

Contents of Volume 134, Number 1

	Pages
R. Dobreńko and Z. Kucharski, On the generalization of the Nielsen number	1-14
N. Marmaridis, One point extensions of trees and quadratic forms	15-35
Zhi-Wei Sun, Finite coverings of groups	37-53
P. Mrozik, Finite-dimensional complement theorems in shape theory and their relation to S -duality	55-72
S. Mardešić and J. Segal, Mapping approximate inverse systems of compacta	73-91

The FUNDAMENTA MATHEMATICAE publishes papers devoted to *Set Theory, Topology, Mathematical Logic and Foundations, Real Functions, Measure and Integration, Abstract Algebra*

Each volume consists of three separate issues

Manuscripts and correspondence should be addressed to:

FUNDAMENTA MATHEMATICAE,

Śniadeckich 8, P. O. Box 137, 00-950 Warszawa, Poland, telex PL 816 112

Papers for publication should be submitted in two typewritten (double spaced) copies and contain a short abstract. A complete list of all handwritten symbols with indications for the printer should be enclosed. Special typfaces should be indicated according to the following code: script letters — by encircling the typed Roman letter in black, German letters — by typing the Roman equivalent and underlining in green, boldface letters — by straight black underlining. The authors will receive 75 reprints of their articles.

Correspondence concerning subscriptions, library exchange and back numbers should be sent to:

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Śniadeckich 8, P. O. Box 137, 00-950 Warszawa, Poland, telex PL 816 112

Copyright by Instytut Matematyczny PAN, Warszawa 1990

Published by PWN — Polish Scientific Publishers

ISBN 83-01-09267-X

ISSN 0016-2736

DRUKARNIA UNIwersytetu Jagiellońskiego w Krakowie

On the generalization of the Nielsen number

by

R. Dobreńko (Olsztyn) and Z. Kucharski (Gdańsk)

Abstract. In this paper we consider the following situation: $f: X \rightarrow Y$ is a continuous map of topological spaces and B is a nonempty subset of Y . If we make additional assumptions (see part I), we can define the nonnegative number $N(f; B)$ called the Nielsen number of f relative to B . We prove that $\text{card}(f^{-1}(B)) \geq N(f; B)$. Moreover, if we assume that X, Y, B are smooth, compact and orientable manifolds such that $\dim X = \dim Y - \dim B$, then using the index theory we obtain the Wecken type theorem: Every continuous map $f: X \rightarrow Y$ is homotopic to a map $f_1: X \rightarrow Y$ such that $\text{card}(f_1^{-1}(B)) = N(f; B)$.

0. Introduction. Let $f: X \rightarrow X$ be a continuous map of a space X into itself. A fixed point of a map f is a solution of the equation $x = f(x)$.

Let $\text{Fix}(f)$ denote the set of all fixed points of a map f . Fixed point theory deals with the properties of $\text{Fix}(f)$ in relation to the properties of the space X and the map f .

The question of existence of fixed points (i.e. whether or not $\text{Fix}(f) \neq \emptyset$) is of interest here, as well as the problem of their number $\# \text{Fix}(f)$. (We denote by $\# S$ the cardinality of the set S .) Theorems on the existence of solutions of various equations are usually reduced to the problem of existence of a fixed point. As an example we can mention theorems on the existence of solutions of elliptic partial differential equations or theorems on the existence of closed orbits for dynamical systems, etc.

In many technical problems a positive answer to the question of existence of solution is not satisfactory. Sometimes we want to know the number of solutions or any estimates of their number, and how it changes under continuous deformations of the initial map.

Questions concerning the behaviour of the set $\text{Fix}(f)$ under continuous deformations of the map f are also important. These problems are the object of the Nielsen theory of fixed points.

In 1927 Jacob Nielsen ([12]) showed how we can attribute to a continuous map $f: X \rightarrow X$ a nonnegative integer $N(f)$, called later the *Nielsen number*, which is a lower estimate of the number of fixed points of f . He has also shown that if $g: X \rightarrow X$ is a map homotopic to f then $N(f) = N(g)$; that means, the Nielsen number is a lower estimate of the number of fixed points of every map homotopic to f .

In 1942 F. Wecken ([17]) proved that if X is a polyhedron such that, for every subpolyhedron L of X of dimension less than two, $X \setminus L$ is connected, then in the homotopy class of any continuous map $f: X \rightarrow X$ there exists a continuous map $g: X \rightarrow X$ such that $N(f) = N(g) = \# \text{Fix}(g)$.

Some generalizations of the above result were obtained by F. Weier ([17]) in 1953, and by G. H. Shi ([14]) in 1966.

In this paper we introduce a new homotopy invariant for a map $f: X \rightarrow Y$ from a compact locally path connected space X into a locally uniformly contractible space Y , given a closed subset B of Y . In particular cases we obtain the classical Nielsen number of fixed points, as well as the Nielsen number of coincidence of two or more maps.

(If $p_1, p_2, \dots, p_s: X \rightarrow Z$, then we consider

$$Y = Z \times Z \times \dots \times Z, B = \{(z, z, \dots, z): z \in Z\}$$

and $f: X \rightarrow Y$ given by $f(x) = (p_1(x), p_2(x), \dots, p_s(x))$. In this case, $f^{-1}(B)$ is the set of all coincidence points of the maps p_1, p_2, \dots, p_s , i.e. points $x \in X$ such that $p_1(x) = p_2(x) = \dots = p_s(x)$).

We call this invariant the *Nielsen number of f with respect to B* .

We will also investigate the set $f^{-1}(B)$. The following question may be posed: how does the set $f^{-1}(B)$ change under continuous deformations of the map f ? We deal with this problem in section 3, where we prove new theorems analogous to the Wecken theorem for fixed points.

If $f: X \rightarrow Y$ is a continuous map, $X, Y, B \subset Y$ are smooth closed oriented manifolds and $\dim X + \dim B = \dim Y$, a local index with respect to B for a map f is defined. This index is a modification of the smooth versions of the fixed point and intersection indices. If it is nonzero then there is a point $x \in X$ such that $f(x) \in B$.

1. Nielsen number relative to B . We make the following assumptions:

Z.1. X is a compact locally path connected space.

