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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

A selection theorem for open-graph multifunctions

by

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Abstract. A selection theorem is proved for open-graph multifunctions, which satisfy some homotopic and dimensional conditions. We obtain corollaries similar to Mc Clendon's theorem [1] and Michael's theorem [2] and we give applications to the theory of LC^n spaces.

The main result of this paper states that if $F_1, \ldots, F_k$: $X \to Y$ is a sequence of open-graph multifunctions such that $F_i(x) \subset F_{i+1}(x)$ and this inclusion induces the trivial map in homotopy groups for every $x \in X$ and $i \leqslant k$, then every partial continuous selection for F_1 can be extended to a continuous selection for F_k (provided X and Y satisfy certain conditions). Taking $F_1 = \ldots = F_k$, we obtain a selection theorem similar to the Mc Clendon theorem [1], but with different assumptions on the domain and the range. If we are only interested in the existence of a continuous selection for an open-graph multifunction with n-connected values, then no homotopic assumptions on the domain and range are needed. We obtain also an open-graph analogue of results of E. Michael [2]. We apply these results to transfer some facts from the theory of retracts to the case of LC^n spaces.

0. Preliminaries. Let X and Y be two topological spaces. A set-valued function F: $X \to 2^Y$ is called a *multifunction* if $F(x) \neq \emptyset$ for every $x \in X$. We write F: $X \to Y$. If A is a subset of X, then we put $F(A) = \bigcup \{F(x)\colon x \in A\}$. We define the graph of F as the set $\Gamma(F) = \{(x, y) \in X \times Y\colon y \in F(x)\}$. F is an open-graph multifunction if $\Gamma(F)$ is open in $X \times Y$. A *selection* for F is a single-valued function f: $X \to Y$ such that $f(x) \in F(x)$, for every $x \in X$.

A map is a continuous function. I denotes the unit interval.

A topological space X is called *n-connected* (briefly: C^n), $n = 0, 1, \ldots$ if every map $\partial I^{i+1} \to X$ can be extended to a map $I^{i+1} \to X$, $0 \leqslant i \leqslant n$. X is infinitely connected (C^∞) if it is n-connected for every $n \in N$. X is *locally n-connected* (briefly: LC^n), $n = 0, 1, \ldots$, if for every $x \in X$ and every nbhd U of x, there is a nbhd $V \subset U$ of x such that every map $\partial I^{i+1} \to V$ can be extended to a map $I^{i+1} \to U$, $0 \leqslant i \leqslant n$.

If $\mathcal{U}$ is a locally finite open covering of a paracompact space X, then $N(\mathcal{U})$ denotes the geometric realization of the nerve of $\mathcal{U}$, $N^i(\mathcal{U})$ its i-skeleton and ξ: $X \to N(\mathcal{U})$ the canonical map.

By *dimension* of a topological space we mean the covering dimension.

Let X and Y be topological spaces and A be a subset of X. If $f, g\colon X \to Y$ are two maps such that $f|A = g|A$, then $f \sim g$ rel A means that f and g are homotopic by a homotopy $\{h_t\}$ such that $h_t|A = f|A$ for each $t \in I$.

1 A selection theorem.

(1.1) THEOREM. *Let X be a metrizable space and A its closed subset such that $\dim(X-A) \leqslant n+1$. Let Y be a metrizable and locally n-connected space. Let $F_0, \ldots, F_{n+1}\colon X \to Y$ be open-graph multifunctions such that $F_i(x) \subset F_{i+1}(x)$ for every $x \in X$, $0 \leqslant i \leqslant n$, and every map $\partial I^{i+1} \to F_i(x)$ can be extended to a map $I^{i+1} \to F_{i+1}(x)$, for each $x \in X$, $0 \leqslant i \leqslant n$. Then every continuous selection $f\colon A \to Y$ for $F_0|A$ can be extended to a continuous selection $g\colon X \to Y$ for F_{n+1}.*

Proof. Let d and d' be metrics on X and Y, respectively. Since Y is LC^n and $\dim(X-A) \leqslant n+1$, we can find an open nbhd V of A and a map $f'\colon V \to Y$ such that $f'|A = f$. Since F_0 has open graph, there is an open nbhd V' of A such that $f'|V'$ is a selection for $F_0|V'$. Let W be an open set such that $A \subset W \subset \overline{W} \subset V'$. By the paracompactness of $\overline{W}-A$ there is a map $r_{n+1}\colon \overline{W}-A \to I-\{0\}$ such that $r_{n+1}(x) \leqslant d(x, A)$ and the ball $B(f'(x), r_{n+1}(x)) \subset F_0(x)$, for every $x \in \overline{W}-A$.

Since $F_0(A)$ is LC^n, we can similarly find $n+1$ maps $r_n, \ldots, r_0\colon \overline{W}-A \to I-\{0\}$ such that $r_i(x) \leqslant r_{i+1}(x)$ and every map $\partial I^{i+1} \to B(f'(x), r_i(x))$ can be extended to a map $I^{i+1} \to B(f'(x), r_{i+1}(x))$, for every $x \in \overline{W}-A$ and $i = 0, \ldots, n$.

