$\{y\}\times B(y, p_{n+1}(y)) \subset U$ and every map $\partial I^{i+1} \to B(y, p_i(y))$ can be extended to a map $I^{i+1} \to B(y, p_{i+1}(y))$, $y \in Y$, $i = 0, \ldots, n$. Put $V = \bigcup_{y\in Y} \{y\}\times B(y, p_0(y))$ and $F_i(x) = B(f(x), p_i(f(x)))$, $x \in X$, $i = 0, \ldots, n+1$, and use Theorem (1.1).

The next fact is a substitute of uniform local contractibility for LC^n spaces.

(3.2) THEOREM. *Let Y be a metrizable LC^n space. Then for every* nbhd $U \subset Y\times Y$ *of the diagonal there is a* nbhd $V \subset U$ *of the diagonal such that, whenever X is metrizable, A closed in X,* $\dim(X-A) \leqslant n$, *$f, g\colon X \to Y$ are continuous, $f|A = g|A$ and $(f(x), g(x)) \in V$ for every $x \in X$, then $f \sim g$* rel A *by a homotopy $\{h_t\}$ such that $(f(x), h_t(x)) \in U$ for every $x \in X$ and $t \in I$.*

Proof. Let V be taken from the previous theorem. Define $F\colon X\times I \to Y$ and $G\colon A\times I \cup X\times\{0, 1\} \to Y$ by $F(x, t) = f(x)$, $G(a, t) = f(a)$, $G(x, 0) = f(x)$, $G(x, 1) = g(x)$, $x \in X$, $a \in A$, $t \in I$. From 3.1 it follows that G can be extended to a homotopy $h\colon X\times I \to Y$.

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Received 22 May 1987;
in revised form 2 November 1987

When a subset of E^n locally lies on a sphere

by

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Abstract. Let X be a continuum in E^n, and let G be a collection of pairwise congruent double cones whose cone angles measure 2θ and whose interiors miss X. If each point of X is the vertex of a cone in G, $n = 2$, and $2\theta > 60°$, then X lies on a 1-sphere because it must be an arc or a simple closed curve. Conditions on 2θ and X sufficient to insure that X locally lies on a 2-sphere are also given for the case $n = 3$.

Consider a subset X of Euclidean n-space E^n such that X is touched from its complement at each of its points by an element of some geometric family of solids. What conditions on the touching objects are sufficient to imply that X locally lies on an $(n-1)$-sphere? For example, for $n = 3$ X will locally lie on a 2-sphere if it can be touched by congruent double tangent balls [L_2]; that is, if there exists $\delta > 0$ such that for each $p \in X$ there exist two 3-balls B and B', each with radius δ, such that $\{p\} = B\cap B'$ and $X\cap \operatorname{Int}(B\cup B') = \emptyset$. In this paper related theorems are proven where the double balls are replaced by double cones. For $n = 2$ the double cones become double triangles and a complete analysis is given. The more difficult problems in E^3 are only partially resolved.

Generalizations of the congruent double tangent ball result [L_2] mentioned above could take several directions. However, the weaker hypothesis that there be just congruent single touching balls will not allow the conclusion that X locally lies on a 2-sphere. An example is given in [L_2]. Clear also is the fact that the uniform radii (pairwise congruence) of the double balls is essential to the theorem. Thus it appears that in a generalization one should retain the dual nature of the touching objects while changing their geometry. This leads one to consider double cones in place of the double tangent balls. The needed uniform size on the touching balls could be captured by requiring that the double cones be pairwise congruent. However a ball has the property that given a point where it touches X there is a unique ball of a given size tangent to the first and not intersecting its interior. This property is partially captured in a double cone by insisting that its two nappes have coincident axes. These considerations lead to the definitions below for double cones in E^n.

A *single cone* in E^n $(n > 1)$ is obtained by coning over a standard $(n-1)$-ball B from a point v where the line (*axis*) through v and the center o of B is orthogonal to the hyperplane containing B. A *double cone* in E^n is the union of two single cones with the same *vertex* v about which it is symmetric. The *cone height* $2h$ is twice the distance between v and o, and the *cone angle* 2θ is twice the angle between the axis and a *lateral edge* (a line from v through a point of $\mathrm{Bd}\,B$). In an abuse of notation 2θ is most often used instead for the degree measure of this angle and sometimes for its radian measure. The context makes it clear. A cone is *degenerate* if $2\theta = 0°$. In E^3 a double cone with cone angle $2\left[\mathrm{Tan}^{-1}\frac{1}{a}\right]$ can also be described as a set isometric to $\{(x, y, z) \mid (ax)^2+(ay)^2-z^2 \leqslant 0, a > 0, \text{ and } -h \leqslant z \leqslant h\}$. Here $2h$ is the cone height. In E^2 a double cone is the union of two congruent solid triangles intersecting only at a vertex and symmetric about this vertex. The two *nappes* of a double cone C in E^n are the closures of the two components of $C-\{v\}$ where v is the vertex of C.

