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DEPARTMENT OF PHILOSOPHY
BROWN UNIVERSITY
Providence, RI 02912

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Finitely generated congruence distributive quasivarieties of algebras *

by

Wiesław Dziobiak (Toruń)

Abstract. We consider congruence distributive quasivarieties of algebras with a special emphasis on those which are finitely generated. There are two main results in the paper. The first states that within a congruence distributive quasivariety of algebras every finitely subdirectly irreducible algebra is finitely subdirectly irreducible in the absolute sense. The second provides necessary and sufficient conditions for a finitely generated and congruence distributive quasivariety of algebras of finite type to be finitely axiomatizable. In particular, applying these results we partially answer a question posed in Tumanov [9].

1. Introduction. A universal sentence whose matrix is of the form $r_0 = s_0 \& \ldots \ldots \& r_k = s_k \to r = s$ is called a *quasiidentity*. A class K of similar algebras is said to be a *quasivariety* if $K = \mathrm{Mod}\Sigma$ for some set Σ of quasiidentities. Equivalently (see [8]), K is a *quasivariety* if K is closed under isomorphisms (I), subalgebras (S), direct products (P) and ultraproducts (P_U). If M is a class of similar algebras then, by a result of [5], $ISP\dot{P}_U(M)$ is the least quasivariety containing M. Sometimes we shall write $Q(M)$ instead of $ISPP_U(M)$. A quasivariety K is said to be *finitely generated* if $K = Q(M)$ for some finite set M of finite algebras, and K is said to be *finitely axiomatizable* if $K = \mathrm{Mod}\Sigma$ for some finite set Σ of quasiidentities. For an algebra A, by $\mathrm{Con}A$ we denote the lattice of congruence relations on A. If K is a quasivariety and $A \in K$ then we put $\mathrm{Con}_K A = \{\Theta \in \mathrm{Con}A\colon A/\Theta \in K\}$. Since the set $\mathrm{Con}_K A$ is closed under arbitrary meets (in $\mathrm{Con}A$), it forms a complete lattice. We say that a quasivariety K is *congruence distributive* if for every $A \in K$ the lattice $\mathrm{Con}_K A$ is distributive. For varieties of algebras (= equational classes) this notion was intensively studied in the literature and many interesting results for it have been obtained. One of them, due to Baker [1], states that every finitely generated and congruence distributive *variety* of algebras of finite type is finitely axiomatizable. Our intention is to extend this result into quasivarieties. We prove (Theorem 4.5) that a finitely generated and congruence distributive quasivariety K of algebras of finite type is finitely axiomatizable iff $\mathrm{Mod}\Sigma \cup \Gamma$ is congruence distributive for some set Σ of identities and a finite set Γ of quasiidentities such that $K \subseteq \mathrm{Mod}\Sigma \cup \Gamma$.

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Applying this result we partially answer a question posed in Tumanov [9], and as corollary we derive a result of Blok and Pigozzi [2].

Another point focusing our attention is the problem of establishing properties shared by congruence distributive quasivarieties. We prove (Theorem 2.3) that in a congruence distributive quasivariety every finitely subdirectly irreducible algebra is finitely subdirectly irreducible in the absolute sense. This result allows us to characterize (Corollary 2.4 and Proposition 2 5) congruence distributive quasivarieties in wide classes of algebras. We also prove (Proposition 2.1) that a quasivariety K is congruence distributive iff for every finite n the lattice $\mathrm{Con}_K F_K(n)$ is distributive, where $F_K(n)$ denotes the free algebra in K with n free generators. Moreover, we point out that in general no finite bound on the number of free generators in free algebras of K cannot be made in order to make K congruence distributive. The last observation is in the contrast with a result of Jónsson [7] stating that a *variety* K is congruence distributive iff $\mathrm{Con}_K F_K(3)$ is distributive.

2. Congruence distributivity. Let us note the following

PROPOSITION 2.1. *For a quasivariety K of algebras the following conditions are equivalent*:

(i) *K is congruence distributive*;
(ii) $\mathrm{Con}_K F_K(\omega)$ *is distributive*;
(iii) *For each finite n,* $\mathrm{Con}_K F_K(n)$ *is distributive.*

Proof. That (i) implies (ii) is obvious. As for each finite n the lattice $\mathrm{Con}_K F_K(n)$ is isomorphic to an interval in $\mathrm{Con}_K F_K(\omega)$, (ii) implies (iii). Assuming (iii) we get that $\mathrm{Con}_K A$ is distributive for every finitely generated A of K. So, by Lemma 2.2 of [4] describing the least element $\Theta_K(H)$ in $\mathrm{Con}_K A$ containing H, where $H \subseteq A \times A$, it follows that $\mathrm{Con}_K A$ is distributive for all A of K. Thus K is congruence distributive, proving that (iii) implies (i).