Z.2. B is a closed subspace of Y such that there exists an open neighbourhood $W \subset Y$ of B which can be deformed to B in Y , i.e. there exists a map $D: W \times I \rightarrow Y$ such that:

- (i) $D(w, 0) = w$,
- (ii) $D(w, 1) \in B$,
- (iii) $D(w, t) = w$ for $w \in B, t \in I$, (I — the unit interval).

Z.3. $f: X \rightarrow Y$ is a given continuous function.

(1.1) Remark. If (Z, d) is a uniformly locally contractible (u.l.c.) metric space (e.g. a compact metric ANR, see [1], Chapter A, Th. 3) then a pair $Y = Z \times Z, B = \Delta$ where $\Delta = \{(z, z): z \in Z\}$ is the diagonal in $Z \times Z$, satisfies the assumption Z.2.

Proof. Given $\sigma > 0$, let $W = W_\sigma(Z) = \{(z, z') \in Z \times Z: d(z, z') < \sigma\}$. It follows from the assumptions of Z being u.l.c. that for every $\varepsilon > 0$ there exists $\delta > 0$ and

a map $\gamma: W \times I \rightarrow Z$ satisfying the following conditions:

$$\gamma((z, z'), 0) = z,$$

$$\gamma((z, z'), 1) = z',$$

$$\gamma((z, z), t) = z,$$

$$\text{diam } \gamma[(z, z') \times I] < \varepsilon.$$

Fix $\varepsilon > 0$. A set W is of course an open neighbourhood of the diagonal Δ .

It is easy to see that a map $D: W \times I \rightarrow Z \times Z$ defined by

$$D((z, z'), t) = \gamma([(z, z'), t], z)$$

satisfies Z.2.

(1.2) DEFINITION. Let $f: X \rightarrow Y$ be a continuous map and let $x_0, x_1 \in f^{-1}(B)$. We say that x_0 and x_1 are in *Nielsen relation with respect to the subset B* if there exists a path $\alpha: I \rightarrow X$ such that $\alpha(0) = x_0, \alpha(1) = x_1$ and the path $f \circ \alpha: I \rightarrow Y$ is homotopic $\text{rel } \{0, 1\}$ to some path $\eta: I \rightarrow Y$ such that $\eta(I) \subset B$. Obviously, this is an equivalence relation in the set $f^{-1}(B)$. We call this relation the *Nielsen relation of the map f with respect to the set B* . Equivalence classes of this relation are called *Nielsen classes of f with respect to B* .

(1.3) THEOREM. The set of Nielsen classes of f with respect to B is finite.

Proof. First we notice that $f^{-1}(B)$ is a closed subset of a compact space X , and hence is compact. So it suffices to show that every Nielsen class is an open set in $f^{-1}(B)$. Let us fix a point $x_0 \in f^{-1}(B)$. Then there exists an open neighbourhood U of x_0 in X such that $f(U)$ is contained in an open set $W \supset B$. Because of local path connectedness of the space X we may assume that U is path connected. Consider a point $x_1 \in f^{-1}(B) \cap U$. There exists a path $\alpha: I \rightarrow U$ such that $\alpha(0) = x_0, \alpha(1) = x_1$.

Define a homotopy

$$H: I \times I \rightarrow Y, \quad H(t, s) = D(f \circ \alpha(t), s) \quad \text{for } t, s \in I.$$

It is easy to check that the homotopy joins $\text{rel } \{0, 1\}$ the path $f \circ \alpha$ to $\eta = H(\cdot, 1) = D(f \circ \alpha(\cdot), 1): I \rightarrow Y$. Regarding assumption Z.2(ii) we obtain that $\eta(I) \subset B$. ■

Let now $F: X \times I \rightarrow Y$ be a homotopy and N an arbitrary Nielsen class of F with respect to B .

For any $0 \leq t_0 \leq t_1 \leq 1$ we denote:

$$N^{t_0, t_1} = N \cap X \times [t_0, t_1], \quad F^{t_0, t_1}: X \times [t_0, t_1] \rightarrow Y,$$

where $F^{t_0, t_1} = F|_{X \times [t_0, t_1]}$ is the restriction of F to the set $X \times [t_0, t_1]$.

(1.4) LEMMA. The set N^{t_0, t_1} is a Nielsen class of the map F^{t_0, t_1} with respect to B or is empty.

Proof. We have to show that:

[points $(x_0, s_0), (x_1, s_1) \in F^{-1}(B) \cap X \times [t_0, t_1]$ are in Nielsen relation of the map F with respect to B] \Leftrightarrow [points $(x_0, s_0), (x_1, s_1)$ are in Nielsen relation of the map F^{t_0, t_1} with respect to B].

\Leftarrow is obvious.

We prove \Rightarrow . Assume that there is a path $\alpha = (\alpha_1, \alpha_2): I \rightarrow X \times I$ (where $\alpha_1: I \rightarrow X$, $\alpha_2: I \rightarrow I$) joining points (x_0, s_0) and (x_1, s_1) such that the path $F \circ \alpha: I \rightarrow Y$ is homotopic rel $\{0, 1\}$ to some path $\eta: I \rightarrow Y$ lying in B (i.e. $\eta(I) \subset B$). For $t \in I$ we define a map $r_t: I \rightarrow I$ as follows:

$$r_t(s) = \begin{cases} (1-t) \cdot s + t \cdot t_0 & \text{for } s \leq t_0, \\ s & \text{for } t_0 \leq s \leq t_1, \\ (1-t) \cdot s + t \cdot t_1 & \text{for } t_1 \leq s. \end{cases}$$

Now we define a homotopy $H: I \times I \rightarrow Y$ putting:

$$H(t, s) = F(\alpha_1(t), r_s \circ \alpha_2(t)).$$

Define $H_s: I \rightarrow Y$ by $H_s = H(\cdot, s) = F \circ (\alpha_1, r_s \circ \alpha_2)$.

Then for $s = 0$

$$H_0 = F \circ (\alpha_1, r_0 \circ \alpha_2) = F \circ (\alpha_1, \alpha_2) = F \circ \alpha, \text{ because } r_0 = \text{id}_I;$$

for $s = 1$

$$H_1 = F \circ (\alpha_1, r_1 \circ \alpha_2) = F^{t_0, t_1} \circ (\alpha_1, r_1 \circ \alpha_2).$$

H is a homotopy rel $\{0, 1\}$.

