {}^*f(b/A_{n_k}) = {}^*f_k$.*

(4) *M is constrained.*

Proof. (1) follows by the choice of B_{-1}.

(2) First we prove that A is algebraically independent. If not, then for some n we have $a_n \in \operatorname{acl}(A_n)$, but then $f(a_n/A_n) = {}^*\mathbf{1}$. On the other hand by Lemma 2.1 (iii), $f(a_n/A_n) = {}^*f_k$ for some k, and 2.1 (i) implies $\neg f_k = {}^*\mathbf{1}$, a contradiction. Similarly we prove that the whole C is algebraically independent.

(3) Fix $b = b_k^m \in B_k$ and let $A_{n_k} \subseteq B \subseteq C \setminus \{b\}$, $|B| < \omega$. By Lemma 1.3 (4) wlog whenever $b_t^r \in B$ then $A_{n_t} \subseteq B$. Let

$$R = \{(i,j)\colon (i,j) = (m,k) \text{ or } b_j^i \in B\}\,.$$

By Lemma 1.3 (5) and by the definition of B_n's there is a formula

$$\varphi(\langle x_j^i\colon (i,j) \in R\rangle) \in L(A \cap B)$$

such that

$$\operatorname{tp}(\langle b_j^i\colon (i,j) \in R\rangle/A \cap B)(\langle x_j^i\colon (i,j) \in R\rangle) \equiv \bigcup \{q_j(x_j^i)\colon (i,j) \in R\} \cup \{\varphi(\langle x_j^i\colon (i,j) \in R\rangle)\}\,.$$

Thus $\operatorname{tp}(b/B)(x_k^m) \equiv q_k(x_k^m) \cup \{\varphi(\langle b_j^i\colon (i,j) \in R \setminus \{(m,k)\}\rangle, x_k^m)\}$. So $\operatorname{tp}(b/B)$ is isolated in q_k, hence by (2) and Lemma 1.3 (5) we have $f(b/B) = {}^*f_k$.

(4) Suppose not. Take any $c \in M$ realizing a modular type $p \in \mathrm{ST}(\varnothing)$. There is a finite set $B \subseteq C$ such that $c \in \operatorname{acl}(B)$. Wlog $B = A_{n_k} \cup \{b_j^i\colon i < k,\ -1 \leqslant j < k \text{ and } B_j \neq \varnothing\}$. We see that either for some $n < n_k$, $\operatorname{stp}(a_n/A_n)$ is modular or for some i, j, $b_j^i \in B$ and $\operatorname{stp}(b_j^i/B \setminus \{b_j^i\})$ is modular. Thus either $f(a_n/A_n) = {}^*\mathbf{1}$ or $f(b_j^i/B \setminus \{b_j^i\}) = {}^*\mathbf{1}$, contradicting Lemma 2.1 or (3).

We are left with proving that $g_M = g$. Let $i < \omega$.

Case 1. $g(i) = 0$. We want to prove $g_M(i) = 0$. So take any finite $A^0 \subseteq p_0(M)$. By Lemma 1.3 (6) wlog $A^0 \subseteq C$. Let j_0 be the minimal j such that $j > i$, $A^0 \cap A \subseteq A_{n_j}$ and if $b_r^t \in A^0$ then $r < j$. Let $B = A^0 \cup A_{n_{j_0}}$. Take any $a \in p_0(M)$ such that $f(a/B) = {}^*f_i$. $a \in \operatorname{acl}(B \cup \{\bar{a}, \bar{b}, \bar{c}\})$, where $\bar{a} \subseteq A$, $\bar{b} \subseteq \bigcup \{B_j\colon -1 \leqslant j < i\}$ and $\bar{c} \subseteq \bigcup \{B_j\colon j > i\}$, and $\bar{a}$, $\bar{b}$, $\bar{c}$ are chosen of minimal possible length.

