Supports of Borel measures *

by

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Abstract. For every countable ordinal α, we construct a regular Borel measure on a completely regular space such that the operation of the restriction to the support can be iterated exactly α times. We give a decomposition of Borel measures according to support-properties; it can be used to characterize the τ-additive part of a measure. There exists a support-concentrated Borel measure the support of which cannot be the support of a τ-additive measure. Let $S(X)$ $(S_c(X))$ be the collection of the supports of all (support-concentrated) Borel measures on a space X. Every (completely regular) space Y is a subspace of an appropriate (completely regular) X with $Y \in S(X)$. Thus $S(X) \setminus S_c(X)$ may be nonempty. For locally compact X, the elements of $S(X)$ can be characterized by a topological condition which is due to Kelley. There are spaces X and closed subsets $A \notin S_c(X)$ such that the property "$A \in S(X)$" is independent of the ordinary axioms of set theory. Relations between the supports of finite and locally finite measures are investigated.

1. Notation and terminology. For an arbitrary set X and a subset A of X we denote by $\mathscr{P}(X)$ the power set of X and by A^c the complement of A in X.

Let X be a topological space and $A \subseteq X$. By $\mathscr{G}(X)$, $\mathscr{F}(X)$, $\mathscr{K}(X)$, $\mathscr{B}(X)$ we denote the collection of all open, closed, compact, Borel subsets of X, by $\bar{A}$ $(\mathring{A})$ the closure (interior) of A and by $\mathscr{G}(A)$ the relative topology of A. Compact, locally compact and completely regular spaces are always assumed to be Hausdorff.

A *locally finite measure* μ on a topological space X is a non-negative, extended real valued, countably additive set function on $\mathscr{B}(X)$ with the property, that every $x \in X$ has a neighbourhood U with $\mu(U) < \infty$. If $\mu(X) < \infty$, μ will be called a *Borel measure* on X. Let 0 be the measure which is identically zero everywhere on $\mathscr{B}(X)$ and δ_x the Dirac measure for $x \in X$. The *support* of a locally finite measure μ is defined as the set

$$\operatorname{supp}\mu := \bigcap \{F \in \mathscr{F}(X)\colon\ \mu(F^c) = 0\}$$
$$= \{x \in X\colon\ \mu(G) > 0 \text{ for every open neighbourhood } G \text{ of } x\};$$

μ is said to be *support-concentrated*, if $\mu(X \setminus \operatorname{supp}\mu) = 0$.

A Borel measure μ is said to be

— (*weakly*) τ-*additive* if $\mu(\bigcup_\alpha G_\alpha) = \sup_\alpha \mu(G_\alpha)$ for every increasing net (G_α) in

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$\mathscr{G}(X)$ (with $\bigcup_{\alpha} G_\alpha = X$);

— *purely σ-additive* if there is no τ-additive Borel measure ν with $0 < \nu \leqslant \mu$;

— *regular* if $\mu(B) = \sup\{\mu(F)\colon F \in \mathscr{F}(X),\ F \subseteq B\}$ for all $B \in \mathscr{B}(X)$;

— *compact-regular* if $\mu(B) = \sup\{\mu(K)\colon K \in \mathscr{K}(X),\ K \subseteq B\}$ for all $B \in \mathscr{B}(X)$ (here we assume X to be Hausdorff).

Obviously, every τ-additive Borel measure is support-concentrated.

A space X is *Borel measure-compact* if every regular Borel measure on X is τ-additive, and *(weakly) Borel measure-complete* if every Borel measure on X is (weakly) τ-additive.

We define the *restriction* μ_A of a Borel measure μ to $A \in \mathscr{B}(X)$ as the measure satisfying $\mu_A(B) = \mu(B \cap A)$ for all $B \in \mathscr{B}(X)$; $\mu^S := \mu_{\operatorname{supp}\mu}$ denotes the restriction of μ to its support.

For every ordinal number α, $\alpha+1$ denotes its immediate successor. Ordinal numbers are always identified with the set of their predecessors; in particular, $\alpha+1 = \alpha \cup \{\alpha\}$ holds for every ordinal. Ω denotes the first uncountable ordinal. Ω and $\Omega+1$, when considered as topological spaces, are endowed with the order topology.

All examples of this paper are collected in Chapter 5.

2. Support-restriction and the decomposition of a measure in a support-concentrated and a support-diffused part.

At the outset of this chapter, we discuss the iteration of support-restriction.

2.1. DEFINITION. Let μ be a Borel measure.

(a) μ^α is defined inductively for every ordinal α as follows:

$$\mu^0 := \mu \text{ and } \mu^\alpha := \mu_{\bigcap_{\beta<\alpha} \operatorname{supp}\mu^\beta} \quad \text{for } \alpha > 0.$$

(b) $\varrho_\mu := \inf\{\alpha\colon \alpha \text{ is an ordinal with } \mu^\alpha = \mu^{\alpha+1}\}$.

From Theorem 2.2 below it will follow that ϱ_μ is a well-defined ordinal. It is in fact exactly the ordinal at which the iteration of support-restriction stops, since on the one hand we have $\mu^\beta = \mu^{\varrho_\mu}$ for every $\beta \geqslant \varrho_\mu$, but $\mu^\beta > \mu^\gamma$ for every $\beta < \gamma \leqslant \varrho_\mu$. Observe that $\mu^{\alpha+1} = (\mu^\alpha)^S$ for every ordinal α.

2.2. THEOREM. (a) *ϱ_μ is a countable ordinal for every Borel measure μ.*

(b) *There is a completely regular space X such that for every countable ordinal α there exists a regular Borel measure μ on X with $\varrho_\mu = \alpha$.*

Theorem 2.2 answers Okada's question [17, Problem 2.10], whether the equality $\operatorname{supp}\mu = \operatorname{supp}\mu^S$ holds for every regular Borel measure on a Hausdorff-space, since in this case ϱ_μ would be smaller or equal to one. A simpler counterexample can be obtained as follows: Let Z be the ordinal space Ω and λ the Dieudonné measure on Z. If we construct Y and ν of Z and λ as we will do in Lemma 2.3, then ν is an example of a regular Borel measure on a Hausdorff-space with $\operatorname{supp}\nu \neq \operatorname{supp}\nu^S$.

2.3. LEMMA. (a) *Starting with an arbitrary topological space Z, we construct a space Y as follows: Let $(Z_n)_{1 \leqslant n < \infty}$ be a sequence of disjoint copies of Z, $z \notin \bigcup_{n=1}^{\infty} Z_n$ and $Y := \bigcup_{n=1}^{\infty} Z_n \cup \{z\}$. Let $\mathscr{G}(Y)$ be generated by $\bigcup_{n=1}^{\infty} (\mathscr{G}(Z_n) \cup \{Y \setminus Z_n\})$.*

Then Y is separable (completely regular, metrizable) if Z is separable (completely regular, metrizable).