Since the path η is homotopic to $F \circ \alpha$ rel $\{0, 1\}$ and since the relation of homotopy is transitive, η is homotopic to the path $F^{t_0, t_1} \circ (\alpha_1, r_1 \circ \alpha_2)$ rel $\{0, 1\}$. ■

(1.5) COROLLARY. Let $f_t: X \rightarrow Y$, $t \in I$, be the map defined by

$$f_t(x) = F(x, t), \quad t \in I, x \in X.$$

Let $N_t = \{x \in X: (x, t) \in N\}$ denote the t -section of the set N . Then N_t is a Nielsen class of the map f_t with respect to B or is empty.

(1.6) DEFINITION. Let $F: X \times I \rightarrow Y$ be a homotopy joining maps $f_0, f_1: X \rightarrow Y$.

Let N_0, N_1 be Nielsen classes of maps f_0 and f_1 with respect to B , respectively. We say that the classes N_0, N_1 are in F -Nielsen relation provided there is a Nielsen class N of the map F with respect to B such that N_0 and N_1 are 0-section and 1-section of the class N .

(1.7) LEMMA. F -Nielsen relation is right and left-univalent.

We omit an easy proof. ■

(A relation R is right-univalent iff $[(q, q') \in R \text{ and } (q, q'') \in R] \Rightarrow [q' = q'']$; left-univalent iff $[(q', q) \in R \text{ and } (q'', q) \in R] \Rightarrow [q' = q'']$).

(1.8) DEFINITION. Let $f: X \rightarrow Y$ be a continuous map and $N_0 \subseteq f^{-1}(B)$ its Nielsen class with respect to B . This class is said to be an *essential Nielsen class*

of f with respect to B if for each homotopy $F: X \times I \rightarrow Y$ joining the map f to some map $f_1: X \rightarrow Y$ there is a Nielsen class N_1 of f_1 with respect to B , which is in F -Nielsen relation to N_0 .

The number of essential Nielsen classes of f with respect to B is called the *Nielsen number of f with respect to B* and is denoted by $N(f; B)$.

The following is an obvious consequence of the above definitions.

(1.9) THEOREM. The Nielsen number $N(f; B)$ is a homotopy invariant. Moreover, every map f' homotopic to f has at least $N(f; B) = N(f'; B)$ points $x \in X$ such that $f'(x) \in B$.

It is quite easy to see (regarding (1.1)) that if X is a locally path connected space and Z is uniformly locally contractible then the following is meaningful:

(1.10) DEFINITION. Let $p, q: X \rightarrow Z$ be given maps. They define a map

$$f = (p, q): X \rightarrow Y = Z \times Z \quad \text{by} \quad f(x) = (p(x), q(x)).$$

The *coincidence Nielsen number* of maps p and q is the number

$$n(p, q) = N(f; \Delta)$$

(i.e. the Nielsen number of f with respect to the diagonal

$$\Delta = \{(z, z) \in Z \times Z: z \in Z\}$$

in $Z \times Z$). It is a lower estimate of the number of coincidence points of maps p and q .

2. The local index. Let M be a smooth compact manifold with boundary ∂M , $\dim M = m$, $A \subset M$ a closed subset of M . Let, moreover, N be a smooth closed manifold (compact without boundary), $\dim N = n$, $P \subset N$ a smooth closed submanifold, $\dim P = k$.

We fix a metric d_N in N and for any two maps $f, f': M \rightarrow N$ we put

$$\varrho_N(f, f') = \sup \{d_N(f(x), f'(x)): x \in M\}.$$

Along with the above assumptions, we will use in this section the following well-known facts:

F.1. If $f: M \rightarrow N$ is a continuous map, smooth on A (i.e. there exists an open nbd V of A such that f is smooth on V), then for each $\varepsilon > 0$ there is a smooth map $f': M \rightarrow N$ such that $f|_A = f'|_A$ and $\varrho_N(f, f') < \varepsilon$.

F.2. If $f: M \rightarrow N$ is a smooth map, transversal to a submanifold P on the set A , then for each $\varepsilon > 0$ there exists a smooth map $f': M \rightarrow N$ transversal to P on the whole of M such that $f|_A = f'|_A$ and $\varrho_N(f, f') < \varepsilon$.

F.3. If $f: M \rightarrow N$ is a smooth map transversal to P and $f|_{\partial M}$ is transversal to P , then $f^{-1}(P)$ is a smooth submanifold of M of dimension $m+k-n$ (if $m+k-n < 0$ then $f^{-1}(P) = \emptyset$) with boundary $\partial f^{-1}(P) = f^{-1}(P) \cap \partial M$.

For details see, for example, [5].

We will assume in the sequel (through Sections 2 and 3) that X, Y, B are smooth, oriented and closed manifolds, B is a submanifold of Y , $\dim X = n$, $\dim B = k$, $\dim Y = n + k$.

Orientations coherent with the fixed orientations of these manifolds will be called shortly positive; the opposite ones — negative.

(2.1) DEFINITION. Let $U \subset X$ be an open subset of X . A continuous map $f: X \rightarrow Y$ is admissible on U with respect to a submanifold B if $f^{-1}(B) \cap \partial U = \emptyset$. Similarly, a homotopy $F: X \times I \rightarrow Y$ is admissible on U with respect to B if $F^{-1}(B) \cap (\partial U \times I) = \emptyset$.

Let $f: X \rightarrow Y$ be a smooth map transversal to B and admissible on U with respect to B , and let $x \in f^{-1}(B)$. Let us fix ordered bases $\{\xi_1, \dots, \xi_n\}$ and $\{\eta_1, \dots, \eta_k\}$ in the tangent spaces $T_x X$ and $T_{f(x)} B$, respectively, which induce positive orientations.

By the assumption of transversality of f to B , we have

$$df_x(T_x X) \oplus T_{f(x)} B = T_{f(x)} Y$$

where $df_x: T_x X \rightarrow T_{f(x)} Y$ denotes the differential of f at point x . A local index $I(f, B, x)$ of f with respect to B in a point x is defined as follows:

$$I(f, B, x) = \begin{cases} 1 & \text{if the basis } \{df_x(\xi_1), \dots, df_x(\xi_n), \eta_1, \dots, \eta_k\} \text{ induces the} \\ & \text{positive orientation in } T_{f(x)} Y, \\ -1 & \text{if the basis induces the negative orientation in } T_{f(x)} Y. \end{cases}$$

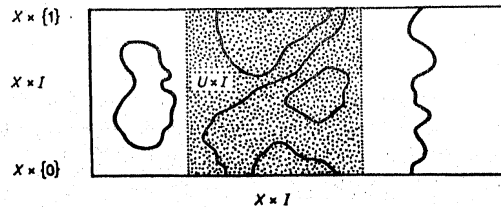
The local index $I(f, B, U)$ of f with respect to B on a set U is defined by

$$I(f, B, U) = \sum_{x \in f^{-1}(B) \cap U} I(f, B, x).$$

This definition is correct since $f^{-1}(B)$ consists of a finite number of points.