LEMMA 2.3. *If $d \in \bar{a}$ then $f(d/B) = {}^*f_{j_0}$ (and $j_0 > i$) hence $\neg f(d/B) = {}^*f_i$.*

Proof. Similar to that of Lemma 2.2 (3). Let $R = \{(i,j)\colon b_j^i \in B\}$. Then

$$\operatorname{tp}(\langle b_j^i\colon (i,j) \in R\rangle^\frown \langle d\rangle/A_{n_{j_0}})(\langle x_j^i\colon (i,j) \in R\rangle^\frown \langle y\rangle) \equiv \bigcup \{q_j(x_j^i)\colon (i,j) \in R\} \cup q_{j_0}(y) \cup \{\varphi(\langle x_j^i\colon (i,j) \in R\rangle^\frown \langle y\rangle)\}\,,$$

for some $\varphi \in L(A_{n_{j_0}})$. So $\operatorname{tp}(d/B)$ is (non-algebraic and) isolated in $q_{j_0} = \operatorname{tp}(d/A_{n_{j_0}})$, and by Lemma 1.3 (5) we are done.

CLAIM 2.4. (1) *If $c \in \bar{c} \cap B_j$ ($j > i$) then $f(c/B) \leqslant {}^*f_{j_1}$ for some $j_1 > i$, hence $\neg f(c/B) = {}^*f_i$.*

(2) *$\bar{b} = \varnothing$.*

Proof. (1) $j_1 = \min\{j_0, j\} > i$. $A_{n_{j_1}} \subseteq B$, so $f(c/B) \leqslant {}^*f(c/A_{n_{j_1}}) = {}^*f_{j_1}$.

(2) If not, take any $b \in \bar{b}$. By Lemma 2.2 (3), $f(b/B \cup \{\bar{a}, \bar{b}, \bar{c}\} \setminus \{b\}) = {}^*f_j$ for some $j < i$. By the minimality of $\bar{a}, \bar{b}, \bar{c}$, we have $b \in \operatorname{acl}(B \cup \{\bar{a}, \bar{b}, \bar{c}, a\} \setminus \{b\})$. So by Lemma 1.4, $f_j \leqslant {}^*f(a/B \cup \{\bar{a}, \bar{b}, \bar{c}\} \setminus \{b\})$ hence $f_i = {}^*f(a/B)^* \geqslant f(a/B \cup \cup \{\bar{a}, \bar{b}, \bar{c}\} \setminus \{b\})^* \geqslant f_j$, and $f_j^* \geqslant f_i$. This implies $f_j = {}^*f_i$, contradicting 2.1 (i).

Clearly 2.3 and 2.4 imply that $g_M(i) = 0$, and we are done in case 1.

Case 2. $g(i) = 1$. We want to prove $g_M(i) = 1$. That means that we want to find a finite $A^0 \subseteq p_0(M)$ such that for every finite $B \subseteq p_0(M)$ containing A^0 there is $a \in p_0(M)$ such that

$f(a/B) = {}^*f_i$ and there are no $b_0, \ldots, b_m \in p_0(M)$ such that

(*) $a \in \operatorname{acl}(B \cup \{b_0, \ldots, b_m\})$ and for each t, $f(b_t/B) \leqslant {}^*f_i$ and $\neg f(b_t/B) = {}^*f_i$.

So let $A^0 = A_{n_i}$. Take any finite $B^0 \subseteq p_0(M)$ with $B^0 \supseteq A^0$. Choose a finite B with $A^0 \subseteq B \subseteq C$ such that $B^0 \subseteq \operatorname{acl}(B)$, and take any $a \in B_i \setminus B$. In order to prove that a

and B_0 satisfy (*), by Lemma 1.3 (4), (6), it suffices to prove that a and B satisfy this condition. By Lemma 2.2 (3), $f(a/B) = {}^*f_i$. Suppose towards a contradiction that there are $b_0, \ldots, b_m \in p_0(M)$ such that $a \in \mathrm{acl}(B \cup \{b_0, \ldots, b_m\})$ and for $t \leqslant m$, $f(b_t/B) \leqslant {}^*f_i$ and $\neg f(b_t/B) = {}^*f_i$.