A subset X of E^n is *touched by congruent double cones* if there exist a positive number $2h$, a nonnegative number 2θ, and a family F of pairwise congruent double cones of height $2h$ and cone angle 2θ such that, for each p of X, there is a cone C_x in F whose vertex is p such that $X \cap \mathrm{Int}\, C_x = \emptyset$. A *continuum* is a compact, connected metric space containing more than one point.

Even with all the restrictions imposed on the double cones to capture what might be viewed as the essential properties of the double tangent balls, it is easy to see that they fail to produce an analogue to Theorem 3.1 of [L_2]. An easy example is to take $2\theta = 0$ and note that every subset of E^n can be touched by congruent double cones that degenerate to straight line segments. Other examples of continua in E^2 and E^3 that do not locally lie on codimension one spheres yet are touched by congruent double cones with $2\theta > 0°$ are given in Sections 2 and 3. To make the cones more like a ball, one increases the cone angle. The extreme case where $2\theta = 0°$ was mentioned above. In the opposite direction, the limiting case where $2\theta = 180°$, the double cones converge to objects containing double cubes I^n sharing a face. Any object touched by these "cones" would locally lie on an $(n-1)$-sphere. Thus the problem of determining when X locally lies on an $(n-1)$-sphere becomes one focusing on the size of the cone angle 2θ.

The critical cone angle for a continuum X in E^2 is proven to be 60°; in fact, a continuum touched by double cones (triangles) with $2\theta > 60°$ must either be an arc or a simple closed curve (see Section 2). The critical cone angle for continua in E^3 remains to be found, but partial results are given in Section 3.

A subset X of E^n is said to *locally lie on an $(n-1)$-sphere at a point $p \in X$* if there exist a neighborhood N of p in E^n and an $(n-1)$-sphere Σ such that $N \cap X$ lies in Σ. If this condition holds for each p in X, then X is said to *locally lie on an $(n-1)$-sphere*. To show that X locally lies on an $(n-1)$-sphere it suffices to show that X locally lies in the interior of an $(n-1)$-cell because open $(n-1)$-cells must locally lie on $(n-1)$-spheres. For $n = 2$ this is easy to see, and for $n = 3$ it follows from Theorem 5 of [B_1]. That the result also holds for $n > 3$ was communicated to me by Ric Ancel. Ancel's theorem, which generalizes Theorem 5B.10 of [D], was proved using [AC] for $n > 4$ and [A] for $n = 4$. Since this paper focuses mostly on $n = 2$ and $n = 3$ there is no need to detail Ancel's proof here.

The fact that open $(n-1)$-cells locally lie on $(n-1)$-spheres provides a convenient way to identify the two sides of $(n-1)$-cells. If J is an $(n-1)$-manifold in E^n and C is a double cone touching J at a point $p \in J$, then C is said to *pierce J at p* if there exist an $(n-1)$-cell D in J and an $(n-1)$-sphere Σ containing D such that $p \in \mathrm{Int}\, D$ and the axis of C pierces Σ at p.

A *collection F of cones is said to be closed* if F contains the limiting set of each convergent sequence of cones in F. If F is a collection of congruent cones, each touching a compact set in E^n, then each sequence from F contains a convergent subsequence [HY, Theorem 2–102], and, because the cones in F are congruent, the limiting set of the convergent subsequence is also congruent to each element in F. If, in addition, F is closed, then this limiting cone will itself belong to F.

The history of double ball and double cone embeddings goes back to Bing [B_2] and Fort [F] who asked questions about the tameness of 2-spheres in E^3 when they were touched by balls and cones. Griffith [G], Bothe [B_0], Loveland [L_1], [L_4], Cannon [C], Daverman and Loveland [DL_1], [DL_2], Loveland and Wright [LW], and Burgess and Loveland [BL], for example, all gave conditions in terms of touching balls or cones (single or double, sometimes pairwise congruent) under which an $(n-1)$-sphere Σ in E^n (usually $n = 3$) would be flat. Summaries of some of this history can be found in [L_3]. This paper does not depend on any in the references because its examples and proofs are quite elementary and because the flatness of the continuum X is not of concern except briefly in Section 4.

1. Sets in E^n touched by cones. Let X be a compact subset of E^n that can be touched by a nondegenerate double cone at each of its points, and let W be the set of all points of X where X does not locally lie on an $(n-1)$-sphere in E^n. From Theorem 1.2 it follows that W cannot contain an $(n-1)$-cell. Notice that the double cones in the hypothesis of Theorem 1.2 are not required to be pairwise congruent nor is there a common height or cone angle, but each double cone has a cone angle with positive measure. Lemma 1.1 is used in the proof of Theorem 1.2 and is referred to several times in the sequel.