(b) *For a Borel measure λ on Z, we define a Borel measure ν on Y: For every natural number n let λ_n be a measure on Y such that $\lambda_n(Z_n^c) = 0$ and such that the restriction of λ_n to Z_n is a copy of λ. Define ν to be the measure $\nu := \sum_{n=1}^{\infty} 2^{-n}\lambda_n$. ν is a regular measure, if λ is regular. If $\operatorname{supp}\lambda = \emptyset$ and $\lambda(Z) > 0$, then $\operatorname{supp}\nu = \{z\}$ and $\nu(\{z\}) = 0$.*

Proof of Theorem 2.2. (a) Suppose $\mu^\alpha \neq \mu^{\alpha+1}$ for every $\alpha \in \Omega$. Then $0 < a_\alpha := \mu^\alpha(X) - \mu^{\alpha+1}(X)$ and $\mu(X) \geqslant \sum_{\alpha \in \Omega} a_\alpha = \infty$.

(b) (I) Consider the space $Z := 2^{\Omega} \setminus \{x_0\}$ and the measure λ on Z as discussed in Example 5.2; λ is a regular Borel measure on the completely regular space Z with $\operatorname{supp}\lambda = \emptyset$. Construct Y and ν of Z and λ as done in Lemma 2.3, then ν is a regular Borel measure on the completely regular and separable space Y with $\nu(Y) = 1$, $\operatorname{supp}\nu = \{z\}$ and $\nu(\{z\}) = 0$.

(II) For every $\beta \in \Omega$ we define $X_\beta := Y^{\Omega \setminus \beta}$ (endowed with the product topology), then X_β is separable. Let A_β be a countable dense subset of X_β and τ_β be a normed Borel measure on X_β which is concentrated on A_β such that $\operatorname{supp}\tau_\beta = X_\beta$. Furthermore we set $\omega^\beta := (a_\gamma)_{0 \leqslant \gamma < \beta}$ where $a_\gamma = z$ for all γ and $X := X_0$ (i.e. $X = Y^\Omega$). X is a completely regular space. For every $\beta \in \Omega$ we now define Borel measures φ_β and ψ_β on X: $\varphi_0 := \tau_0$, and ψ_0 is the canonical extension of the product measure $\nu \otimes \tau_1$ on $\mathscr{B}(X)$ (which exists, since τ_1 is a weighted sum of Dirac measures; the extensions of the product measures below exist from the same reason); for $\beta > 0$ the measure φ_β is the canonical extension of $\delta_{\omega^\beta} \otimes \tau_\beta$ on $\mathscr{B}(X)$ and ψ_β the canonical extension of $\delta_{\omega^\beta} \otimes \nu \otimes \tau_{\beta+1}$ on $\mathscr{B}(X)$. For $a \in A_{\beta+1}$, the canonical extension χ_a of $\delta_{\omega^\beta} \otimes \nu \otimes \delta_a$ on $\mathscr{B}(X)$ is a regular measure; since $\psi_\beta = \sum_{a \in A_{\beta+1}} \tau_{\beta+1}(\{a\}) \cdot \chi_a$, the measure ψ_β is regular. Obviously, (i) and (ii) hold:

(i) $\operatorname{supp}\varphi_0 = X$ and $\operatorname{supp}\varphi_\beta = \{\omega^\beta\} \times X_\beta$ for $\beta > 0$;

(ii) $\operatorname{supp}\psi_0 = \{z\} \times X_1$ and $\operatorname{supp}\psi_\beta = \{\omega^{\beta+1}\} \times X_{\beta+1}$ for $\beta > 0$.

(III) We are now able to construct the desired measure μ:

For $\alpha \in \Omega \setminus \{0\}$, there exist real numbers $b_\beta > 0$, $\beta < \alpha$ with $\sum_{\beta<\alpha} b_\beta < \infty$.

We then define μ as $\mu := \sum_{\beta<\alpha} b_\beta \cdot \psi_\beta + \varphi_\alpha$.

μ is a regular measure. In order to show the remaining properties, we set $\nu_\beta := \sum_{\beta \leqslant \gamma < \alpha} b_\gamma \cdot \psi_\gamma + \varphi_\alpha$ for $\beta < \alpha$, then $\operatorname{supp}\nu_\beta = \{\omega^{\beta+1}\} \times X_{\beta+1}$.

It is easy to show by transfinite induction

$$\mu^{\beta} = \begin{cases} \varphi_\alpha & \text{for } \beta \geqslant \alpha \\ \nu_\beta & \text{for } \beta < \alpha, \end{cases}$$

which yields $\mu^{\alpha} = \mu^{\alpha+1}$ and $\mu^{\beta} \neq \mu^{\beta+1}$ for $\beta < \alpha$.

2.4. Remark. For metrizable spaces, we can give the following answer to Okada's question: The equality $\operatorname{supp}\mu = \operatorname{supp}\mu^S$ holds for every Borel measure on a metrizable space X if X has a dense subset the cardinality of which is of measure zero (a cardinal m is said to have *measure zero* if every finite measure defined for all subsets of any set of power m and vanishing for one-point sets vanishes identically). Conversely, assume that m does not have measure zero. Then there exists a metrizable space X of power m and a Borel measure μ on X with $\operatorname{supp}\mu \neq \operatorname{supp}\mu^S$. Moreover it is possible to construct a metrizable space X with the following properties: The ordinary axioms of set theory together with the continuum hypothesis imply that $\operatorname{supp}\mu = \operatorname{supp}\mu^S$ for every Borel measure μ on X. On the other hand, if every set of real numbers is Lebesgue-measurable (which can be assumed under certain axioms of set theory, see Solovay [21]), X admits a Borel measure ν with $\operatorname{supp}\nu \neq \operatorname{supp}\nu^S$.

Proof. If X has a dense subset the cardinality of which is of measure zero then every Borel measure is concentrated on a separable subset of X, hence τ-additive and support-concentrated (Marczewski and Sikorski [13, Theorem III]). On the other hand, let m be a cardinal which is not of measure zero. Then there exists a set Z of power m and a measure λ on $\mathscr{P}(Z)$ with $\lambda(Z) = 1$ and $\lambda(\{z\}) = 0$ for every $z \in Z$. On Z, endowed with the discrete topology, λ is a Borel measure with empty support. Z is metrizable. Construct Y and ν of Z and λ as done in Lemma 2.3, then Y is a metrizable space of power m and ν is a Borel measure on Y with $\operatorname{supp}\nu \neq \operatorname{supp}\nu^S$.

If we set $Z := \boldsymbol{R}$ with the discrete topology and construct Y as above, then ZFC+CH implies $\operatorname{supp}\mu = \operatorname{supp}\mu^S$ for every Borel measure μ on Y. If every set of real numbers is Lebesgue-measurable, then there exists a measure λ on Z with $\lambda(Z) = 1$ and $\operatorname{supp}\lambda = \emptyset$. For the measure ν constructed as above, we have $\operatorname{supp}\nu \neq \operatorname{supp}\nu^S$.