For each $t \leqslant m$ choose $\bar{a}_t$, $\bar{b}_t$, $\bar{c}_t$ of minimal possible length such that $b_t \in \mathrm{acl}(B \cup \{\bar{a}_t, \bar{b}_t, \bar{c}_t\})$, $\bar{a}_t \subseteq A$, $\bar{b}_t \subseteq \bigcup \{B_j\colon -1 \leqslant j \leqslant i\}$ and $\bar{c}_t \subseteq \bigcup \{B_j\colon j > i\}$. So $a \in \mathrm{acl}(B \cup \{\bar{a}_t, \bar{b}_t, \bar{c}_t\colon t \leqslant m\})$ and by Lemma 2.2 (2), for some t we have $a \in \bar{b}_t$. However we have

CLAIM 2.5. $\bar{b}_t = \emptyset$ *for any* $t \leqslant m$.

Proof. If not, take any $b \in \bar{b}_t$. So $b \in \mathrm{acl}(B \cup \{\bar{a}_t, \bar{b}_t, \bar{c}_t\} \cup \{\bar{b}_t\} \setminus \{b\})$. By Lemma 1.4, $f(b/B \cup \{\bar{a}_t, \bar{b}_t, \bar{c}_t\} \setminus \{b\}) \leqslant {}^*f(b_t/B \cup \{\bar{a}_t, \bar{b}_t, \bar{c}_t\} \setminus \{b\}) \leqslant {}^*f(b_t/B) \leqslant {}^*f_i$. On the other hand, by Lemma 2.2 (3) (as $\bar{b}_t \subseteq \bigcup \{B_j\colon j \leqslant i\}$ and $A_{n_i} \subseteq B$), we have

$$f(b/B \cup \{\bar{a}_t, \bar{b}_t, \bar{c}_t\} \setminus \{b\}) = {}^*f(b/B) = {}^*f_j \quad \text{for some } j \leqslant i.$$

Thus we get

$$f_j \leqslant {}^*f(b_t/B) \leqslant {}^*f_i,$$

which implies $j = i$ and $f(b_t/B) = {}^*f_i$, a contradiction.

This contradiction finishes the proof in case 2.

The concept of assigning to $\mathrm{tp}(a/A)$ the function $f(a/A)$ (which is essential in the above proof) is similar to the concept of ordinal rank. $f(a/A)$ can be regarded as an "f-rank" of $\mathrm{tp}(a/A)$, where the values of f-rank are not ordinals but functions, or as a "topological shape" of $\mathrm{tp}(a/A)$. The core of the above proof is of course Lemma 1.4. We did not use its full strength. The group structure on $\mathfrak{C}$ induces a group structure on $\mathrm{ST}(\emptyset)$. Lemma 1.4 implies that definable functions preserve the ideal of sets of Haar measure zero on $\mathrm{ST}(\emptyset)$. Although Vaught's conjecture for w.m. theories is proved, several questions concerning the proof arise. The main is whether there is a T which fails to satisfy 1.5 (B), and is small. We state some result connected with this problem in the next section. Another kind of problem is how CB-rank and f-rank are related (whether there is any relation at all).

§ 3. acl* for small w.m. 1-dimensional not ω-stable T with finite $F(T)$. In this section we assume that T is small, w.m., 1-dimensional and not ω-stable. We shall show what impact on acl* has the assumption of smallness and $|F(T)| < \omega$. First let us say what acl* is (see [N]). acl* is a dependence relation on $\mathrm{ST}(\emptyset)$. For $R \subseteq \mathrm{ST}(\emptyset)$ and $p \in \mathrm{ST}(\emptyset)$ we define $p \in \mathrm{acl}^*(R)$ iff $p(\mathrm{acl}(\bigcup \{r(\mathfrak{C})\colon r \in R\})) \neq \emptyset$. We have

FACT 3.1 [N]. *The following are equivalent for* $R \subseteq \mathrm{ST}(\emptyset)$ *and* $p \in \mathrm{ST}(\emptyset)$.

(1) $p \in \mathrm{acl}^*(R)$.

(2) p *is modular or for every selector* C *from* $\{r(\mathfrak{C})\colon r \in R\}$, *we have* $p(\mathrm{acl}(C)) \neq \emptyset$, *i.e.* p *is realized in* $\mathrm{acl}(C)$.

So Lemma 0.2 says just that $\mathrm{acl}^*(R)$ is finite for every finite R.