LEMMA 1.1. *If F is a closed collection of nondegenerate, congruent double cones, X is a subset of E^n that contains an $(n-1)$-cell J, and X can be touched by cones in F at each point of J, then J is pierced by a cone in F at each point of* $\mathrm{Int}\, J$.

Proof. Let P be the set of all points p in J such that J is pierced at p by some cone in F. The strategy is to prove that P is dense in $\mathrm{Int}\, J$, because then the conclusion follows from the hypothesis that F is a closed collection of congruent double cones. By way of contradiction, suppose there exists an $(n-1)$-cell D_1 in $\mathrm{Int}\, J$ such that $D_1 \cap P = \emptyset$. To deal with the definition of a cone piercing D_1, let Σ be an $(n-1)$-sphere and D_2 an $(n-1)$-cell in D_1 such that $D_2 \subset \Sigma$. For $n = 3$ the existence of Σ and D_2 follows Theorem 5 of [B_1], and Ancel has generalized Bing's theorem to $n > 3$ (see the comments in the introduction). Choose an $(n-1)$-cell D_3 such that

$D_3 \subset \operatorname{Int} D_2$, and, for notational convenience, uniformly decrease the height of all cones in F to length less than $d(\Sigma - \operatorname{Int} D_2, D_3)$ without changing cone angles or cone axes. Let this collection of shortened, congruent cones be F', let $p \in \operatorname{Int} D_3$, and let C_p be a cone in F' touching J at p. Since $D_3 \cap P = \emptyset$, C_p does not pierce D_3 at p, which, because of the height of C_p, means $\Sigma \cap \operatorname{Int} C_p = \emptyset$. Let $I = \{p \in D_3 |$ there exists a cone C_p in F' touching J at p such that $\operatorname{Int} C_p \subset \operatorname{Int} \Sigma\}$, and let $E = D_3 - I$. Since $D_3 \cap P = \emptyset$, it follows that for each $p \in E$ there exists a cone $C_p \in F'$ such that $\operatorname{Int} C_p \subset \operatorname{Ext} \Sigma$. Because F, and hence F', is closed and the cones are congruent, it is clear that I is closed. Therefore E is an open subset of D_3.

Suppose $E \neq \emptyset$, and let D_4 be an $(n-1)$-cell in $D_3 \cap E$. Let $x \in \operatorname{Int} D_4$, and choose a point q of $\operatorname{Int} \Sigma$ such that

$$d(q, x) < d(q, \Sigma - \operatorname{Int} D_4) .$$

Then the ball B_q centered at q with radius $d(q, x)$ intersects Σ only in $\operatorname{Int} D_4$. Shrink the radius of B_q to obtain a ball B centered at q such that $\operatorname{Int} B \subset \operatorname{Int} \Sigma$ and $\operatorname{Bd} B$ contains a point y of D_4. Since $y \in E$ there exists a cone C_y in F' touching J at y such that $\operatorname{Int} C_y \subset \operatorname{Ext} \Sigma$. Then the axis of C_y is tangent to B at y. However the cone C_y is nondegenerate, and $(\operatorname{Int} C_y) \cap \operatorname{Int} B = \emptyset$. This contradiction shows that $E = \emptyset$.

Since $E = \emptyset$, $I = D_3$. This time choose a point q in $\operatorname{Ext} \Sigma$ and a ball B with center at q such that $\operatorname{Int} B \subset \operatorname{Ext} \Sigma$ and there is a point y in $(\operatorname{Bd} B) \cap (\operatorname{Int} D_3)$, as in the previous paragraph. There must exist a nondegenerate cone C_y in F' touching D_3 at y such that $\operatorname{Int} C_y \subset \operatorname{Int} \Sigma$ while B touches Σ from $\operatorname{Ext} \Sigma$, so the same contradiction results. This means D_1 must intersect P, so P is dense in $\operatorname{Int} J$.