We now want to decompose a measure in two measures with extremely different support-properties.

2.5. Definition. (a) For every Borel measure μ, define

$$C_\mu := \{\nu\colon \nu \text{ is a support-concentrated Borel measure with } \nu \leqslant \mu\}.$$

(b) A Borel measure μ is called *support-diffused*, if $C_\mu = \{0\}$.

2.6. Remark. (a) Every Borel measure μ with $\mu(\operatorname{supp}\mu) = 0$ is support-diffused.

(b) If μ is a support-diffused Borel measure, then there exists a Borel measure ν with $\operatorname{supp}\nu = \operatorname{supp}\mu$ and $\nu(\operatorname{supp}\nu) = 0$.

(c) There exists a nontrivial support-diffused Borel measure on X iff X is not Borel measure-complete.

Proof. (a) is obvious.

(b) Define $\nu := \mu_{(\operatorname{supp}\mu)^c}$, then $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$ and $\nu(\operatorname{supp}\nu) \leqslant \nu(\operatorname{supp}\mu) = 0$. Assume $\operatorname{supp}\nu \subsetneqq \operatorname{supp}\mu$. Then there exists a set $G \in \mathscr{G}(X)$ with $G \cap \operatorname{supp}\mu \neq \emptyset$ and $\mu(G \setminus \operatorname{supp}\mu) = 0$, thus $\mu(G \cap \operatorname{supp}\mu) > 0$. For $A := G \cap \operatorname{supp}\mu$, the measure $\lambda := \mu_A$ is therefore a nondegenerate minorant of μ. If $y \in \bar{A}$ and $y \in U \in \mathscr{G}(X)$, $U \cap G$ is a nonempty open set with $U \cap G \cap \operatorname{supp}\mu \neq \emptyset$. Since $\mu(G \setminus \operatorname{supp}\mu) = 0$, we have $\lambda(U) = \mu(U \cap G \cap \operatorname{supp}\mu) = \mu(U \cap G) > 0$, and therefore $y \in \operatorname{supp}\lambda$. Since $\lambda(\bar{A}^c) = 0$, λ is a nontrivial support-concentrated minorant of μ, which is a contradiction to μ support-diffused.

(c) If X is not Borel measure-complete, then there exists a non τ-additive Borel measure ϱ on X and a set $B \in \mathscr{B}(X)$ with $\varrho(B) > 0$ such that every $x \in B$ has a neighbourhood G with $\varrho(B \cap G) = 0$ (Gardner [4, Theorem 5.3]). Then $\operatorname{supp}\mu \cap B = \emptyset$ holds for the measure $\mu := \varrho_B$, and therefore μ is a nontrivial Borel measure with $\mu(\operatorname{supp}\mu) = 0$. The assertion follows from part (a) of this remark.

2.7. Theorem. *Let μ be a Borel measure on a space X.*

(a) *There exist Borel measures μ^c and μ^d on X having the following properties:*

(i) $\mu = \mu^c + \mu^d$,

(ii) $\mu^c = \max C_\mu$ *(especially μ^c support-concentrated),*

(iii) μ^d *is support-diffused.*

(b) *If $\mu = \mu_1 + \mu_2$ where μ_1 is support-concentrated and μ_2 is a support-diffused measure, then* $\operatorname{supp}\mu_1 = \operatorname{supp}\mu^c$.

(c) $\overset{\circ}{\overline{\operatorname{supp}\mu}} \subseteq \operatorname{supp}\mu^c \subseteq \operatorname{supp}\mu$.

Proof. (a) With $A := \overline{\bigcup_{\nu \in C_\mu} \operatorname{supp}\nu}$, $\mu^c := \mu_A$ and $\mu^d := \mu_{A^c}$ we have $\mu = \mu^c + \mu^d$. Obviously, μ^d is support-diffused, and in order to verify, that μ^c is support-concentrated, it remains to show that $A \subseteq \operatorname{supp}\mu_A$. For $G \in \mathscr{G}(X)$ with $G \cap A \neq \emptyset$ there exists a measure $\nu \in C_\mu$ with $G \cap \operatorname{supp}\nu \neq \emptyset$. Consequently $\mu_A(G) \geqslant \nu(G \cap A) = \nu(G) > 0$ (the equality $\nu(G \cap A) = \nu(G)$ holds, because ν is support-concentrated).

(b) μ_1 is support-concentrated, consequently $\mu_1 \leqslant \mu^c$, $S := \operatorname{supp}\mu_1 \subseteq \operatorname{supp}\mu^c$ and $\mu_{S^c} \leqslant \mu_2$. We now assume, that $B := \operatorname{supp}\mu^c \setminus S \neq \emptyset$. In this case we prove, that μ_B is a nondegenerate support-concentrated measure. If the intersection of a $G \in \mathscr{G}(X)$ with $\bar{B}$ is not empty, then $G \cap S^c$ is a nonempty open set with $(G \cap S^c) \cap \operatorname{supp}\mu^c \neq \emptyset$, hence $\mu^c(G \cap S^c) > 0$. But μ^c is support-concentrated, and thus we obtain $\mu_B(G) = \mu(G \cap S^c \cap \operatorname{supp}\mu^c) = \mu^c(G \cap S^c \cap \operatorname{supp}\mu^c) = \mu^c(G \cap S^c) > 0$. It follows, that $\bar{B} \subseteq \operatorname{supp}\mu_B$. As $\operatorname{supp}\mu_B \subseteq \bar{B}$, we get $\operatorname{supp}\mu_B = \bar{B}$; clearly $\mu_B(\bar{B}^c) = 0$. Since $B \neq \emptyset$, μ_B is a nontrivial support-concentrated measure. But observe, that $\mu_B \leqslant \mu_{S^c} \leqslant \mu_2$, which is a contradiction to μ_2 support-diffused.

(c) To prove (c), we show, that the measure μ_T is support-concentrated for $T := \overset{\circ}{\overline{\operatorname{supp}\mu}}$. Clearly $\operatorname{supp}\mu_T \subseteq T$ and $\mu_T(T^c) = 0$. Suppose, $G \cap T \neq \emptyset$ for a $G \in \mathscr{G}(X)$, then $\tilde{G} := G \cap \overset{\circ}{\widehat{\operatorname{supp}\mu}}$ is an open set with $\tilde{G} \cap \operatorname{supp}\mu \neq \emptyset$, thus $0 < \mu(\tilde{G}) = \mu(G \cap \overset{\circ}{\widehat{\operatorname{supp}\mu}}) \leqslant \mu(G \cap T) = \mu_T(G)$. Thus $T \subseteq \operatorname{supp}\mu_T$.

2.8. Corollary. (a) *The support of a support-diffused measure is nowhere dense.*

(b) *The sum of two support-diffused measures is support-diffused.*

Proof. (a) immediately follows from Theorem 2.7 (c).