FACT 3.2 [N]. *If* $U \subseteq \mathrm{ST}(\emptyset)$ *and* $\mathrm{int}(U) \neq \emptyset$ *then* $\mathrm{acl}^*(U)$ *is clopen.*

The author believes that investigating acl* may be basic in proving that in Theorem 1.5 always (B) holds. Here we show that if $F(T)$ is finite then acl* is "regular". Now we define precisely what we mean by this. We say that acl* is *regular* if for every decreasing sequence of clopen sets $U_n \subseteq \mathrm{ST}(\emptyset)$, $n < \omega$, with $R = \bigcap_n U_n$ finite, we have $\mathrm{acl}^*(R) = \bigcap_n \mathrm{acl}^*(U_n)$.

For a finite A, every non-algebraic $p \in S(A)$ determines a closed set $[p] = \{q \in \mathrm{ST}(\emptyset)\colon q \text{ is consistent with } p\}$. If moreover p is isolated then $[p]$ is clopen. By Lemma 0.6, for isolated p, $[p]$ is clopen also in the space of *all* strong 1-types over $\emptyset$.

THEOREM 3.3. *If T is small, w.m., 1-dimensional, not ω-stable and $F(T)$ is finite, then acl* is regular.*

Proof. By the proof of Proposition 5.1.1 from [H], acl is modular on the union of 0-definable w.m. sets in $\mathfrak{C}^{eq}$. As an immediate corollary we get that acl* is modular on $\mathrm{ST}_0^{eq}(\emptyset)$ (= the set of w.m. non-algebraic, non-modular strong 1-types over $\emptyset$ in T^{eq}).

Indeed, suppose that for some finite, acl*-independent sets $P, Q \subseteq \mathrm{ST}_0^{eq}(\emptyset)$, we have

(1) $\dim(P) + \dim(Q) > \dim(P \cup Q) + \dim(\mathrm{acl}^*(P) \cap \mathrm{acl}^*(Q))$,

where for $X \subseteq \mathrm{ST}_0^{eq}(\emptyset)$, $\dim(X)$ is the acl*-dimension of X, and it is well defined, because by Fact 3.1, acl*-dependence satisfies the exchange principle on $\mathrm{ST}_0^{eq}(\emptyset)$. Let $Q = \{q_0, \ldots, q_n\}$.

Let X be a selector from $\{p(\mathfrak{C})\colon p \in P\}$ and choose $Y = \{y_0, \ldots, y_n\}$ so that $y_i \in q_i(\mathfrak{C})$ and whenever q_i is realized in $\mathrm{acl}(X \cup \{y_0, \ldots, y_{i-1}\})$, then

$$y_i \in \mathrm{acl}(X \cup \{y_0, \ldots, y_{i-1}\}).$$

For $A \subseteq \mathfrak{C}$ let $\dim(A)$ be the usual acl-dimension of A, and let acl_0 be acl restricted to 0-definable w.m. sets in T^{eq}.

By the choice of X, Y, for every $q \in P \cup Q$, $\dim\big(q(\mathrm{acl}(X \cup Y))\big) \leqslant 1$. Hence by [B1], for every $r \in \mathrm{ST}_0^{eq}(\emptyset)$, $\dim\big(r(\mathrm{acl}(X \cup Y))\big) \leqslant 1$. It follows that

$$\dim(X) = \dim(P), \quad \dim(Y) = \dim(Q), \quad \dim(X \cup Y) = \dim(P \cup Q) \quad \text{and}$$
$$\dim(\mathrm{acl}_0(X) \cap \mathrm{acl}_0(Y)) = \dim(\mathrm{acl}^*(P) \cap \mathrm{acl}^*(Q)).$$

So (1) translates into

$$\dim(X) + \dim(Y) > \dim(X \cup Y) + \dim(\mathrm{acl}_0(X) \cap \mathrm{acl}_0(Y)),$$

contradicting the modularity of acl.

Another proof of modularity of acl* on $\mathrm{ST}_0^{eq}(\emptyset)$ (without reference to [H]) is implicit in [N]. We shall not distinguish clearly between acl* and acl*^{eq}. Now we prove

(2) *If A is finite, $p \in S(A)$ is isolated, $q \in [p]$ and $r_0 \in \mathrm{ST}(\emptyset|) \setminus \mathrm{acl}_A^*(q)$ then for some clopen U containing q, we have $r_0 \notin \mathrm{acl}_A^*(U)$.*

Here acl^*_A is acl* regarded in $T(A)$. $\mathrm{ST}(A)$ can be canonically identified with $\mathrm{ST}(\emptyset)$.