THEOREM 1.2. *If the compact subset X of E^n can be touched at each of its points by a nondegenerate double cone, then the set W of points at which X fails to locally lie on an $(n-1)$-sphere contains no $(n-1)$-cell.*

Proof. Suppose the closed set W contains an $(n-1)$-cell D, and, for each integer $i > 1$, let W_i be the set of all points x of D such that there exists a double cone C_x touching X at x with cone angle π/i radians and height $1/i$. The uniform size of these cones insures that each W_i is closed. Because the hypothesized cones are nondegenerate W is the union of the compact sets W_i. A Baire category theorem yields the existence of an $(n-1)$-cell U in D and an integer N such that $U \subset W_N$. Let F be the collection of all double cones of height $1/N$ and cone angle π/N that touch X at a point of U, let S be a unit sphere at the origin, and let Q be a countable set of points $\{r_1, r_2, \ldots\}$ which is dense in S. Each r_i is thought of as a direction or line through the origin. Because F is a closed collection of cones over the compact set U, the set of cone axes of cones in F can be thought of as a compact subset of S. Adjust each cone in the family F to form a new family F' of congruent double cones touching X at each point of U so that each cone in F' has its axis parallel to some direction r_i. In accomplishing this adjustment the cone angles might have to be uniformly reduced in size to some constant radian measure less than π/N, but, by adding limiting sets of convergent sequences of cones in F' to F' if necessary, the collection F' can be constructed to also be a closed collection.

Let D_i be the set of all points of U for which there is a touching cone in F' with axis in the direction r_i. Again $U = \bigcup_1^\infty D_i$ and each D_i is closed, so a Baire theorem yields an $(n-1)$-cell U' lying in U and an integer M such that $U' \subset D_M$. This means X can be touched at each point of U' by a cone from a pairwise congruent, closed family F'' of nondegenerate double cones whose axes are parallel. From Lemma 1.1 it may be assumed that each cone in F'' intersects both sides of U', so that the union of the cones of F'' clearly contains an open n-cell V which intersects U'. Because the interiors of the cones in F'' do not intersect X, $V \cap X$ must lie in the $(n-1)$-cell U'. However, because $U' \in W$, this contradicts the fact that X cannot locally lie on an $(n-1)$-sphere at any point of U'.

The hypothesis that there be double touching cones rather than just single cones is essential in Theorem 1.2. It is easy to construct an example of a continuum X in E^2 that can be touched by single cones such that X contains an arc of points where X does not locally lie on a 1-sphere.

2. Continua in E^2 touched by cones. A *triod* T is a set homeomorphic to $[(-1, 0), (1, 0)] \cup [(0, 0), (0, 1)]$ in the xy-coordinate plane. Its *vertex* v is the image of the origin, and the closures of the images of the three components of $T - \{v\}$ are called its *legs*. The most obvious examples of continua that can be touched with congruent double cones but do not locally lie on a 1-manifold are certain embeddings of the triod in E^2.

EXAMPLE 2.1. Let T be the troid in E^2 with vertex v at $(0, 0)$ whose legs are the three straight line segments $[v, (0, 1)]$, $[v, (-\sqrt{3}/2, -1/2]$, and $[v, (\sqrt{3}/2, -1/2)]$. Each pair of legs forms an angle of 120°. It is easy to see that T can be touched by congruent double cones having cone angles as large as 60°.

Example 2.1 is the best possible example in the sense that any continuum touched by congruent double cones with cone angles larger than 60° must be a 1-manifold, as stated in Theorem 2.4. This section is devoted to proving this result and some other theorems about the nature of the non-manifold set in a continuum having double touching cones.

LEMMA 2.2. *If a continuum X in E^2 is touched by congruent nondegenerate double cones, then X is locally connected.*

Proof. A continuum that is not locally connected must contain a continuum of convergence [W, p. 209] which can be shown to contradict the hypothesis.

LEMMA 2.3. *If T is a triod in E^2 having vertex v_0, $h > 0$, $\theta > 0$, G is the set of all double cones of height $2h$ and cone angle 2θ whose interiors fail to intersect T, and for each point x of T there exists a cone in G with vertex x, then G contains three cones with vertex v_0 whose interiors are pairwise disjoint.*

Proof. Let D be a round disk centered a v_0 with radius h. By reducing the radius of D or shortening the legs L_1, L_2, and L_3 of T one may assume $T \subset D$ and that $D-T$ is the union of three pairwise disjoint open disks D_{12}, D_{13}, and D_{23} where the two subscripts refer to the subscripts of the two legs L_i and L_j of T lying in $\mathrm{Bd}\,D_{ij}$.

Let $\{v_i\}$ be a sequence of points of L_1 converging to v_0 such that, for each i, there exists a cone in G touching T at v_i and intersecting both sides of L_1 (see Lemma 1.1). Passing to subsequences if necessary, assume there is a sequence $\{C_i\}$ of cones of G converging to a cone C_1' such that v_0 is the vertex of C_1' and v_i is the vertex of C_i for $i = 1, 2, \ldots$ For each i the nappes of C_i fail to intersect D_{23}, so the same is true for C_1'. In a similar manner, using legs L_2 and L_3, one obtains cones C_2' and C_3' in G with vertices at v_0 such that C_2' and C_3' fail to intersect D_{13} and D_{12}, respectively. From this it is not difficult to see that the interiors of the cones C_1', C_2' and C_3' are disjoint.