If K is a *variety* then due to a result of Jónsson [7] we know that K is congruence distributive iff the lattice $\mathrm{Con}_K F_K(3)$ is distributive. The analogous result for quasivarieties is not true. This follows from the following example.

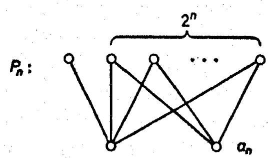

Fig. 1

Let P_n, $n \geqslant 3$, denote the poset depictured in Figure 1; the partial order of P_n is ascending as one moves upwards on the figure. Let $H(P_n)$ denote the Heyting algebra of all increasing subsets of P_n, and let $K = Q(H(P_n))$. We claim that the lattice $\mathrm{Con}_K F_K(n)$ is distributive while K is not congruence distributive. Since $H(P_n)$

is subdirectly irreducible in K and $H(P_n)$ is not subdirectly irreducible in the absolute sense, then by Theorem 2.3 below we obtain that K is not congruence distributive. On the other hand, as $H(R_n)$, where R_n is the subposet of P_n with universe $\{x: a_n \leqslant x\}$, belongs to K and every n-generated subdirectly irreducible algebra of $V(K)$ is embeddable into $H(R_n)$, it follows that every congruence relation on $F_K(n)$ is an element of $\mathrm{Con}_K F_K(n)$. Thus the lattices $\mathrm{Con} F_K(n)$ and $\mathrm{Con}_K F_K(n)$ coincide. Hence, by congruence distributivity of $V(K)$, the lattice $\mathrm{Con}_K F_K(n)$ is distributive.

The above example also shows that in general no finite bound on the number of free generators in free algebras of a quasivariety K cannot be made in order to make K congruence distributive. Although in the case when K is finitely generated a certain finite bound is possible. It depends on the cardinalities of generators of K. More precisely, if $K = Q(M)$ where M is a finite set of finite algebras and $m = \max\{|A|: A \in M\}$ then one can show that K is congruence distributive iff the lattice $\mathrm{Con}_K F_K(3+m)$ is distributive. Thus for a given finite set M of finite algebras of finite type the problem whether the quasivariety $Q(M)$ is congruence distributive is decidable. These results and others concerning congruence distributivity of quasivarieties will be published elsewhere jointly with Janusz Czelakowski.

Let K be a quasivariety and $A \in K$. An element Θ of $\mathrm{Con}_K A$ is said to be *finitely meet irreducible in* $\mathrm{Con}_K A$ if for all $\Theta_0, \Theta_1 \in \mathrm{Con}_K A$, $\Theta = \Theta_0 \cap \Theta_1$ implies $\Theta = \Theta_0$ or $\Theta = \Theta_1$. If the identity relation on A, denoted ω_A, is finitely meet irreducible in $\mathrm{Con}_K A$ then A is said to be *finitely subdirectly irreducible in K*. By K_{FSI} we denote the class of all finitely subdirectly irreducible members of K. We say that A is *finitely subdirectly irreducible in the absolute sense* if ω_A is finitely meet irreducible in $\mathrm{Con} A$.

LEMMA 2.2. *For a quasivariety K of algebras the following conditions are equivalent*:

(i) *K is congruence distributive*;
(ii) *For every $A \in K$, $\Theta_0, \Theta_1 \in \mathrm{Con} A$ and $\psi \in \mathrm{Con}_K A$: if ψ is finitely meet irreducible in $\mathrm{Con}_K A$ and $\Theta_0 \wedge \Theta_1 \leqslant \psi$ then $\Theta_0 \leqslant \psi$ or $\Theta_1 \leqslant \psi$.*