Proof of (2). Wlog $A=\emptyset$. By modularity, $\mathrm{ST}^{\mathrm{eq}}(\emptyset)$ with acl*-dependence relation can be regarded as a projective space over a division ring D, and as $F(T)$ is finite, it is easy to prove that $D=F(T)$ (this is true also for $F(T)$ infinite, but the proof is harder). So we can choose in $\mathrm{ST}^{\mathrm{eq}}(\emptyset)$ types $r_1,\ldots,r_{n-1}$ such that

(3) $\mathrm{acl}^*(q,r_0)=\dot{\bigcup}_{i<n}\mathrm{acl}^*(r_i)\,\dot{\cup}\,\mathrm{acl}^*(q)$.

(This union is not strictly disjoint, but we neglect modular types by Fact 3.1.)

Choose for $i<n$ a clopen V_i with $r_i\in V_i$ and $\mathrm{acl}^*(q)\cap V_i=\emptyset$ (here we use Lemma 0.2, i.e. smallness). Extend the signature by adding some parameters from $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ to make every V_i 0-definable, and let the present p be the completion of the old p in the new signature such that still $q\in[p]$. Hence, for every $q'\in[p]$ and $i<n$ we have

(4) $\mathrm{acl}^*(q')\cap V_i=\emptyset$.

We treat V_i both as a formula and as a clopen subset of $\mathrm{ST}^{\mathrm{eq}}(\emptyset)$. Choose $a_n\in q(\mathfrak{C})$ and $a_i\in r_i(\mathfrak{C})$ for $i<n$ so that for every $i<n$, $a_i\in\mathrm{acl}(a_0,a_n)$. By (3) we have $\dim(\{a_0,\ldots,a_n\})=2$ and any 2 elements from $\{a_0,\ldots,a_n\}$ are independent. So choose a formula $h(\bar{x})\in L(\emptyset)$ such that

(a) $h(a_0,\ldots,a_n)$ holds,

(b) for $i<j<n$, if $\bar{x}'=\{x_0,\ldots,x_n\}\setminus\{x_i,x_j\}$ and $\bar{a}'=\{a_0,\ldots,a_n\}\setminus\{a_i,a_j\}$ then $h(\bar{x}',a_i,a_j)$ isolates $\mathrm{tp}(\bar{a}'/\{a_i,a_j\})$, *and*

(c) $h(\bar{x})\vdash p(x_n)\,\&\bigwedge_{i<n}V_i(x_i)$.

For $i<n$ let $h_i(x_i,x_n)=(\exists x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_{n-1})h(\bar{x})$. $h_i(a_i,a_n)$ holds, hence $h_i(x_i,a_n)$ is non-algebraic. Choose $E\in\mathrm{FE}(\emptyset)$ such that for every $i<n$ and $b,c\in C$,

either $E(\mathfrak{C},c)\cap h_i(\mathfrak{C},b)$ or $E(\mathfrak{C},c)\cap\neg h_i(\mathfrak{C},b)$ is finite,

and wlog if $c\in V_i(\mathfrak{C})$ then $E(\mathfrak{C},c)\subseteq V_i(\mathfrak{C})$.

It follows that whenever $b\in h_i(\mathfrak{C},a_n)\setminus\mathrm{acl}(\emptyset)$ then

$$E(\mathfrak{C},b)\setminus h_i(\mathfrak{C},a_n)\subseteq\mathrm{acl}(a_n)\cap V_i(\mathfrak{C})\subseteq\mathrm{acl}(\emptyset)\qquad(\text{by }(4)).$$