THEOREM 2.4. *If X is a continuum in E^2, $h > 0$, G is the collection of all double cones of height $2h$ and cone angle greater than $60°$ whose interiors fail to intersect X, and each point of X is the vertex of a cone in G, then X is either an arc or a simple closed curve.*

Proof. By Lemma 2.2, X is locally connected. This means X is either an arc, a simple closed curve, or X contains a triod [M]. Suppose X contains a triod T with vertex v, and conclude from Lemma 2.3 that there exist three cones of G with disjoint interiors, all with vertex v. The cones of G are double cones and each nappe has an angle greater than $60°$. But six times $60°$ is $360°$, the most allowed at v.

Theorem 2.4 identifies the critical cone angle as $60°$ for continua in E^2. It provides a nice model to attempt duplicating in higher dimensional Euclidean spaces. Those wishing to see partial analogues in E^3 should skip to Section 3.

Suppose X is a continuum in E^2 that can be touched by congruent double cones of height $2h$ and cone angle 2θ. If $2\theta > 60°$, the set W of points where X fails to locally lie on a 1-sphere is empty (Theorem 2.4). If $2\theta = 0°$, W could be 2-dimensional as when X is a 2-cell. These are the two extremes. What is the nature of W when $0 < 2\theta \leqslant 60°$? Theorem 1.2 shows W cannot contain a 1-cell as long as the touching double cones are nondegenerate. The next example shows that W can be an infinite set in the case where X is touched by double cones with the same height and with positive cone angles.

EXAMPLE 2.5. In Figure 1 a continuum X is pictured that contains an infinite sequence $\{p_i\}$ of nonmanifold points, yet X can be touched by nondegenerate double cones of uniform height. The continuum X is the union of chords of circle S from p_{i-1} to p_i, together with the point p to which $\{p_i\}$ converges, and infinitely many straight line segments S_i, one at each p_i, sticking toward $\mathrm{Ext}\,S$. Some of the touching double cones are pictured. Notice that, as in the proof of Lemma 2.3, there must exist for each i a cone C_i that locally separates S_i from the two chords meeting at p_i. This sequence $\{C_i\}$ of cones can be chosen to converge to a line segment tangent to S at p. The cone C_i is constructed with small enough cone angle to allow S_{i-1} to exist in its complement.

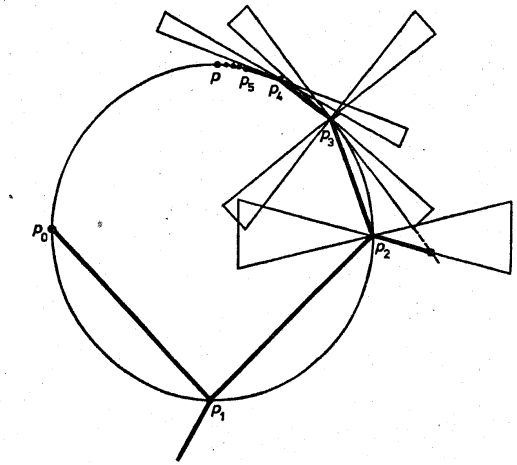

Fig. 1

Of course if the cone angles are bounded away from 0 this is the same as having all the cone angles positive and pairwise congruent. With this condition on 2θ and with uniform height, Theorem 2.7 says that W must be finite.

LEMMA 2.6. *Let T be a triod in E^2 and G be a set of double cones satisfying all the hypotheses of Lemma 2.3. If L, R, and F are the legs of T, then there exists a cone C in G with vertex v_0 such that the cone axis of C separates $F-\{v_0\}$ from $L\cup R-\{v_0\}$ in E^2.*

Proof. The proof of Lemma 2.3 applies here as well.

THEOREM 2.7. *If $\theta > 0$, $h > 0$, and X is a continuum in E^2 that is touched by congruent double cones of height $2h$ and cone angle 2θ, then X contains a finite set W such that X locally lies on a 1-sphere at each point of $X-W$.*

Proof. Let G be the collection of all double cones of height $2h$ and cone angle 2θ whose interiors miss X and whose vertices lie in X, and let W be the set of all points of X where X fails to locally lie on a 1-sphere. Suppose W is infinite, and let p be a limit point of the compact set W. Let V be an open set containing p such that $\mathrm{diam}\, V < h$ and $V\cap X$ is arcwise connected (see Lemma 2.2 and [W, 31.2]), let $\{p_i\}$ be a sequence of points in $V\cap W$ converging to p, and let A be an arc in $V\cap X$ with one endpoint p. The next paragraph shows that A may be chosen to contain infinitely many p_i.