(b) Let μ_1, μ_2 be support-diffused Borel measures, define $\mu := \mu_1 + \mu_2$ and $A := \operatorname{supp}\mu^c$. Then $\mu^c = \mu_A$, as demonstrated in the proof of Theorem 2.7 (a). Denote by $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}$ the restrictions of μ_1, μ_2 and μ to the subspace A of X. Then it is easy to see, that $\tilde{\mu}_i$ is support-diffused for $i = 1, 2$, $\tilde{\mu}$ is support-concentrated with $\operatorname{supp}\tilde{\mu} = A$ and $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_2$. It follows from Corollary 2.8 (a), that $\operatorname{supp}\tilde{\mu}_i$ is nowhere dense in $\mathscr{G}(A)$. On the other hand it holds that $A = \operatorname{supp}\tilde{\mu} = \operatorname{supp}\tilde{\mu}_1 \cup \operatorname{supp}\tilde{\mu}_2$ (compare Lotz [12, Lemma 2.2]). Thus A is nowhere dense in its relative topology; consequently $A = \emptyset$ and $C_\mu = \{0\}$.

The decomposition $\mu = \mu_1 + \mu_2$ given in part (b) of Theorem 2.7 needs not to be unique, as Claim 1 in Example 5.2 shows. In the next section we will show, that the uniqueness of this decomposition is equivalent to the τ-additivity of μ^c.

For a measure μ now consider the iteration of support-restriction as defined in 2.1. The result is a decreasing ordinal-sequence (μ^α) of measures, which converges pointwise to μ^c, as the following proposition shows:

2.9. Proposition. *Let μ be a Borel measure. Then $\mu^\alpha = \mu^c + (\mu^d)^\alpha$ holds for every ordinal α, and $(\mu^d)^\alpha$ vanishes iff $\alpha \geqslant \varrho_\mu$.*

Proof. It is enough to show by transfinite induction that $\mu^\alpha = \mu^c + (\mu^d)^\alpha$ holds for every ordinal α. This equality obviously holds for $\alpha = 0$; assume that it is true for all $\beta < \alpha$. Then with $\operatorname{supp}\mu^\beta = \operatorname{supp}\mu^c \cup \operatorname{supp}(\mu^d)^\beta$ it follows that $\bigcap_{\beta<\alpha} \operatorname{supp}\mu^\beta = \operatorname{supp}\mu^c \cup \bigcap_{\beta<\alpha} \operatorname{supp}(\mu^d)^\beta$ and

$$\mu^\alpha = \mu_{\bigcap_{\beta<\alpha}\operatorname{supp}\mu^\beta} = \mu_{\operatorname{supp}\mu^c \cup \bigcap_{\beta<\alpha}\operatorname{supp}(\mu^d)^\beta} = (\mu_{\operatorname{supp}\mu^c} + \mu_{(\operatorname{supp}\mu^c)^c})_{\operatorname{supp}\mu^c \cup \bigcap_{\beta<\alpha}\operatorname{supp}(\mu^d)^\beta}$$

$$= \mu_{\operatorname{supp}\mu^c} + (\mu_{(\operatorname{supp}\mu^c)^c})_{\bigcap_{\beta<\alpha}\operatorname{supp}(\mu^d)^\beta} = \mu_{\operatorname{supp}\mu^c} + \mu^d_{\bigcap_{\beta<\alpha}(\operatorname{supp}\mu^d)^\beta} = \mu^c + (\mu^d)^\alpha.$$

3. Relations to τ-additivity. Support-concentration and τ-additivity are closely related. Pym [18] investigates the connections for certain functionals, and similar results hold for Borel measures: Every τ-additive measure is support-concentrated, but the converse is not true, in general (see Adamski [1, Example 1], and Lotz [12, Example 2.7]). Deeper relations can be established using a construction given implicitly by Knowles [10, p. 146] and explicitly by Moran [15, Theorem 2.1] for Baire measures; for Borel measures cf. Gardner [4, Theorem 4.1 and Theorem 5.3]: Every non-(weakly) τ-additive measure possesses a nontrivial minorant the support of which is a null set (empty). Thus τ-additivity of a measure can be described by support-concentration (see Adamski [1, Proposition 1] and Lotz [12, Theorem 2.5]), and every Borel measure on a space X is support-concentrated iff X is Borel measure-complete.

It is a well-known fact that every Borel measure μ can be uniquely represented as the sum of a τ-additive measure μ^τ and a purely σ-additive measure μ^σ. The relation to the decomposition of μ according to Theorem 2.7 is given by the following theorem:

3.1. Theorem. *Let μ be a Borel measure and μ^τ its τ-additive part. Then $\mu^\tau = \inf\{\mu_1 \colon \mu = \mu_1 + \mu_2$ with μ_1 support-concentrated and μ_2 support-diffused$\}$.*

Proof. Define

$$M := \{\mu_1 \colon \mu = \mu_1 + \mu_2 \text{ with } \mu_1 \text{ support-concentrated and } \mu_2 \text{ support-diffused}\}$$

and $\varrho := \inf M$. For $\mu_1 \in M$ and $\mu_2 := \mu - \mu_1$, the measure $\lambda := \inf\{\mu_2, \mu^\tau\}$ is a minorant of μ^τ, hence τ-additive and thus support-concentrated. Since $\lambda \leqslant \mu_2$ holds and since μ_2 is support-diffused, λ is identical to zero everywhere on $\mathscr{B}(X)$. It follows, that for an appropriate $A \in \mathscr{B}(X)$ we have $(\mu^\tau)_A = \mu^\tau$ and $\mu_2(A) = 0$; hence $\mu_1 \geqslant (\mu_1)_A = \mu_A \geqslant (\mu^\tau)_A = \mu^\tau$. Since μ_1 was an arbitrary element of M, $\mu^\tau \leqslant \varrho$ holds.

Now assume $\mu^\tau \lneqq \varrho$. Then $\varrho - \mu^\tau$ is a nontrivial measure which is not τ-additive, and we can show that there exists a support-diffused Borel measure χ with $0 < \chi < \varrho - \mu^\tau$ (see the proof of Remark 2.6(c)).

Let $\mu_1 \in M$ and $B \in \mathscr{B}(X)$ be such that $\mu_1(B) - \varrho(B) < \chi(B)$. By 2.7 $\mu_1 - \chi = \nu^c + \nu^d$ where ν^c is support-concentrated and ν^d support-diffused. $\mu = \nu^c + (\nu^d + \chi + \mu_2)$. By 2.8(b) $\nu^c \in M$ and $\nu^c(B) \leqslant \mu_1(B) - \chi(B) < \varrho(B)$, so ϱ is not $\inf M$. The contradiction.

It follows, that the decomposition of μ in a support-concentrated and a support-diffused part is unique iff μ^c is τ-additive.