Hence for $i<n$, there is $\bar{c}_i\subseteq C/E$ such that for every $r\in\mathrm{ST}^{\mathrm{eq}}(\emptyset)\cap V_i$, $r(x)\cup\{h_i(x,a_n)\}$ is consistent iff for $c\in r(\mathfrak{C})$, $c/E\in\bar{c}_i$. Extend the signature by $\mathfrak{C}/E$, and extend p to a complete type over $\emptyset$ in this new signature so that still $q\in[p]$. So for $i<n$ there are 0-definable clopen subsets $V_i^0\subseteq V_i$ with $r_i\in V_i^0$ such that for every $q'\in[p]$ and every $i<n$, if $r_i'\in V_i^0$ then for some $a_0',\ldots,a_n'$ with $a_n'\in q'(\mathfrak{C})$, $a_i'\in r_i'(\mathfrak{C})$ and $a_j'\in V_j(\mathfrak{C})$ for $i\neq j<n$, we have $h(a_0',\ldots,a_n')$ holds. It follows that $a_j'\notin\mathrm{acl}(\emptyset)$ for $j\leqslant n$, hence $a_j'\in V_j^0(\mathfrak{C})$ for $j<n$. We get that whenever $q'\in[p]$ and for some $i<n$, $r_i'\in V_i^0$, then there are $r_j'\in V_j^0$ for $i\neq j<n$ such that

(5) $\mathrm{acl}^*(q',r_i')=\mathrm{acl}^*(q')\cup\bigcup_{j<n}\mathrm{acl}^*(r_j')$.

Now we prove that $\mathrm{acl}^*([p])\cap V_i^0=\emptyset$ for every i. If not, take the minimal m such that for some $q_0,\ldots,q_m\in[p]$, for some $i<n$ and $r_i'\in V_i^0$, we have $r_i'\in\mathrm{acl}^*(q_0,\ldots,q_m)$. By (5) and modularity of acl*, we see that there is $j<n$ and $r_j'\in\mathrm{acl}^*(r_i',q_0)\cap V_j^0\cap\mathrm{acl}^*(q_1,\ldots,q_m)$, which contradicts the minimality of m.

In particular we get $r_0\notin\mathrm{acl}^*([p])$, and (2) is proved.

Now we approach the general case. Let $R\subseteq\mathrm{ST}(\emptyset)$ be finite and $r_0\notin\mathrm{acl}^*(R)$. Let A be a selector from $\{r(\mathfrak{C})\colon r\in R\}$. Take an isolated non-algebraic $p\in S(A)$, and $q\in[p]\setminus\mathrm{acl}^*(R\cup\{r_0\})$. In particular $r_0\notin\mathrm{acl}^*(R\cup\{q\})$. By (2) there is a clopen U containing q such that $r_0\notin\mathrm{acl}^*_A(U)$. Thus also, by Fact 3.1 (or just by the definition of acl*), $r_0\notin\mathrm{acl}^*(R\cup U)$. By Fact 3.2, $\mathrm{acl}^*(R\cup U)$ contains a clopen neighbourhood V and R, so we get $r_0\notin\mathrm{acl}^*(V)$.

Now, take any decreasing sequence U_n, $n<\omega$, of clopen subsets of $\mathrm{ST}(\emptyset)$ with $\bigcap_n U_n=R$. We see that for some n, $U_n\subseteq V$, hence $r_0\notin\mathrm{acl}^*(U_n)$, and we are done.

LEMMA 3.4. *If* acl* *is regular then in Theorem* 1.5 *condition* (B) *holds.*

Proof. We keep the notation from Theorem 1.5. Take $a\in p_0(\mathfrak{C})$. Hence, over a, p_0 splits into infinitely many types $q_n\in S(a)$, $n<\omega$, and let $q_0\|\mathrm{stp}(a/\emptyset)$. Let us work in $T(a)$. By regularity of acl*, notice that as any $[q_n]$ is closed, if $\mathrm{acl}^*(q_0)\cap[q_n]=\emptyset$ then for some clopen $U\ni q_0$, $\mathrm{acl}^*(U)\cap[q_n]=\emptyset$. Using this, we can choose types q_{n_i}, $i<\omega$, such that for every i, $\mathrm{acl}^*(\bigcup\{[q_{n_j}]\colon j>i\})\cap[q_{n_i}]=\emptyset$. In particular, $\{q_{n_i}\colon i<\omega\}$ is an almost orthogonal family of types.

COROLLARY 3.5 [B3]. *If* $F(T)$ *is finite then in Theorem* 1.5 *condition* (B) *holds.*

CONJECTURE. If T is small, 1-dimensional, w.m., not ω-stable, then acl* is regular.

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