(2.2) LEMMA. Let $f_0, f_1: X \rightarrow Y$ be smooth maps transversal to B and admissible with respect to B on an open set $U \subset X$. Moreover, let there exist a continuous homotopy admissible with respect to B on U , joining f_0 to f_1 . Then $I(f_0, B, U) = I(f_1, B, U)$.

Proof. Making use of F.1 and F.2, we may assume without loss of generality that the admissible homotopy $F: X \times I \rightarrow Y$ joining f_0 to f_1 is smooth and transversal to B . It follows from F.3 that $F^{-1}(B)$ is a disjoint union of components homeomorphic to a circle or to an interval with ends in $X \times \{0\} \cup X \times \{1\}$. Moreover, every component lies out of $\partial U \times I$ (see the picture).



Notice that each orientation of the manifold $X \times I$ induces an orientation of the boundary $\partial(X \times I) = X \times \{0\} \cup X \times \{1\}$ (see the details in Milnor [10]).

Let $(x, t) \in \partial(X \times I)$. We choose in $T_{(x,t)}(X \times I)$ an oriented basis $\{\xi_0, \xi_1, \dots, \xi_n\}$ giving a positive orientation in such a way that vectors $\xi_1, \dots, \xi_n \in T_{(x,t)} \partial(X \times I)$. We accept the agreement that vectors $\xi_1, \dots, \xi_n \in T_{(x,t)} \partial(X \times I)$ induce positive orientation if ξ_0 is 'directed outside' of $X \times I$. Let us fix an orientation in $X \times I$ such that $X \times \{0\}$ is oriented coherently with a fixed orientation of X . Then of course $X \times \{1\}$ is oriented in opposite way. Consider an arbitrary 1-manifold γ which is a component of $F^{-1}(B)$.

We orient this manifold in a canonical way. Let $(x, t) \in \gamma$ and let

$$\xi_0(x, t), \xi_1(x, t), \dots, \xi_n(x, t) \in T_{(x,t)}(X \times I)$$

be an oriented basis of $T_{(x,t)}(X \times I)$ giving a positive orientation of this space. Assume that $\xi_0(x, t) \in T_{(x,t)} \gamma$. Then the differential $dF_{(x,t)}: T_{(x,t)}(X \times I) \rightarrow T_{F(x,t)} Y$ of the map F at the point $(x, t) \in \gamma \subset X \times I$ sends a vector $\xi_0(x, t) \in T_{(x,t)} \gamma \subset T_{(x,t)}(X \times I)$ to a vector $dF_{(x,t)}(\xi_0(x, t)) \in T_{F(x,t)} B$, whereas vectors

$$dF_{(x,t)} \xi_1(x, t), \dots, dF_{(x,t)} \xi_n(x, t)$$

together with a basis of the space $T_{F(x,t)} B$ span the whole space $T_{F(x,t)} Y$.

Let vectors $\eta_1, \dots, \eta_k \in T_{F(x,t)} B$ form an oriented basis inducing a positive orientation of the manifold B at the point $F(x, t)$. We make the following agreement:

A vector $\xi_0(x, t) \in T_{(x,t)} \gamma$ gives the positive orientation of a manifold γ when the oriented basis

$$dF_{(x,t)} \xi_1(x, t), \dots, dF_{(x,t)} \xi_n(x, t), \eta_1, \dots, \eta_k \in T_{F(x,t)} Y$$

gives the positive orientation of the manifold Y . The components of $F^{-1}(B)$, which are closed curves or which are curves outside of $U \times I$ (see the picture), have no influence on the numbers $I(f_0, B, U)$ and $I(f_1, B, U)$.

Let then γ be a 1-manifold in $U \times I$ homeomorphic to an interval with ends in $U \times \{0\} \cup U \times \{1\}$. Let $\xi_0(x, t), \xi_1(x, t), \dots, \xi_n(x, t) \in T_{(x,t)}(X \times I)$ be an oriented basis defining a positive orientation in $X \times I$. Furthermore, let the vector $\xi_0(x, t)$ define an orientation of γ . If points $(x_0, t_0), (x_1, t_1) \in U \times \{0\} \cup U \times \{1\}$ are ends of a curve γ then one of the vectors $\xi_0(x_0, t_0), \xi_1(x_1, t_1)$ is directed outside and the other inside of the manifold $X \times I$.

We may assume that:

$$\xi_1(x_0, t_0), \dots, \xi_n(x_0, t_0) \in T_{(x_0, t_0)} \partial(X \times I),$$

$$\xi_1(x_1, t_1), \dots, \xi_n(x_1, t_1) \in T_{(x_1, t_1)} \partial(X \times I).$$

Hence the ordered bases

$$\{\xi_1(x_0, t_0), \dots, \xi_n(x_0, t_0)\} \text{ and } \{\xi_1(x_1, t_1), \dots, \xi_n(x_1, t_1)\}$$

define opposite orientations in $\partial(X \times I)$.