Let $F_i'\colon X-A \to Y$ be given by

$$F_i'(x) = \begin{cases} B(f'(x), r_i(x)) & \text{if } x \in \overline{W} \\ F_i(x) & \text{if } x \notin \overline{W}, \end{cases}$$

$0 \leqslant i \leqslant n$. Observe that each F_i' has open graph, $F_i'(x) \subset F_{i+1}'(x)$ and every map $\partial I^{i+1} \to F_i'(x)$ can be extended to a map $I^{i+1} \to F_{i+1}'(x)$ for every $x \in X-A$ and $i = 0, \ldots, n$. Moreover, if $g_0\colon X-A \to Y$ is a continuous selection for F_{n+1}', then the map $g\colon X \to Y$, given by the formula

$$g(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g_0(x) & \text{if } x \notin A, \end{cases}$$

is a continuous selection for F_{n+1}, because $r_{n+1}(x) \leqslant d(x, A)$ for every $x \in X-A$. Therefore we must only prove that F_{n+1}' has a continuous selection. This follows from the following two facts.

(1.1.1) LEMMA. *There exists a locally finite open covering $\mathcal{U}$ of $X-A$ of order $\leqslant n+1$ and a $u\colon \mathcal{U} \to Y$ such that $u(U) \in F_0'(x)$, whenever $x \in U \in \mathcal{U}$.*

(1.1.2) LEMMA. *Let $\mathcal{U}$ be a locally finite open covering of $X-A$ of order $\leqslant n+1$ and let $u\colon N^i(\mathcal{U}) \to Y$ be a map such that $u(\sigma \cap N^i(\mathcal{U})) \subset F_i'(x)$ for every simplex σ with vertices $U_1, \ldots, U_r$ and for every $x \in U_1 \cap \ldots \cap U_r$, $i = 0, \ldots, n$. Then there exists a locally finite open covering $\mathcal{V}$ of $X-A$ of order $\leqslant n+1$ and a map $v\colon N^{i+1}(\mathcal{V}) \to Y$ such that $v(\sigma \cap N^{i+1}(\mathcal{V})) \subset F_{i+1}'(x)$, for every simplex σ with vertices $V_1, \ldots, V_s$* and every $x \in V_1 \cap \ldots \cap V_s$.

The proofs of Lemmas (1.1.1) and (1.1.2) are easy modifications of the proofs of Lemmas 7.1 and 7.3 in [2] and will be omitted.

Now the proof can be finished as in [2]. Namely, we can find an open, locally finite covering $\mathcal{W}$ of $X-A$ of order $\leqslant n+1$ and a map $w\colon N^{n+1}(\mathcal{W}) = N(\mathcal{W}) \to Y$ such that $w(\sigma) \subset F_{n+1}'(x)$ for every simplex σ with vertices $W_1, \ldots, W_t$ and for every $x \in W_1 \cap \ldots \cap W_t$. Let $\xi\colon X-A \to N(\mathcal{W})$ be the canonical map. Then $w \circ \xi$ is a continuous selection for F_{n+1}'.

Remark (1.2) One can easily see that the same proof goes over if we assume only that X is paracompact, A has a metrizable nbhd, Y is an arbitrary topological space and $F_0(A)$ is metrizable and LC^n.

2. Some corollaries on the existence of selections.

Taking $F_0 = \ldots = F_{n+1}$ in Theorem (1.1), we obtain immediately:

(2.1) COROLLARY. *Let X, Y and A be the same as in Theorem (1.1). Let $F\colon X \to Y$ be an open-graph multifunction with n-connected values. Then every continuous selection for $F|A$ can be extended to a continuous selection for F.*

Using Remark (1.2) and taking $A = \emptyset$, $F_0 = \ldots = F_{n+1}$, we obtain:

(2.2) COROLLARY. *Let X be a paracompact space of covering dimension $\leqslant n+1$. Then every open-graph multifunction with n-connected values from X to a topological space Y has a continuous selection.*

The next fact, which is an open-graph analogue of a result of E. Michael [2], shows that the assumption of $F(A)$ being LC^n is essential in (2.1).

(2.3) COROLLARY. *Let $\mathcal{K}$ be a family of open subsets of a metrizable space Y. Then the following conditions are equivalent:*

(2.3.1) *Every element of $\mathcal{K}$ is n-connected and locally n-connected.*

(2.3.2) *Whenever X is metrizable, A closed in X, $\dim(X-A) \leqslant n+1$, and $F\colon X \to Y$ is an open-graph multifunction with values belonging to $\mathcal{K}$, then there exists a continuous selection for F extending f.*

Proof. It follows from (2.1) and from the well-known characterization of C^n and LC^n spaces (see, e.g. Prop. 10.1 in [2]).