For each i let C_i be a double cone in G having p_i as its vertex. By taking subsequences if necessary it may be assumed that $\{C_i\}$ converges to a cone C in G having p as its vertex [HY, p. 102]. By again taking subsequences if necessary and possibly renaming it may be assumed that all the points p_i lie in the same component of $E^2-\{\text{axis of } C\}$. Now choose A in $V\cap X$ with endpoints p and p_1. Uniformly shrink the cone angles slightly so that $\{C_i\}$ still converges to C, $C \cap X = \{p\}$, and $C_i \cap X = \{p_i\}$. Since $\operatorname{diam} V < h$, $C_i \cap X = \{p_i\}$, $C \cap X = \{p\}$, and $\{C_i\}$ converges to C there must exist infinitely many integers k such that C_k separates p_1 from p in V. For each such k, p_k must belong to A since $C_k \cap X = \{p_k\}$. For the remainder of the proof assume the sequence $\{p_i\}$ has been renamed so that A contains every p_i.

Since $p_i \in W$ and $V \cap X$ is locally arcwise connected, there must exist, for each i, a triod T_i in $V \cap X$ with vertex q_i and with legs L_i, R_i, and F_i such that $L_i \cup R_i \subset A$, $F_i \cap A = \{q_i\}$, and T_i lies in a $1/i$-neighborhood of p_i. From Lemma 2.6 there exists for each i, a cone $C(F_i)$ in G with vertex q_i such that the cone axis Z_i of $C(F_i)$ separates $F_i - \{q_i\}$ from $L_i \cup R_i - \{q_i\}$ in V. By passing to subsequences if necessary, it may be assumed that $\{C(F_i)\}$ converges to a cone $C(F)$ which must also lie in G. Of course $C(F)$ has p as its vertex, and the lines $\{Z_i\}$ converge to the cone axis of $C(F)$. Since p does not belong to any Z_i, there is an integer m such that Z_m separates p from q_1. Since Z_m also separates $L_m \cup R_m - \{q_m\}$ from $F_m - \{q_m\}$ in E^2 and $L_m \cup R_m \subset A$, it follows that A does not cross Z_m at q_m. But $Z_m \cap A = \{q_m\}$ and the arc A contains points p and q_1 on opposite sides of Z_m This contradiction shows W cannot be infinite.

Used often in the proof of Theorem 2.7, the uniform height requirement on the double cones is essential. It is easy to construct an example of a graph X satisfying all the other hypotheses of Theorem 2.7 but having infinitely many points in W.

Notice that Theorem 1.2 applies to show that W contains no 1-cell even when no uniform size condition is imposed on the cones except that the cones do not degenerate to straight line segments.

3. Continua in E^3 that are touched by cones. Consider a continuum X in E^3 that is touched by congruent double cones with cone angle 2θ, and let W denote the set of all points of X at which X does not locally lie on a 2-sphere. If $\theta = 0°$ W can contain a 3-cell. In the other limiting case where $2\theta = 180°$ one may interpret the "touching cones" to be two 3-dimensional half spaces sharing a plane in their boundaries. Whatever the interpretation X would have congruent touching ball pairs so that W would be empty [L_2, Theorem 3.1]. If $2\theta > 0$, Theorem 1.2 states that W contains no 2-cell. An example where $2\theta > 0$ and W contains a 1-cell is easily constructed.

EXAMPLE 3.1 A 3-*page book* B is a set homeomorphic to the product of an interval I with a triod T. If T has legs L_1, L_2, and L_3, then the three pages of B are the 2-cells $L_i \times I$. The desired example is a 3-page book X whose pages are planar and form dihedral angles of measure 120°. Then X is easily seen to be touched by congruent double cones whose cone angles measure 60°. From Theorem 3.3 it follows that X cannot be touched by such cones with larger angles.

Taken with the lower dimensional result of Theorem 2.6, Example 3.1 suggests that a continuum in E^3 should also locally lie on a 2-sphere if it can be touched by congruent double cones with cone angles satisfying $2\theta > 60°$. Perhaps a counter example can be produced by constructing an unusual topological embedding of the umbrella space in the next example, but none has been found.

EXAMPLE 3.2. Let M be the union of the two straight line segments $[(0,0,0),(0,0,1)]$ and $[(0,0,0),(0,\sqrt{3}/2,-1/2)]$ in E^3. Rotate M about the z-axis to obtain the umbrella X. The verification that X is touched by congruent double cones with $2\theta = 60°$ is left to the reader. It is also true that no such cones touch X if $2\theta > 60°$.