It is not possible to replace "inf" by "min" in Theorem 3.1; see Example 5.2, Claim 2 and Example 5.3. Thus support-concentration and τ-additivity are essentially different properties (in the examples given by Adamski [1, Example 1] and Lotz [12, Example 2.7] for non τ-additive, but support-concentrated measures, support-concentration was due to a τ-additive minorant).

Investigating the question raised by Okada (see Section 2), one may look for topological conditions which assure that the equality $\operatorname{supp}\mu = \operatorname{supp}\mu^S$ holds for every (regular) Borel measure μ. Let us call a space X "*locally Borel measure-complete*" ("*locally Borel measure-compact*"), if every $x \in X$ possesses a Borel measure-complete (Borel measure-compact) neighbourhood (considered as a subspace). We claim, that every (regular) purely σ-additive Borel measure on a locally Borel measure-complete (locally Borel measure-compact) space has empty support: Let X be locally Borel measure-compact and μ be a purely σ-additive, regular Borel measure on X. Assume that $\operatorname{supp}\mu \neq \emptyset$ and choose a point $x \in \operatorname{supp}\mu$. Then there exists a Borel measure-compact neighbourhood U of x and a set $G \in \mathscr{G}(X)$ with $x \in G \subset U$. Since $x \in \operatorname{supp}\mu$, we have $\mu(G) > 0$. The measure μ_G, considered as a measure on U, is regular and thus τ-additive, therefore μ_G is also a τ-additive measure if considered as a measure on X. But now we have found a nontrivial τ-additive minorant of μ, in contradiction to μ purely σ-additive.

Consequently for every (regular) Borel measure μ on a locally Borel measure-complete (locally Borel measure-compact) space the following properties hold:

(i) $\mu^\tau + \mu^\sigma$ is a decomposition of μ in a support-concentrated and a support-diffused part, especially $\operatorname{supp}\mu = \operatorname{supp}\mu^c = \operatorname{supp}\mu^\tau$;

(ii) $\operatorname{supp}\mu = \operatorname{supp}\mu^S$.

For a further discussion see Example 5.1 and Example 5.2, Claim 3 and 4.

Most of the results of Chapter 2 and 3 remain true, if we consider measures on a σ-algebra which is generated by a basis of the underlying topology. The proofs are more complicated, since supports need no longer be measurable. The advantage of this approach is, that it covers also Baire measures on completely regular spaces and products of Borel measures (which themselves need not to be Borel measures on the product space).

4. Topological properties of supports. For a space X, we denote by $S(X)$ the set of the supports of all Borel measures on X, and by $S_c(X)$ the subset of $S(X)$ consisting of the supports of support-concentrated measures.

Since Borel measures are finite, the countable chain condition (ccc) is necessary for "$X \in S(X)$". The relation to the existence of a locally finite measure with full support is as follows:

4.1. PROPOSITION. *Suppose that there is a locally finite measure μ on a space X with* $\operatorname{supp}\mu = X$. *Then $X \in S(X)$ if and only if X satisfies the countable chain condition.*

Proof. Suppose that X satisfies the countable chain condition. Denote by $\mathscr{U}$ the set of all $U \in \mathscr{G}(X)$ with $0 < \mu(U) < \infty$. There is a maximal subfamily $\overline{\mathscr{U}}$ of $\mathscr{U}$ consisting of pairwise disjoint sets. $\overline{\mathscr{U}}$ is countable say $\overline{\mathscr{U}} = \{U_n: n \in N\}$. $\nu := \sum_{n\in N} 2^{-n}(\mu(U_n))^{-1}\mu_{U_n}$ is a finite measure with $\operatorname{supp}\nu = X$.

A necessary condition for the existence of a locally finite measure μ on a space X with $\operatorname{supp}\mu = X$ is, that every $x \in X$ possesses an open neighbourhood U with $U \in S(U)$ (where U is endowed with its relative topology). In general, this condition is not sufficient, as the ordinal space Ω shows (Example 5.1). It is however sufficient for paracompact spaces:

4.2. PROPOSITION. *Let X be a paracompact space with the property that every $x \in X$ possesses an open neighbourhood U with $U \in S(U)$. Then there is a locally finite measure μ on X with* $\operatorname{supp}\mu = X$.

Proof. Choose for every $x \in X$ an open neighbourhood U_x with $U_x \in S(U_x)$. $(U_x: x \in X)$ is then an open cover of X, and since X is paracompact, there exists a locally finite open cover $(B_i: i \in I)$ such that for every $i \in I$ there is a point x_i with $B_i \subseteq U_{x_i}$. Every U_{x_i} admits a Borel measure ν_i with $\operatorname{supp}\nu_i = U_{x_i}$. Then $\nu_i(B_i) > 0$ holds, and we define for every $i \in I$ a Borel measure μ_i on X by $\mu_i(A) := \nu_i(A \cap B_i)$ $(A \in \mathscr{B}(X))$. Since $(B_i: i \in I)$ is locally finite, the measure $\mu := \sup_{i\in I}\mu_i$ is a locally finite measure on X with $\operatorname{supp}\mu = X$.

For a closed subset A of a space X, the property "$A \in S_c(X)$" depends only on the relative topology of A. For such a set A it holds true that every point finite subset of $\mathscr{G}(A)$ is countable (see Grömig [7, Theorem 1.12]). A modification of this statement implies, that every Borel measure on a metacompact space possesses a Lindelöf support. This is proved essentially by Moran [16, Theorem 5.1] and explicitly by Okada [17, Theorem 3.1]. The latter discusses in detail those spaces on which every measure has a Lindelöf support. As a consequence, a closed subset A of a metrizable space X is an element of $S(X)$ iff it is separable.

In contrast to "$A \in S_c(X)$", the property "$A \in S(X)$" is to a great extent independent of the relative topology of A:

4.3. THEOREM. *Every topological space Y can be topologically embedded into a space X such that $Y \in S(X)$. If Y is completely regular, the space X can also be chosen to be completely regular.*

Proof. Let Y be an arbitrary space and define X to be the disjoint union of Y and the ordinal space Ω. Denote by $\mathscr{U}$ the family of the intersections with Ω of open neighbourhoods of the point Ω in the space $\Omega+1$, and define

$$\mathscr{G}(X) := \mathscr{G}(\Omega) \cup \{G \cup U: G \in \mathscr{G}(Y) \text{ and } U \in \mathscr{U}\}.$$