After simple calculations we obtain that if points $(x_0, t_0), (x_1, t_1)$ lie in the same level; say in $X \times \{0\}$, then $I(f_0, B, x_0) = I(f_1, B, x_1)$. (If e.g. $t_0 = 0$ and $t_1 = 1$ then $I(f_0, B, x_0) = I(f_1, B, x_1)$). Summing the indices over points, we finish the proof. ■

(2.3) Remark. A compact manifold is an ANR-space. Therefore, for each $\varepsilon > 0$ there exists $0 < \delta_\varepsilon < \varepsilon$ such that, for any continuous maps $f, f': X \rightarrow Y$, if $\varrho_Y(f(x), f'(x)) < \delta_\varepsilon$ then f and f' are ε -homotopic, i.e. there is a homotopy $F: X \times I \rightarrow Y$ joining f to f' and satisfying the condition

$$\text{diam}\{F(x, t) : t \in I\} < \varepsilon \quad \text{for each } x \in X. \quad \blacksquare$$

Let now $f: X \rightarrow Y$ be a continuous map admissible on $U \subset X$ with respect to $B \subset Y$. Take $\varepsilon < 1/2 \inf\{d_Y(f(x), y) : x \in \partial U, y \in B\}$. It follows from F.1 and F.2 that there exists a map $f': X \rightarrow Y$ transversal to B on X and such that

$$\varrho_Y(f(x), f'(x)) < \delta_\varepsilon$$

(see 2.3); then of course, f' is admissible on U with respect to B . We can define the index $I(f, B, U)$ as the index of the map f' on U with respect to B .

(2.4) DEFINITION. $I(f, B, U) = I(f', B, U)$.

(2.5) LEMMA. Definition 2.4 is correct, i.e. the number $I(f, B, U)$ does not depend on the choice of an approximation f' of the map f .

Proof. This follows from the fact that any two approximations are 2ε -homotopic (hence admissibly homotopic) and from Lemma 2.2.

It is easy to verify the following properties of the local index (by reducing the problem to maps which are transversal to B).

(2.6) HOMOTOPY INVARIANCE. If $f_0, f_1: X \rightarrow Y$ are continuous maps admissibly homotopic on an open set $U \subset X$, then $I(f_0, B, U) = I(f_1, B, U)$.

(2.7) EXCISION. If $V \subset U \subset X$ are open sets and if $f: X \rightarrow Y$ is a continuous map such that $f^{-1}(B) \cap \text{Cl}(U) = f^{-1}(B) \cap V$ then $I(f, B, U) = I(f, B, V)$.

(2.8) ADDITIVITY. If $U, V \subset X$ are open, disjoint sets and $f: X \rightarrow Y$ is a continuous map such that $f^{-1}(B) \cap (\partial U \cup \partial V) = \emptyset$ then

$$I(f, B, U \cup V) = I(f, B, U) + I(f, B, V).$$

(2.9) INTERSECTION. If $f: X \rightarrow Y$ is a map admissible on an open set $U \subset X$ with respect to $B \subset Y$, then $I(f, B, U) \neq 0 \Rightarrow f^{-1}(B) \cap U \neq \emptyset$.

We omit easy proofs of these properties. ■

(2.10) DEFINITION. Let $N \subset X$ be a Nielsen class with respect to B of a continuous map $f: X \rightarrow Y$. We define the index $I(N, B)$ of this class putting

$$I(N, B) = I(f, B, U)$$

where $U \subset X$ is an open set such that $f^{-1}(B) \cap \text{Cl}(U) = N = f^{-1}(B) \cap U$.

The correctness of this definition follows immediately from (2.7).

(2.11) THEOREM. If $N_0 \subset X$ is a Nielsen class with respect to B of a continuous map $f_0: X \rightarrow Y$ and $I(N_0, B) \neq 0$ then N_0 is essential in the sense of Definition 1.8.

Proof. Assume that $F: X \times I \rightarrow Y$ is a homotopy starting from f_0 and let $N \subset F^{-1}(B)$ be a Nielsen class of F with respect to B such that N_0 is the 0-section of N .

By Corollary (1.5), the t -section N_t of the class N is a Nielsen class of the map $f_t: X \rightarrow Y$, where $f_t(x) = F(x, t)$, $t \in I$, $x \in X$. Let $U \subset X$ be an open set such that $N_t \subset U$ and $f^{-1}(B) \cap \text{Cl}(U) = N_t$. One easily checks that there exists $\varepsilon > 0$ such that $F^{-1}(B) \cap \text{Cl}(U) \times [t - \varepsilon, t + \varepsilon] = N \times [t - \varepsilon, t + \varepsilon] = F^{-1}(B) \cap U \times [t - \varepsilon, t + \varepsilon]$. It follows from the homotopy invariance of the index that for any integer k the set $\{t \in I: I(N_t, B) = k\}$ is open in I . Because of the connectedness of the interval I we obtain that $I(N_t, B) = I(N_0, B)$ for each $t \in I$.

The class N_0 is then essential by (2.9). ■

(2.12) Remark. Similarly as in Section 1 all the facts exposed in Section 2 can be translated into facts concerning coincidence.

Set $Y = Z \times Z$, where Z is a closed smooth oriented manifold of dimension n .

To say that a map $f = (p, q): X \rightarrow Y$ (where $p, q: X \rightarrow Z$) is admissible on U means that there are no coincidence points on ∂U .

Similarly, for an admissible homotopy $F = (F_1, F_2): X \times I \rightarrow Z \times Z = Y$ on U there are no coincidence points of F_1 and F_2 on $\partial U \times I$.

Therefore we define the local coincidence index $i(p, q, U)$ by

$$i(p, q, U) = I(f, \Delta, U)$$

where $\Delta = \{(z, z) \in Z \times Z: z \in Z\}$.

Of course, this index satisfies conditions (2.6)–(2.10).

3. Wecken's Theorem. Let P, Q be smooth submanifolds of a smooth oriented manifold R , $\dim P = p \geq 3$, $\dim Q = q \geq 3$, $\dim R = r$ and $r = p + q$. Let $x \in P \cap Q$ be a transversal point of intersection of P and Q (i.e. $T_x P \oplus T_x Q = T_x R$). Assume that (ξ_1, \dots, ξ_p) , (η_1, \dots, η_q) are oriented bases in $T_x P$ and $T_x Q$ determining positive orientations in P and Q , respectively. Denote

$$\text{ind}(P: Q, x) = \begin{cases} 1 & \text{if the basis } (\xi_1, \dots, \xi_p, \eta_1, \dots, \eta_q) \text{ of the space } T_x R \\ & \text{determines a positive orientation of a manifold } R, \\ -1 & \text{in the other case.} \end{cases}$$

The number $\text{ind}(P: Q, x)$ is called the *intersection index of submanifolds P and Q at a point x* .

In the sequel we make use of well-known Whitney's theorem.

(3.1) THEOREM (Whitney). Assume that $x, y \in P \cap Q$ are transversal intersection points of manifolds P and Q such that $\text{ind}(P: Q, x) = -\text{ind}(P: Q, y)$.