3. Applications.

We give two applications to the theory of LC^n spaces. These facts are analogues of some theorems on retracts.

The first fact deals with the extension of approximations.

(3.1) THEOREM. *Let Y be a metrizable LC^n space. Then for every* nbhd $U \subset Y \times Y$ *of the diagonal there is a* nbhd $V \subset U$ *of the diagonal such that, whenever X is metrizable, A closed in X, $\dim(X-A) \leqslant n+1$, $f\colon X \to Y$ and $g\colon A \to Y$ are continuous and $(f(a), g(a)) \in V$ for every $a \in A$, then there is a map $\bar{g}\colon X \to Y$ such that $\bar{g}|A = g$ and $(f(x), \bar{g}(x)) \in U$ for every $x \in X$.*

Proof. Let d be a metric on Y and $B(y, r) = \{z \in Y\colon d(y, z) < r\}$. Since Y is LC^n, there are $n+2$ maps $p_0, \ldots, p_{n+1}\colon Y \to (0, 1]$ such that $p_0(y) \leqslant \ldots \leqslant p_{n+1}(y)$,

$\{y\}\times B(y, p_{n+1}(y)) \subset U$ and every map $\partial I^{i+1} \to B(y, p_i(y))$ can be extended to a map $I^{i+1} \to B(y, p_{i+1}(y))$, $y \in Y$, $i = 0, \ldots, n$. Put $V = \bigcup_{y\in Y} \{y\}\times B(y, p_0(y))$ and $F_i(x) = B(f(x), p_i(f(x)))$, $x \in X$, $i = 0, \ldots, n+1$, and use Theorem (1.1).

The next fact is a substitute of uniform local contractibility for LC^n spaces.

(3.2) THEOREM. *Let Y be a metrizable LC^n space. Then for every* nbhd $U \subset Y\times Y$ *of the diagonal there is a* nbhd $V \subset U$ *of the diagonal such that, whenever X is metrizable, A closed in X,* $\dim(X-A) \leqslant n$, *$f, g\colon X \to Y$ are continuous, $f|A = g|A$ and $(f(x), g(x)) \in V$ for every $x \in X$, then $f \sim g$* rel *A by a homotopy $\{h_t\}$ such that $(f(x), h_t(x)) \in U$ for every $x \in X$ and $t \in I$.*

Proof. Let V be taken from the previous theorem. Define $F\colon X\times I \to Y$ and $G\colon A\times I \cup X\times\{0, 1\} \to Y$ by $F(x, t) = f(x)$, $G(a, t) = f(a)$, $G(x, 0) = f(x)$, $G(x, 1) = g(x)$, $x \in X$, $a \in A$, $t \in I$. From 3.1 it follows that G can be extended to a homotopy $h\colon X\times I \to Y$.

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When a subset of E^n locally lies on a sphere

by

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Abstract. Let X be a continuum in E^n, and let G be a collection of pairwise congruent double cones whose cone angles measure 2θ and whose interiors miss X. If each point of X is the vertex of a cone in G, $n = 2$, and $2\theta > 60°$, then X lies on a 1-sphere because it must be an arc or a simple closed curve. Conditions on 2θ and X sufficient to insure that X locally lies on a 2-sphere are also given for the case $n = 3$.

Consider a subset X of Euclidean n-space E^n such that X is touched from its complement at each of its points by an element of some geometric family of solids. What conditions on the touching objects are sufficient to imply that X locally lies on an $(n-1)$-sphere? For example, for $n = 3$ X will locally lie on a 2-sphere if it can be touched by congruent double tangent balls [L_2]; that is, if there exists $\delta > 0$ such that for each $p \in X$ there exist two 3-balls B and B', each with radius δ, such that $\{p\} = B\cap B'$ and $X \cap \operatorname{Int}(B\cup B') = \emptyset$. In this paper related theorems are proven where the double balls are replaced by double cones. For $n = 2$ the double cones become double triangles and a complete analysis is given. The more difficult problems in E^3 are only partially resolved.

Generalizations of the congruent double tangent ball result [L_2] mentioned above could take several directions. However, the weaker hypothesis that there be just congruent single touching balls will not allow the conclusion that X locally lies on a 2-sphere. An example is given in [L_2]. Clear also is the fact that the uniform radii (pairwise congruence) of the double balls is essential to the theorem. Thus it appears that in a generalization one should retain the dual nature of the touching objects while changing their geometry. This leads one to consider double cones in place of the double tangent balls. The needed uniform size on the touching balls could be captured by requiring that the double cones be pairwise congruent. However a ball has the property that given a point where it touches X there is a unique ball of a given size tangent to the first and not intersecting its interior. This property is partially captured in a double cone by insisting that its two nappes have coincident axes. These considerations lead to the definitions below for double cones in E^n.

A *single cone* in E^n $(n > 1)$ is obtained by coning over a standard $(n-1)$-ball B from a point v where the line (*axis*) through v and the center o of B is orthogonal to