Proof. (i)⇒(ii): First, we show that (i) implies (a). For every $A \in K$, $\Theta_0, \Theta_1 \in \mathrm{Con} A$ and $\psi \in \mathrm{Con}_K A$: if ψ is finitely meet irreducible in $\mathrm{Con}_K A$, $\{\Theta_0, \Theta_1\} \cap \mathrm{Con}_K A \neq \emptyset$ and $\Theta_0 \wedge \Theta_1 \leqslant \psi$ then $\Theta_0 \leqslant \psi$ or $\Theta_1 \leqslant \psi$. Suppose otherwise. Then on a certain algebra A of K there exist congruence relations Θ_0, Θ_1 and ψ for which (a) fails. Let us assume that $\Theta_0 \in \mathrm{Con}_K A$; in the case $\Theta_1 \in \mathrm{Con}_K A$ we proceed similarly. Let $B = \{(a, b) \in A \times A: (a, b) \in \Theta_1\}$. Evidently, B is a subalgebra of $A \times A$ and hence $B \in K$. We shall show that $\mathrm{Con}_K B$ is not distributive which would finish the proof that (i) implies (a). To this effect it suffices to find three elements α, β and γ of $\mathrm{Con}_K B$ such that $\alpha \wedge \beta \leqslant \gamma$, γ is finitely meet irreducible in $\mathrm{Con}_K B$ and neither $\alpha \leqslant \gamma$ nor $\beta \leqslant \gamma$. Let $\pi_1, \pi_2: B \to A$ denote the projections of B onto A. As $\pi_1^{-1}(\Theta_1) = \pi_2^{-1}(\Theta_1)$, we have $\pi_1^{-1}(\Theta_0) \wedge \pi_2^{-1}(\omega_A) \leqslant \pi_1^{-1}(\Theta_0) \wedge \pi_2^{-1}(\Theta_1) = \pi_1^{-1}(\Theta_0) \wedge \pi_1^{-1}(\Theta_1) = \pi_1^{-1}(\Theta_0 \wedge \Theta_1) \leqslant$ (by $\Theta_0 \wedge \Theta_1 \leqslant \psi$) $\leqslant \pi_1^{-1}(\psi)$. Thus

1. $\pi_1^{-1}(\Theta_0) \wedge \pi_2^{-1}(\omega_A) \leqslant \pi_1^{-1}(\psi)$.

Since $\Theta_0 \nleqslant \psi$ and $\pi_1(B) = A$, we have

2. $\pi_1^{-1}(\Theta_0) \nleqslant \pi_1^{-1}(\psi)$.

By $\Theta_1 \nleqslant \psi$, $\Theta_1 \setminus \psi \neq \emptyset$. Let $(a, b) \in \Theta_1 \setminus \psi$. Then $(a, b) \equiv (b, b)(\pi_2^{-1}(\omega_A))$ because $(a, b), (b, b) \in B$. Evidently, $(a, b) \not\equiv (b, b)(\pi_1^{-1}(\psi))$. Thus we also have

3. $\pi_2^{-1}(\omega_A) \nleqslant \pi_1^{-1}(\psi)$.

As B and A/Θ_0 belong to K and $B/\pi_1^{-1}(\Theta_0) \cong A/\Theta_0$, we get that $\pi_1^{-1}(\Theta_0)$ is an element of $\mathrm{Con}_K B$. Similarly, $\pi_2^{-1}(\omega_A)$ and $\pi_1^{-1}(\psi)$ belong to $\mathrm{Con}_K B$. Moreover, $\pi_1^{-1}(\psi)$ is finitely meet irreducible in $\mathrm{Con}_K B$. Thus, by (1), (2) and (3), as α, β and γ we can take $\pi_1^{-1}(\Theta_0)$, $\pi_2^{-1}(\omega_A)$ and $\pi_1^{-1}(\psi)$, respectively.

Repeating the arguments used above it is easy now to show that (a) yields (ii). Thus (i) implies (ii).

(ii)⇒(i): Let $A \in K$ and $\Theta_0, \Theta_1, \Theta_2 \in \mathrm{Con}_K A$. Assume $\Theta_0 \wedge \Theta_1 +_K \Theta_0 \wedge \Theta_2 \leqslant \psi$ where ψ is a finitely meet irreducible element of $\mathrm{Con}_K A$ and $+_K$ denotes the lattice join formed in $\mathrm{Con}_K A$. By (ii), $\Theta_0 \wedge (\Theta_{1+K}\Theta_2) \leqslant \psi$. Therefore, as in $\mathrm{Con}_K A$ every element is the meet of finitely meet irreducibles over it, we conclude that $\Theta_0 \wedge (\Theta_1 +_K \Theta_2) \leqslant \Theta_0 \wedge \Theta_1 +_K \Theta_0 \wedge \Theta_2$. The converse inequality is immediate. Thus K is congruence distributive, proving that (ii) implies (i).