Let μ be the Dieudonné measure on Ω, considered in a canonical way as a measure on X. Then $\operatorname{supp}\mu = Y$. For completely regular Y, the space X constructed as above is not Hausdorff. In this case we proceed as follows: There exists a topological embedding $\Phi: Y \to \hat{Y} := [0,1]^\Psi$ for an appropriate Ψ. The product measure $\bigotimes_{\psi\in\Psi} \nu_\psi$, where ν_ψ is the Lebesgue measure on $[0,1]$ for every $\psi \in \Psi$, can be extended to a τ-additive Borel measure $\hat{\lambda}$ on $\mathscr{B}(\hat{Y})$ (Mařik [14]). Then $\operatorname{supp}\hat{\lambda} = \hat{Y}$. Let $\hat{\nu}$ be the Dieudonné measure on the ordinal space $\Omega+1$ and let $\tilde{\mu}$ be an extension of $\hat{\lambda}\otimes\hat{\nu}$ to $\mathscr{B}(\tilde{Y})$, where $\tilde{Y}$ is the topological product of $\hat{Y}$ and $\Omega+1$ (such an extension exists, see Ressel [20, proof of Theorem 1]). Then $\operatorname{supp}\tilde{\mu} = \hat{Y}\times\{\Omega\}$ and $\tilde{\mu}(\operatorname{supp}\tilde{\mu}) = 0$ holds. Define X as the subspace $(\hat{Y}\times\Omega) \cup (\Phi[Y]\times\{\Omega\})$ of $\tilde{Y}$ and μ as the Borel measure given by $\mu(B) = \tilde{\mu}(B \cap [\hat{Y}\times\Omega])$ for $B \in \mathscr{B}(X)$, then X is completely regular and $\operatorname{supp}\mu = \Phi[Y]\times\{\Omega\}$. But $\Phi[Y]\times\{\Omega\}$ and Y are clearly homeomorphic.

4.4. COROLLARY. *There is a completely regular space X with $S(X)\setminus S_c(X) \neq \emptyset$.*

Proof. Let Y be an uncountable, discrete space. According to Theorem 4.3, Y can be embedded into a completely regular space X such that $Y \in S(X)$. On the other hand, $Y \notin S_c(X)$, since Y does not satisfy the countable chain condition.

Although there are locally compact spaces which admit support-diffused measures, the equality $S(X) = S_c(X)$ holds for every locally compact space X, as the next theorem will show (in fact, for every $A \in S(X)$ there exists a compact-regular Borel measure μ with $\operatorname{supp}\mu = A$). Thus we can give a topological characterization of the elements of $S(X)$, using a condition which is due to Kelley [9]:

4.5. Definition. A family H of subsets of a set X is called a *positive collection*, if there is a positive real number a such that for every finite sequence $S = (A_1, \ldots, A_n)$ in H there exists a subsequence $(A_{i_1}, \ldots, A_{i_k})$ with $\frac{k}{n} > a$ and $\bigcap_{j=1}^{k} A_{i_j} \neq \emptyset$.

A topological space is said to satisfy the *Kelley-condition*, if the family of its nonempty, open subsets is the union of a countable family of positive collections.

The Kelley-condition is for an arbitrary space X necessary and for compact X also sufficient for $X \in S(X)$, see Hebert and Lacey [8, Theorem 1.6] or Comfort and Negrepontis [2, Lemma 6.2(a) and Theorem 6.4]. Kelley's condition implies the countable chain condition, but a famous counterexample given by Gaifman [3] shows that there is a compact space that satisfies the countable chain condition but does not satisfy the Kelley-condition and thus is not the support of a Borel measure. For completely regular X, the Kelley-condition is no longer sufficient for $X \in S(X)$. More about this subject and other chain conditions can be found in the monograph of Comfort and Negrepontis [2].

4.6. Theorem. *For any locally compact space X, we have $S(X) = S_c(X)$ $= \{A \subset X\colon A$ is closed and $\mathscr{G}(A)$ satisfies the Kelley-condition$\}$.*

Proof. (1) $S(X) \subseteq S_c(X)$: Let A be an element of $S(X)$ and ν be a Borel measure with $\operatorname{supp}\nu = A$. Denote by $\tilde{X}$ the Alexandroff-compactification of X ($=X$, if X is compact) and by $\tilde{\nu}$ the canonical extension of ν to $\tilde{X}$. The restriction of $\tilde{\nu}$ to the Baire σ-algebra defines a τ-additive Baire measure on $\tilde{X}$, which can be extended to a τ-additive and hence support-concentrated Borel measure $\tilde{\mu}$ on $\tilde{X}$ (Mařík [14]). Since the open Baire sets generate $\mathscr{G}(\tilde{X})$, $\operatorname{supp}\tilde{\mu} = \operatorname{supp}\tilde{\nu}$ holds. The restriction μ of $\tilde{\mu}$ to X is then a support-concentrated (in fact compact-regular) Borel measure on X with $\operatorname{supp}\mu = A$.

(2) For every $A \in S_c(X)$, $A \in S(A)$ holds and therefore A satisfies the Kelley-condition.

(3) Let A be a closed subset of X, A is locally compact. If A is not compact, let $\tilde{A} := A \cup \{\omega\}$ be its Alexandroff-compactification. If A satisfies the Kelley-condition, $\tilde{A}$ also does: $\mathscr{U} := \{G \in \mathscr{G}(\tilde{A})\colon \omega \in G\}$ is a positive collection and so $\mathscr{G}(\tilde{A}) \setminus \{\emptyset\} = (\mathscr{G}(A) \setminus \{\emptyset\}) \cup \mathscr{U}$ is a countable family of positive collections. Thus there exists a Borel measure $\tilde{\nu}$ on $\tilde{A}$ with $\operatorname{supp}\tilde{\nu} = \tilde{A}$. Denote by μ the canonical extension to $\mathscr{B}(X)$ of the restriction of $\tilde{\nu}$ to A, then μ is a support-concentrated Borel measure with $\operatorname{supp}\mu = A$.

In spaces with $S(X) \setminus S_c(X) \neq \emptyset$ it is not possible to give a characterization of $S(X)$ without using support-diffused measures. Furthermore, Example 5.4 shows that it is in some cases impossible to decide within the ordinary axioms of set theory, whether a given set $A \notin S_c(X)$ is an element of $S(X)$ or not. For another conjecture on supports which is undecidable in ZFC, see Gardner [5, Theorem 14.4].

5. Examples.

5.1. *The ordinal spaces Ω and $\Omega+1$.* Ω is locally Borel measure-complete, but admits a non τ-additive Borel measure (the Dieudonné measure).

$\Omega+1$ is Borel measure compact, but not locally Borel measure complete.

For the space Ω the following properties hold:

(i) *For every $\alpha \in \Omega$ there exists an open neighbourhood U such that $U \in S(U)$.*

(ii) *There is no locally finite measure μ on Ω with $\operatorname{supp}\mu = \Omega$.*

Proof. (i) holds, because for every $\alpha \in \Omega$ the set $[0, \alpha]$ is an open, countable neighbourhood of α.