THEOREM 3.3. *If X is a finite polyhedron in E^3 with no local cut points, and X is touched by congruent double cones with $2\theta > 60°$, then X locally lies on a 2-sphere.*

Proof. The touching cone hypothesis rules out 3-simplices lying in X. It also implies that X contains no 3-page book, for suppose X contains three 2-simplices σ_1, σ_2, and σ_3 and a 1-simplex σ such that $\sigma_i \cap \sigma_j = \sigma$ for $i \neq j$. For $i = 1, 2$ and 3, let σ_i' be the reflection of σ_i in σ, and let P_i be the plane containing $\sigma_i \cup \sigma_i'$. One of these three planes, say P_1, is distinct from the other two. An argument is now given to show $P_2 \neq P_3$. Let $\{p_n\}$ be a sequence of distinct points of σ_1 converging to a point p in $\operatorname{Int}\sigma$, and let C_n be a double cone as described in the hypothesis so that p_n is the vertex of C_n for each n. By taking subsequences of $\{p_n\}$ if necessary, one may assume $\{C_n\}$ converges to a double cone C^1 with vertex p. Let B be a ball centered at p such that $\sigma_i \cup \sigma_i'$ separates B for each i, and note that C_n lies in the closure S of the σ_1-side of $\sigma_2 \cup \sigma_3$ in B for each n. This means C^1 must also lie in S, and it shows that P_2 and P_3 form a dihedral angle with positive measure so that $P_2 \neq P_3$.

The argument above produced a cone C^1 at p where C^1 was between the planes P_2 and P_3. Similar arguments using sequences in σ_2 and σ_3 produce double cones C^2 and C^3 between P_1 and P_3 and between P_1 and P_2, respectively. In fact each of the six components of $B - \{P_1 \cup P_2 \cup P_3\}$ contains a nappe of one of the double cones C^1, C^2, and C^3, and each such nappe is a cone whose angle measures more than 60°.

Consider now two planes P and Q intersecting in a line L, and a single cone K with vertex at $p \in L$ and cone angle 2θ such that $\operatorname{Int} K$ does not intersect $P \cup Q$. Fix K, but squeeze P and Q closer together by rotating them about L until they both contain a lateral edge of C. The claim is that the dihedral angle β made by $P \cup Q$ is greater than or equal to 2θ and that $\beta = 2\theta$ only if the axis of C is orthogonal to L. One sees this clearly by thinking of rotating the cone axis from its orthogonal position toward the line L. As the cone moves it must spread the planes apart because it remains tangent to each plane.

Thus each of the six dihedral angles between the planar boundaries of the individual components of $B - \bigcup\{P_1, P_2, P_3\}$ must be equal to or larger than 2θ. Since $2\theta > 60°$ this is impossible, and it follows that X contains no 3-page book.

Let p be an arbitrary point of X. If p lies in the interior of a 2-simplex, the conclusion follows. If p lies in the interior of a 1-simplex σ, then, because X contains no 3-page book, σ lies in the boundary of at most two 2-simplices and again the conclusion follows. The last case is where p is a vertex of X, and the only interesting situation is when p is not an isolated point of X. Let $L(p)$ denote the link of p in X. The fact that p does not locally separate X implies that $L(p)$ is connected. Because X contains no 3-page book, $L(p)$ contains no triod. Then from [M] it follows that the locally connected, compact set $L(p)$ is either an arc, a simple closed curve, or a point. In any case it is clear that X locally lies on a 2-sphere at p because the cone from p to $L(p)$ contains a neighborhood of p in X.

THEOREM 3.4. *If X is a finite polyhedron with a triangulation T consisting of closed 2-simplices, no two simplices of T intersect at a point, and X is touched by congruent double cones with $2\theta > 60°$, then each component of X is either a 2-cell or a tetrahedron.*

Proof. Let M be a component of X. Theorem 3.3 applies to show that M is a compact 2-manifold. From the hypothesis that no two 2-simplices intersect at a point it is easy to see that M is the union of one, two, three, or four 2-simplices in T, that M is a 2-cell if it contains three or fewer 2-simplices, and that M is a tetrahedron if it contains four.

THEOREM 3.5. *If a subset X of E^3 is touched by congruent double cones with cone angles 2θ, $2\theta > 90°$, and D is an open 2-cell lying in X, then no point of D is a limit point of $X-D$.*

Proof. Suppose p is a point of D that is a limit point of $X-D$, and let Σ be a 2-sphere containing a disk D' such that $p \in \operatorname{Int} D' \subset D$. If D is locally polyhedral at p the sphere Σ is easily obtained, but, in any case, Σ can be obtained from Theorem 5 of [B$_1$]. The sphere Σ is used to identify the two sides of D near p. Let $U = \operatorname{Int}\Sigma$, and let G be the collection of all double cones congruent to the hypothesized ones whose interiors miss X. From Lemma 1.1 G contains a double cone C with nappes C_1 and C_2 such that p is the vertex of C, a neighborhood of p in C_1 lies U, and a neighborhood of p in C_2 lies in $\operatorname{Ext} S$; that is, C pierces D at p.