Further suppose that there exist smooth paths $\alpha, \beta: I \rightarrow R$ and a smooth homotopy $h: I \times I \rightarrow R \setminus \{0, 1\}$ such that

- (i) $\alpha(0) = x = \beta(0)$,
- (ii) $\alpha(1) = y = \beta(1)$,
- (iii) $\alpha(I) \subset P$, $\beta(I) \subset Q$,
- (iv) $h(\cdot, 0) = \alpha$, $h(\cdot, 1) = \beta$,
- (v) $\{\alpha(I) \cup \beta(I)\} \cap (P \cap Q) = \{x, y\}$.

Then for any open neighbourhood $U \subset P$ of $\alpha(I)$ there exists a smooth isotopy $H: P \times I \rightarrow R$ with a compact support contained in U , transforming the manifold P onto a manifold P' such that $P \cap Q \setminus \{x, y\} = P' \cap Q$.

(Cf. J. W. Milnor [11] and B. J. Jiang [9].) ■

Recall that X, Y are closed oriented smooth manifolds, B is a closed oriented smooth submanifold of Y and $\dim X = \dim Y - \dim B$ and let $\dim X \geq 3$.

(3.2) **A procedure of cancelling points with image in B .** Assume that

- (1) $f: X \rightarrow Y$ is a smooth map such that $f^{-1}(B)$ consists of a finite number of points.
- (2) $x_0, x_1 \in f^{-1}(B)$ are points where the map f is transversal to B .
- (3) $I(f, B, x_0) + I(f, B, x_1) = 0$.
- (4) $x_0, x_1 \in N$, where N is a Nielsen class of the map f with respect to B .

It is easy to check that then the points $(x_0, f(x_0)), (x_1, f(x_1)) \in X \times Y$ are transversal intersection points of submanifolds $\Gamma_f = \{(x, f(x)) \in X \times Y: x \in X\}$ and $X \times B$ in the manifold $X \times Y$, and $I(f, B, x_i) = (-1)^{\text{ind}}(\Gamma_f: X \times B, (x_i, f(x_i)))$ for $i = 0, 1$. $X \times Y, X \times B$ have the product orientations and the orientation of Γ_f is induced from X by the diffeomorphism $(1, f): X \rightarrow \Gamma_f$.

(3.2.1) Since $x_0, x_1 \in N$ are in the same Nielsen class, there are paths $\sigma: I \rightarrow X$ and $\mu: I \rightarrow Y$ such that $\sigma(0) = x_0$, $\sigma(1) = x_1$, $\mu(0) = f(x_0)$, $\mu(1) = f(x_1)$, $\mu(I) \subset B$ and there is a homotopy $H_t: f \circ \sigma \sim \mu \text{ rel } \{0, 1\}$ such that $H_0 = f \circ \sigma$, $H_1 = \mu$.

We reduce the situation to Whitney's Theorem (3.1), defining:

$$P = \Gamma_f, \quad Q = X \times B, \quad R = X \times Y.$$

We define smooth paths $\alpha, \beta: I \rightarrow R$ as follows:

DEFINITION of α . A path σ is replaced by a smooth path $\hat{\sigma}: I \rightarrow X$ such that $\hat{\sigma}(I) \cap f^{-1}(B) = \{x_0, x_1\}$; this is possible, since $f^{-1}(B)$ contains of a finite number of points and $\dim X \geq 3$.

A smooth path $\alpha: I \rightarrow X \times Y = R$ is defined as $\alpha(t) = (\hat{\sigma}(t), f \circ \hat{\sigma}(t))$.

DEFINITION of β . We first define a path $\hat{\beta}: I \rightarrow X \times Y = R$ putting

$$\hat{\beta}(t) = (\hat{\sigma}(t), \mu(t)),$$

where μ is the path from (3.2.1). Next, the path $\hat{\beta}$ is replaced in the same homotopy class by a smooth path $\beta: I \rightarrow X \times Y = R$ arbitrarily close to $\hat{\beta}$ and such that

- (1) $\beta(I) \subset X \times B$,
- (2) $\beta(0) = (x_0, f(x_0))$, $\beta(1) = (x_1, f(x_1))$,
- (3) $\beta(I) \cap \Gamma_f = \{(x_0, f(x_0)), (x_1, f(x_1))\}$.

This is possible, since $\Gamma_f \cap X \times B$ consists of a finite number of points and $\dim X \times B \geq 3$.

DEFINITION of a smooth homotopy h_t joining α and β . Let $g_t: \hat{\sigma} \sim \sigma$ be a homotopy joining the smooth path $\hat{\sigma}$ to the path σ (i.e. $g_0 = \hat{\sigma}$, $g_1 = \sigma$).

Then $G_t = f \circ g_t$ is a homotopy joining the smooth path $f \circ \hat{\sigma}$ to $f \circ \sigma$.

We define a homotopy

$$W_t = \begin{cases} G_{2t} & \text{for } 0 \leq t \leq 1/2, \\ H_{2t-1} & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

This homotopy joins the smooth path $f \circ \hat{\sigma}$ to a path μ contained in B .

Let us notice that the homotopy $H_t = (\hat{\sigma}, W_t)$ joins α to $\hat{\beta}$. Consider the homotopy:

$$F = \begin{cases} H_{2t} & \text{for } 0 \leq t \leq 1/2, \\ B_{2t-1} & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

where $B_t: \hat{\beta} \sim \beta \text{ rel } \{0, 1\}$ is a homotopy joining the path $\hat{\beta}$ to β . The homotopy F_t joins α and β but may be nonsmooth.

Replacing F_t by a smooth homotopy h_t , we obtain a homotopy satisfying the assumption of Whitney's theorem.

Therefore all assumptions in Whitney's theorem are satisfied. That means, for each open neighbourhood U of the path α , that there is a smooth isotopy $H: \Gamma_f \times I \rightarrow X \times Y$ with a support in U and such that

$$\Gamma' \cap X \times B = \Gamma_f \cap X \times B \setminus \{(x_0, f(x_0)), (x_1, f(x_1))\}$$

where $\Gamma' = H(\Gamma_f, 1)$. Let $h: X \rightarrow \Gamma_f$ be the diffeomorphism defined as follows:

$$h(x) = (x, f(x)).$$

Put $\varphi = (\varphi_1, \varphi_2): X \times I \rightarrow X \times Y$, $\varphi(x, t) = H(h(x), t)$ (H denotes an isotopy from Whitney's theorem).