(ii) Suppose there exists a locally finite measure μ on Ω with $\operatorname{supp}\mu = \Omega$. Then $0 < \mu(\{\alpha+1\}) < \infty$ for every $\alpha \in \Omega$ (since the set $\{\alpha+1\}$ is always open), and therefore there exists a natural number n such that the set

$$C := \left\{\alpha+1\colon \alpha \in \Omega,\ \mu(\{\alpha+1\}) \geqslant \frac{1}{n}\right\}$$

is infinite. The ordinal $\beta_0 := \sup\{\beta \in \Omega\colon \beta+1 \cap C$ is finite$\}$ is a countable ordinal (since C contains a countable subset) with the property, that for every ordinal $\gamma < \beta_0$ the interval $(\gamma, \beta_0]$ contains infinitely many elements of C (since otherwise $\beta_0+2 \cap C = (\gamma+1 \cap C) \cup ((\gamma, \beta_0] \cap C) \cup (\{\beta_0+2\} \cap C)$ would be finite). But this implies that $\mu(U) = \infty$ for every open neighbourhood U of β_0.

5.2. *Subspaces of 2^{Ω}.* $2^{\Omega}(= \{0, 1\}^{\Omega})$, endowed with the product topology, is a separable Hausdorff-space; let $A = \{a_n\colon n \in N\}$ be a countable dense subset and ϱ a Borel measure concentrated on A with $\varrho(\{a_n\}) > 0$ for every $n \in N$ (for instance $\varrho = \sum_{n \in N} 2^{-n}\delta_{a_n}$). Then $\operatorname{supp}\varrho = 2^{\Omega}$. There is a topological embedding $h\colon \Omega+1 \to 2^{\Omega}$ (see Le Roy Peterson [11, Embedding Theorem 4]) with $x_0 := h(\Omega) \notin A$. Since $\Omega+1$ is compact, $H^+ := h[\Omega+1]$ is a closed subset of 2^{Ω} and $H := h[\Omega]$ is closed in $2^{\Omega} \setminus \{x_0\}$. Denote by λ the image measure of the Dieudonné measure on $\Omega+1$ under the mapping h. Then $\operatorname{supp}\lambda = \{x_0\}$, and the restriction of λ to $2^{\Omega} \setminus \{x_0\}$ is a regular measure (since the Dieudonné measure on Ω is regular and H is closed in $2^{\Omega} \setminus \{x_0\}$) with empty support.

Claim 1. *The decomposition of a measure in a support-concentrated and a support-diffused part needs not be unique*: For $X := 2^{\Omega}$ and $\mu := \varrho+\lambda$, $\mu = \mu+0 = \varrho+\lambda$ are two distinct decompositions of μ in a support-concentrated and a support-diffused part.

Claim 2. *There exist regular and antiregular, purely σ-additive, support-concentrated measures with nonempty supports on completely regular spaces*: $X := 2^{\Omega}$ can be considered as a topological group, so for every $n \in N$ there is a homeomorphism $j_n\colon X \to X$ with $j_n(x_0) = a_n$. The measure $\mu := \sum_{n \in N} 2^{-n} j_n(\lambda)$ is a purely σ-additive measure with $\operatorname{supp}\mu = X$; since X is compact, μ is an antiregular measure. The restriction ν of μ to $X \setminus A$ is a regular measure on $X \setminus A$ with $\operatorname{supp}\nu = X \setminus A$.

CLAIM 3. *$X := A \cup H$ is locally Borel measure-complete, admits a non τ-additive measure and in addition a (τ-additive) measure μ with* $\operatorname{supp}\mu = X$.

CLAIM 4. *For $X := 2^{\Omega}\setminus\{x_0\}$ the following properties hold*:

(1) *For every $x \in X$ there exists a support-diffused Borel measure μ on X with $\operatorname{supp}\mu = \{x\}$, thus X cannot be locally Borel measure-complete.*

(2) *X admits a regular, non τ-additive Borel measure, so X is not Borel measure-compact.*

(3) *$X = \operatorname{supp}\nu$ for a purely σ-additive measure ν.*

(4) *X is locally compact, hence locally Borel measure-compact.*

5.3. *A Hausdorff-space X with the property, that $X = \operatorname{supp}\mu$ for an appropriate purely σ-additive measure μ on X, but that $\operatorname{supp}\nu \neq X$ for every τ-additive Borel measure ν on X.*

For the construction we continue Example 5.2; in particular, j_n is defined as in Claim 2.

Define $H_n^+ := j_n[H^+]$ and $H_n := j_n[H]$, where we can assume that $H_n \cap A = \emptyset$ for every n (since A is countable and H_n is homeomorphic to the ordinal space Ω, we can replace H_n by a subspace that is homeomorphic to H_n and disjoint from A). Furthermore, we can assume, that the sets H_n^+ are pairwise disjoint: Suppose that $H_1^+, \ldots, H_m^+$ are pairwise disjoint. $\tilde{H} := \bigcup_{i=1}^{m} H_i^+$ is closed and $a_{m+1} \notin \tilde{H}$. Since 2^{Ω} is a completely regular space, there exist disjoint open neighbourhoods U and V of $\tilde{H}$ and a_{m+1}. V contains a subspace of H_{m+1}^+ which is homeomorphic to H_{m+1}^+ and disjoint from $\tilde{H}$.

Let $\mathscr{G}(Y)$ be the relative topology of $Y := \bigcup_{n \in N} H_n$, $(\alpha, \Omega) := \{\beta \in \Omega\colon \beta > \alpha\}$ for every ordinal α and $U_{(\alpha_n)} := \bigcup_{n \in N} j_n \circ h[(\alpha_n, \Omega)]$ for every sequence $(\alpha_n)_{n \in N} \in \Omega^N$. The space X is then defined as Y, equipped with the topology generated by $\mathscr{G}(Y) \cup \{U_{(\alpha_n)}\colon (\alpha_n) \in \Omega^N\}$.

On every H_n there exists a copy λ_n of the Dieudonné measure on Ω, which can be extended in a natural way to a measure on X with $\lambda_n(H_n^c) = 0$. Thus the measure $\mu := \sum_{n=1}^{\infty} 2^{-n}\lambda_n$ is a Borel measure on X with $\operatorname{supp}\mu = X$.

On the other hand, let ν be a τ-additive measure on X. For every $n \in N$, the restriction of ν to H_n is a τ-additive measure and thus concentrated on a countable subset of H_n (see Bhaskara-Rao et al. [19]). Therefore there exists a sequence $(a_n) \in \Omega^N$ with $\nu(U_{(\alpha_n)}) = \sum_{n \in N} \nu_{H_n}(j_n \circ h[(\alpha_n, \Omega)]) = 0$; and since $U_{(\alpha_n)} \neq \emptyset$, we have $\operatorname{supp}\nu \neq X$.

5.4. *A class of completely regular spaces Z_X and of subsets A of Z_X with the property that in some cases it is impossible to decide within* ZFC, *whether $A \in S(Z_X)$ does or does not hold.*

1. Define Y to be the set of real numbers with the rational sequence topology: If Q denotes the set of rational and $\mathscr{I}$ the set of irrational numbers, then every $q \in Q$ is open. For each $x \in \mathscr{I}$ we choose a sequence $(x_n)_{n \in N}$ in Q converging to x in the Euclidean topology and define the sets $U_n(x) := \{x_i\}_{i=n}^{\infty} \cup \{x\}$, $n \in N$, as a local basis at x. Y is completely regular, Q is a countable dense subset and $\mathscr{I}$ is an uncountable discrete subset of Y (see Steen–Seebach [22, Example 65]).