Since p is a limit point of $X-D$, there must exist a sequence $\{q_i\}$ of points of $X-D$ converging to p. For convenience assume q_i belongs to U for each i. For each i there exists a cone C_i in G with vertex q_i, and, by passing to subsequences if necessary, the sequence $\{C_i\}$ may be assumed to converge to a cone C' at p. Because each C_i locally lies in U at q_i and because $p \in \operatorname{Int} D' \subset X \cap \Sigma$, it follows that, near its vertex p, C' lies in the closure of U. Then $\operatorname{Int} C$ and $\operatorname{Int} C'$ are disjoint, which yields a contradiction because X cannot have two disjoint double cones at p, each with a cone angle larger than 90°.

The following are among the questions left unanswered.

QUESTION 3.6. Does a subset X of E^3 locally lie on a 2-sphere if it is touched by congruent double cones with $2\theta > 60°$?

QUESTION 3.7. Can Theorem 3.5 be strengthened by replacing 90° by a smaller number?

QUESTION 3.8. Is it possible for a continuum X in E^3 to contain either a topological 3-page book or a topological umbrella space and to be touched by congruent double cones whose angles measure more than 60°?

4. Sets locally lying on flat spheres. Since questions about touchings sets with cones and balls originally focused on obtaining conditions under which the set would be locally flat it seems appropriate to conclude this paper with a brief section on this subject. The paper has dealt with cone conditions sufficient to make X locally lie on an $(n-1)$-sphere. Of course every 1-sphere in E^2 is flat, but for $n > 2$ this is not the case. The Fox–Artin wild arc [FA] can be embedded in the 3-page book of Example 3.1 (see [L$_4$]) to provide an example of a subset X of a 2-sphere in E^3 that is touched by double congruent cones with $2\theta = 60°$ yet does not locally lie on a flat 2-sphere. Theorem 2.1 of [L$_4$] says that a subset X of a 2-sphere in E^3 must locally lie on a flat 2-sphere except at a finite number of points if X is touched by congruent single cones with $2\theta > 2\tan^{-1} 3$.

In Theorem 3.5 X is seen to locally lie on a 2-sphere at a point p of X if X is touched by congruent double cones with $2\theta > 90°$ and if some 2-cell D lies in X and has p in the interior. Under these hypotheses it follows from Lemma 1.1 and [C] that D locally lies on a flat 2-sphere at each of its interior points. This means that $\operatorname{Int} D$ can be pushed a little to one side to obtain another disk D' such that $D \cup D'$ is a 2-sphere. This establishes the following generalization of Theorem 3.5.

THEOREM 4.1. *If D is a 2-cell in a subset X of E^3, and X is touched by congruent double cones with $2\theta > 90°$, then D lies on a 2-sphere that is locally flat modulo* $\operatorname{Bd} D$.

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Received 22 May 1987;
in revised form 20 January 1988

The existence of universal invariant measures on large sets

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Abstract. We consider countably additive, nonnegative, extended real-valued measures which vanish on singletons. Such a measure is *universal* on a set X iff it is defined on all subsets of X. We prove, in particular, that there exists a universal σ-finite measure on X which is invariant with respect to a given group G of bijections of X iff there exists a universal σ-finite measure on X such that for every subgroup H of G of cardinality ω_1 the set of all points of X with uncountable H-orbits has measure zero.

0. Terminology. Our set-theoretic notation and terminology are standard. Ordinals are identified with the sets of their predecessors and cardinals are defined as initial ordinals. If A is a set, then $P(A)$ denotes the family of all subsets of A, and $|A|$ is the cardinality of A. If f: $X \to Y$ is a function and $A \subset X$, then $f[A]$ denotes the image of A.

All measures considered in this paper are assumed to be:

— nonnegative extended real-valued;
— countably additive;
— vanishing on singletons;
— assuming at least one positive finite value.

A measure is called *universal* on a set X iff it is defined on $P(X)$. We adopt the convention that the phrase "measure on X" always means "universal measure on X".

Let $\varkappa$, λ be infinite cardinals. A measure μ on X is called:

— $\varkappa$*-additive* iff every union of less than $\varkappa$ sets of measure 0 has measure 0;
— *finite* iff $\mu(X) < +\infty$;
— λ*-finite* iff every set of positive measure is the union of less than λ pairwise disjoint subsets of positive finite measure.

Notice that if $\varkappa > \lambda$ and μ is $\varkappa$-additive, then it is λ-finite iff X is the union of less than λ sets of positive finite measure. Following traditional terminology, we write "σ-finite" instead of "ω_1-finite".

By an *ideal* on a set X we mean here a family $I \subset P(X)$ which contains all single-