Consider the map $\varphi_2 = \pi_Y \circ \varphi$, where $\pi_Y: X \times Y \rightarrow Y$ is the projection onto the second coordinate. The map φ_2 is a homotopy joining the map f to the map $f_2 = \varphi_2(\cdot, 1): X \rightarrow Y$ such that $f^{-1}(B) \setminus \{x_0, x_1\} = f_2^{-1}(B)$.

We can demand that the support of the homotopy φ_2 be contained in an arbitrarily small open neighbourhood U of $\sigma(I)$.

(3.3) **A procedure of creating new points with image in B .** Let $f: X \rightarrow Y$ be a smooth map, $x_0 \in f^{-1}(B)$ an isolated point in $f^{-1}(B)$ and $v \neq 0$ an arbitrary integer.

We will construct a map $\varphi: X \rightarrow Y$ homotopic to f and such that:

- (1) $\varphi^{-1}(B) = f^{-1}(B) \cup \{x_1, x_2, \dots, x_{|v|}, y\}$ ($| \cdot |$ denotes the modulus),
- (2) the points $x_0, x_1, x_2, \dots, x_{|v|}, y$ belong to the same Nielsen class of φ with respect to B ,
- (3) the points $x_1, x_2, \dots, x_{|v|}$ are points where the map φ is transversal to the submanifold B ,
- (4) y is a point for which $I(\varphi, B, \Omega) = v$ for an open neighbourhood Ω of y such that $\varphi^{-1}(B) \cap \text{Cl}(\Omega) = \{y\}$,
- (5) the support of homotopy joining maps f and φ is contained in an arbitrarily small open neighbourhood of x_0 .

Thus let $U \subset X$ be an arbitrary neighbourhood of x_0 . Fix an open tubular neighbourhood $T \subset Y$ of the submanifold B . There exists an open neighbourhood $V \subset U$ of x_0 in X which is diffeomorphic to \mathbb{R}^n and $f(V) \subset T$ and $f^{-1}(B) \cap V = \{x_0\}$.

Let $\xi: T \rightarrow B$ denote the projection of a bundle with fibre \mathbb{R}^n . Without loss of generality we may assume that $f(V) \subset \xi^{-1}(W)$, where W is a neighbourhood of the point $f(x_0)$ in B diffeomorphic to \mathbb{R}^k . Then $\xi|_{\xi^{-1}(W)}: \xi^{-1}(W) \rightarrow W$ is a trivial bundle. We can identify $\xi^{-1}(W)$ with $\mathbb{R}^k \times \mathbb{R}^n$ where $\xi^{-1}(W) \cap B = \mathbb{R}^k \times \{0\}$ and consider the restriction $f|_V: V \rightarrow \xi^{-1}(W)$ as a map $(f_1, f_2): \mathbb{R}^n \rightarrow \mathbb{R}^k \times \mathbb{R}^n$.

We will assume that $2 < \|x_0\| < 3$. Let us regard \mathbb{R}^n as $\mathbb{C} \times \mathbb{R}^{n-2}$ (\mathbb{C} denotes the complex plane) and fix $|v|$ different points $z_1, z_2, \dots, z_{|v|} \in \mathbb{C}$ such that $0 < |z_i| < 1$ for $i = 1, 2, \dots, |v|$. Let $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ be given by the formula

$$\gamma(z) = \begin{cases} z^\lambda(z-z_1)(z-z_2) \dots (z-z_{|v|}) & \text{for } \lambda > 0, \\ z^{-\lambda}(z-z_1)(z-z_2) \dots (z-z_{|v|}) & \text{for } \lambda < 0, \end{cases}$$

where $\lambda = (-1)^{n-k} \cdot v$.

We define a map $h: \mathbb{C} \times \mathbb{R}^{n-2} \rightarrow \mathbb{C} \times \mathbb{R}^{n-2}$ putting

$$h(z, w) = (\gamma(z), w) \quad \text{for } z \in \mathbb{C}, w \in \mathbb{R}^{n-2}.$$

Then the points $x_1 = (z_1, 0), \dots, x_{|v|} = (z_{|v|}, 0) \in \mathbb{R}^n$ are isolated zeros of degree ± 1 (depending on $\text{sign } \lambda$).

The point $y = (0, 0) \in \mathbb{C} \times \mathbb{R}^{n-2} = \mathbb{R}^n$ is a zero of degree λ of the map h . Therefore the map h considered as a map $h|_{S^1}: S^1 \rightarrow \mathbb{R}^n \setminus \{0\}$ (where $S^1 \subset \mathbb{R}^n$ is the unit sphere) is an essential map (i.e. a map of degree 0).

The same can be said about the map f_2 restricted to the sphere of radius 2.

Hence, there exists a map $f_2^*: \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the conditions:

$$\begin{aligned} f_2^*(x) &= h(x) & \text{for } \|x\| \leq 1, \\ f_2^*(x) &\neq 0 & \text{for } 1 \leq \|x\| \leq 2, \\ f_2^*(x) &= f_2(x) & \text{for } \|x\| \geq 2. \end{aligned}$$

We define a map $\varphi_Y: \mathbb{R}^n \rightarrow \mathbb{R}^k \times \mathbb{R}^n$ putting $\varphi_Y(x) = (f_1(x), f_2^*(x))$.

This map can be obviously extended to a map $\varphi: X \rightarrow Y$ by putting $\varphi(x) = f(x)$ for $x \notin V$. Notice that φ is smooth on an open neighbourhood of points in $\varphi^{-1}(B)$

and $I(\varphi, B, U_x) = (-1)^{nk} \deg(f_2^*, x)$, where U_x denotes an open neighbourhood of x such that $\text{Cl}(U_x) \cap \varphi^{-1}(B) = \{x\}$, for $x = x_1, x_2, \dots, x_{|v|}, y$.

It is not difficult to see that φ satisfies the desired conditions. ■

(3.4) THEOREM. Let X, Y be closed, oriented, smooth manifolds, B be a closed oriented smooth submanifold of Y and $\dim X = \dim Y - \dim B \geq 3$. Then for every continuous map $f: X \rightarrow Y$ there exists a map $g: X \rightarrow Y$ homotopic to f and such that

$$N(f, B) = N(g, B) = \#g^{-1}(B).$$

Proof. Without loss of generality we may assume that $f: X \rightarrow Y$ is a smooth map transversal to the submanifold B . We cancel every Nielsen class of zero index by using Procedure (3.2), since every Nielsen class of zero index consists of the same number of points of index $+1$ and -1 .