Let X be a locally compact, but not compact space. A space Z_X and a subset $A \in \mathscr{F}(Z_X)$ are then constructed as follows: Denote by $\tilde{X} = X \cup \{\omega\}$ the Alexandroff-compactification of X, define $\tilde{Z} := Y \times \tilde{X}$ (endowed with the product topology), $A := \Psi \times \{\omega\}$, $B := Q \times \{\omega\}$ and $Z_X := \tilde{Z}\setminus B$. Z_X is a completely regular space, A is a closed subset of Z_X with $A \notin S_c(Z_X)$, since A is uncountable and discrete.

Assume now (Gardner and Pfeffer [6]), that X is in addition a hereditarily metalindelöf space that satisfies the countable chain condition locally, and that the cardinality of each discrete subset of X is of measure zero. Let us abbreviate Martin's axiom as MA and the continuum hypothesis as CH, then the following statements hold:

(a) $\mathrm{MA} + \neg\mathrm{CH} \Rightarrow A \notin S(Z_X)$.

(b) *Under* CH *there exists a space X_0 that satisfies all the conditions above such that $A \in S(Z_{X_0})$.*

Proof of statement (a): Suppose $A = \operatorname{supp}\mu$ for a Borel measure μ on Z_X.

(i) Then $\mu(Q \times X) > 0$: Consider μ as a measure on $\tilde{Z}$ and denote by ν the image measure under the projection to Y. Now if $\mu(Q \times X) = 0$, then ν is a support-concentrated measure with $\operatorname{supp}\nu = \Psi$, which is impossible, since Ψ does not satisfy the countable chain condition.

(ii) Since Q is countable, there exists a point $q \in Q$ with $\mu(\{q\} \times X) > 0$. $\{q\} \times X$ is a copy of X, thus under $\mathrm{MA} + \neg\mathrm{CH}$ the restriction ϱ of μ to $\{q\} \times X$ is compact-regular (see Gardner and Pfeffer [6, Theorem 2.1]). Thus $\operatorname{supp}\varrho \neq \emptyset$, and it follows, that $\operatorname{supp}\mu\setminus A \neq \emptyset$, which is a contradiction to $A = \operatorname{supp}\mu$.

Proof of (b). Under CH, there exists a space X_0 with the desired properties and a Borel measure μ on X_0 with $\mu(X_0) = 1$ and $\mu(K) = 0$ for every compact set $K \subset X_0$ (see Gardner and Pfeffer [6, Remark 2.13. (γ)]). Since X_0 is locally compact, it follows that $\operatorname{supp}\mu = \emptyset$. Let ϱ be a Borel measure on Y, concentrated on Q, with $\operatorname{supp}\varrho = Y$. The product measure $\varrho \otimes \mu$ can be extended to a Borel measure $\bar{\nu}$ on $Y \times X_0$; let ν be the image measure under the canonical injection $j\colon Y \times X_0 \to Z_{X_0}$.

Since $\operatorname{supp}\mu = \emptyset$, we have $\operatorname{supp}\bar{\nu} = \emptyset$ and thus $\operatorname{supp}\nu \subseteq A$. On the other hand, if U is an open neighbourhood of $x \in A$, then $U \cap (Y \times X_0)$ can be without loss of generality written as $U_1 \times U_2$ with $U_1 \in \mathscr{G}(Y)$ and U_2 the complement of a compact subset of X_0. Then $\varrho(U_1) > 0$ and $\mu(U_2) = 1$, hence $\nu(U) > 0$. It follows that $\operatorname{supp}\nu = A$.

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Functions provably total in $I^{-}\Sigma_n$

by

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Abstract. The main theorem of the paper can be formulated as follows. *Let $n \geq 2$. If, in the proof of the totality of a recursive function $f\colon N \to N$ we use only m different axioms of Σ_n-induction without parameters and additionally only axioms of* PA^{-}, *then f can be bounded (almost everywhere) by a function H_β in Hardy's hierarchy, where $\beta = \omega_{n-1}^{\omega^m \cdot k}$ for some $k \in \omega$.*

§ 1. Introduction. The aim of this paper is a generalization to the case of $I^{-}\Sigma_n$ of the following theorem proved in [A–B], where $I^{-}\Sigma_n$ denotes the theory of Σ_n-induction without parameters.

1.1. THEOREM [A–B]. *If, in the proof of the totality of a recursive function $f\colon N \to N$ in the theory $I^{-}\Sigma_1$ we use only m different axioms of Σ_1-induction, then f can be bounded by the $2m$-th function in the Grzegorczyk hierarchy starting from 2^x.*

Let F_m denote the mth function in Grzegorczyk hierarchy. Thus $F_0(x) = 2^x$, $F_{m+1}(x) = F_m^{x+1}(x)$ for $m \in \omega$. Let us remark that a slight modification of estimation of growth in the proof of the theorem just quoted shows that f can be bounded by an iterate F_m^k of the function F_m for some $k \in N$. This estimation can be formalized in the theory $I\Delta_0 + \exp + \forall x \exists y F_m(x) = y$.

We say that a function $f\colon N \to N$ is *provably total* and *of the class* Σ_r in a theory T iff there exists $\varphi(x, y) \in \Sigma_r$ such that φ defines f in N and $T \vdash \forall x \exists! y \varphi(x, y)$. We will speak shortly: f is *total* and Σ_r in T. Functions which are total and Σ_1 in T are often called functions *provably recursive* in T.

Before we formulate our theorem, which generalizes Theorem 1.1, we need some notation. By H_α, for $\alpha < \varepsilon_0$, let us denote the Hardy's function with index α (see [W]); compare the definition of G_α below. Ordinal numbers ω_n^α, where $n \in \omega$, are defined by the following inductive conditions: $\omega_0^\alpha = \alpha$, $\omega_{n+1}^\alpha = \omega^{\omega_n^\alpha}$ for $n \in \omega$. If $f, g\colon N \to N$ then $f \leq g$ means that $\exists n \forall m \geq n\, f(m) \leq g(m)$; we say that g *dominates* f. If φ is a formula of the language of arithmetic with one free variable then let $\mathrm{Ind}\,\varphi$ denote the formula $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$. The theory $I\Delta_0 + \exp + \mathrm{Ind}\,\varphi_1 + \ldots + \mathrm{Ind}\,\varphi_m$ will be denoted by $I^{-}[\varphi_1, \ldots, \varphi_m]$.

1.2. THEOREM. *Let $n \geq 1$ and let $\varphi_1, \ldots, \varphi_m$ be Σ_n-formulas. If $f\colon N \to N$ is provably total and Σ_1 in $I^{-}[\varphi_1, \ldots, \varphi_m]$ then there exists a $k \in \omega$ such that f is do-*