If the index of the class is $v \neq 0$ then using Procedure (3.3) we can deform our map in such a way that it will have in its Nielsen class additional $|v|$ points of index ± 1 and one point y whose index with respect to B is equal to v . We can now apply Procedure (3.2) to all points with index ± 1 and the class 'reduces' to the single point y .

(3.5) COROLLARY (Wecken's Coincidence Theorem). If X and Z are smooth, closed, orientable manifolds of the same dimension ≥ 3 and $p, q: X \rightarrow Z$ are continuous maps, then there exist continuous maps $p', q': X \rightarrow Z$ homotopic respectively to p and q and possessing exactly $n(p, q) = n(p', q')$ points of coincidence.

Proof. We obtain these maps by putting in Theorem (3.4) $Y = Z \times Z, B = \Delta$ and $f = (p, q)$. ■

References

- [1] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Co., Glenview, Illinois, London 1971.
- [2] — *On the Nielsen Fixed Point Theorem*, Duke J. Math. 36 (1969), 699–708.
- [3] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1965.
- [4] E. Fadell, *Recent Results in the Fixed Point Theory of Continuous Maps*, Bull. Amer. Math. Soc. 76, (1975), 10–29.
- [5] M. Golubitsky, V. Guillemin, *Stable Mappings and Their Singularities*, Graduate Texts in Mathematics 14, Springer-Verlag, New York, Heidelberg, Berlin, 1973.
- [6] W. Greub, S. Halperin, R. Vanstone, *Connections, Curvature and Cohomology*, Academic Press, New York, London 1972.
- [7] B. J. Jiang, *Estimation of the Nielsen Numbers*, Acta Math. Sinica 14 (1964), 304–312 (= Chinese Math. Acta 5 (1964), 330–339).
- [8] — *On the Least Number of Fixed Points*, American J. of Math. 102, No 4 (1980), 749–763.
- [9] — *Fixed Point Classes from Differential Viewpoint*, in Fixed Point Theory (Sherbrooke), Lecture Notes in Math. 886 (1981), 163–170.
- [10] J. W. Milnor, *Topology from the Differentiable Viewpoint*, Univ. of Virginia Press, Charlottesville, Virginia, 1965; 2nd ed. in Polish, translated from the English by K. Sieklucki, PWN Warszawa 1969.
- [11] — *Lectures on the h-cobordism Theorem*, Princeton University Press, 1965.

- [12] J. Nielsen, *Über die Minimalzahl der Fixpunkte bei Abbildungstypen der Ringflächen*, Math. Ann. 82 (1921), 83–93.
- [13] — *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen, I*, Acta Math. 50 (1927), pp. 189–358; *II*, 53 (1929), pp. 1–76; *III*, 58 (1932), 87–167.
- [14] G. H. Shi, *On the least number of fixed points and Nielsen numbers*, Acta Math. Sinica 16 (1966), 223–232. (= Chinese Math. Acta 8 (1966), 234–243.)
- [15] — *Least number of fixed points of identity class*, Acta Math. Sinica 18 (1975), 192–202.
- [16] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, New Jersey 1964; (Russian transl.: Mir, Moscow 1970).
- [17] F. Wecken, *Fixpunktklassen, I*, Math. Ann. 117 (1941), 659–671; *II*, 118 (1942), 216–234; *III*, 118 (1942), 544–577.
- [18] J. Weier, *Fixpunkttheorie in topologischen Mannigfaltigkeiten*, Math. Z. 59 (1953), 171–190.
- [19] H. Whitney, *Geometric Integration Theory*, Princeton University Press, 1957.

MATHEMATICS DEPARTMENT,
NORMAL SCHOOL, ul. Żołnierska 14
10-558 Olsztyn, Poland

INSTITUTE OF MATHEMATICS,
GDAŃSK UNIVERSITY, ul. Wita Stwosza 57
80-952 Gdańsk, Poland

Received 16 March 1987;
in revised form 28 September 1987 and 17 December 1987

One point extensions of trees and quadratic forms

by

Nikolaos Marmaridis (Ioannina)

Abstract. Let T be any tree with underlying graph one of the graphs $D_n, \tilde{D}_n, n \geq 4, E_m, \tilde{E}_m, m = 6, 7, 8$. Let A be one point extension of the path-algebra kT by an indecomposable preinjective kT -module M . Using methods of tilting theory and of vector space categories, we prove that A is of (infinite) tame representation type if and only if the Tits-form t_A of A is weakly non-negative.

Introduction. Let T be an oriented tree with underlying graph one of the graphs $D_n, \tilde{D}_n, n \geq 4, E_m, \tilde{E}_m, m = 6, 7, 8$. Let

$$A = \begin{bmatrix} k & M \\ 0 & kT \end{bmatrix}$$

be any one point extension of kT by an indecomposable preinjective kT -module M . The aim of the present paper is to prove the following:

THEOREM A. *The algebra A is of (infinite) tame representation type if and only if the Tits-form t_A of A is weakly non-negative.*

In representation theory of finite dimensional algebras it is common to associate to an algebra A a quadratic form in order to study the representation type of A or other invariants of $\text{mod } A$. We refer, for example, to [1] and also to the long list of papers cited there, which are dealing with related questions. Moreover, we refer to [11] for a detailed study of relations between quadratic forms and various module categories. Finally, we like to mention [9] received during the preparation of the present paper. In this work J. A. de la Peña proves an analogue to Theorem A for the so-called “hyperbolic algebras”.

The present paper is divided as follows: In the first section we recall preliminary results. Any notion used in our paper and not defined in Section 1 can be found in [3] or [11]. In the second section we introduce sequences of triangular matrix algebras A induced by tilting functors. These sequences behave nicely in relation to the Tits-form t_A and to the representation type of A . In the third section using the above mentioned sequences we reduce the proof of Theorem A to the study of the so-called A' -maximal tame algebras. For any Tits-form associated to an A' -maximal tame algebra there is a convenient presentation such that one can see easily that the Tits-