THEOREM 2.3. *Within a congruence distributive quasivariety of algebras every finitely subdirectly irreducible member is finitely subdirectly irreducible in the absolute sense.*

Proof. Let K be congruence distributive and $A \in K_{FSI}$. By Lemma 2.2, for any $\Theta_0, \Theta_1 \in \mathrm{Con} A$ with $\omega_A = \Theta_0 \wedge \Theta_1$ we have $\omega_A = \Theta_0$ or $\omega_A = \Theta_1$. Thus A is finitely subdirectly irreducible in the absolute sense.

The condition expressed in the above theorem is not sufficient. A counterexample is the quasivariety of all semilattices. However, in some cases it becomes also sufficient. Namely, we have

COROLLARY 2.4. *Suppose K is a congruence distributive quasivariety of algebras and L is a quasivariety contained in K. Then L is congruence distributive iff every finitely subdirectly irreducible algebra in L is finitely subdirectly irreducible in the absolute sense.*

Proof. The "only if" part follows from Theorem 2.3. To prove the "if" part let $A \in L$, $\Theta_0, \Theta_1 \in \mathrm{Con} A$ and let $\psi \in \mathrm{Con}_L A$ be finitely meet irreducible in $\mathrm{Con}_L A$ with $\Theta_0 \wedge \Theta_1 \leqslant \psi$. Evidently, $\psi \in \mathrm{Con}_K A$ and, as A/ψ is finitely subdirectly irreducible in the absolute sense, ψ is finitely meet irreducible in $\mathrm{Con}_K A$. Therefore, by congruence distributivity of K and Lemma 2.2, $\Theta_0 \leqslant \psi$ or $\Theta_1 \leqslant \psi$. So, by Lemma 2.2, L is congruence distributive.

Let K be a finitely generated and congruence distributive *variety* of algebras with K_{FSI} being a universal class. Such a variety can be found among Heyting algebras, Nelson algebras, interior algebras (or more generally, modal algebras), distributive double p-algebras, double Heyting algebras, Ockham lattices, Sugihara algebras and among many other classes of algebras. Let L be a quasivariety contained in K. By Corollary 2.4, we can state that L is congruence distributive iff L is generated by a subset of K_{FSI}. Indeed, by Corollary 2.4, the "if" part needs only a verification. Let $L = Q(M)$ where M is a subset of K_{FSI}. By Jónsson Lemma (see [7, Corollary 3.2], or Lemma 3.1 below), we can assume that M is a finite set of finite algebras. Hence $L = ISP(M)$. So, as K_{FSI} is universal, every member of L_{FSI} is finitely subdirectly irreducible in the absolute sense. Thus, by Corollary 2.4, L is congruence distributive. With the help of this observation we can produce many examples of congruence distributive quasivarieties which are not varieties. However, in a wide class of algebras congruence distributive quasivarieties are just varieties. For instance, we have

PROPOSITION 2.5. *Suppose K is a locally finite and semi-simple congruence distributive variety, and let L be a quasivariety contained in K. Then L is congruence distributive iff L is a variety.*

Proof. The "if" part is obvious. Assume that L is congruence distributive. Let $A \in L$ and $\Theta \in \mathrm{Con} A$. We need to show that A/Θ belongs to L. As L is locally finite, we can assume that A is finite. Moreover, as L is closed under subdirect products, we can also assume that Θ is finitely meet irreducible in $\mathrm{Con} A$. By Theorem 2.3, there exist finitely meet irreducible elements $\Theta_0, \ldots, \Theta_{n-1}$ of $\mathrm{Con} A$ with $\omega_A = \bigwedge (\Theta_i\colon i < n)$ and $A/\Theta_i \in L$ for all $i < n$. Hence $\bigwedge (\Theta_i\colon i < n) \leqslant \Theta$ which, by congruence distributivity of K, yields $\Theta_i \leqslant \Theta$ for some $i < n$. So, by semi-simplicity of K, we get $\Theta_i = \Theta$. Thus $A/\Theta \in L$, proving that L is a variety.

From Proposition 2.5 it follows that among finitely generated quasivarieties of modular lattices every congruence distributive quasivariety is a variety. This is

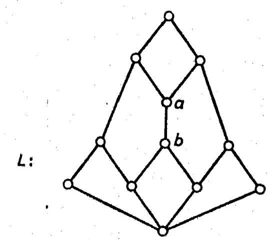

Fig. 2

not true for non-modular lattices. Applying Corollary 2.4 one can verify that the quasivariety $Q(L)$ generated by the lattice L of Figure 2 is congruence distributive. On the other hand, as $L/\Theta(a, b) \notin Q(L)$, $Q(L)$ does not coincide with $V(L)$.

3. Two lattice aspects. We begin with the following

LEMMA 3.1 (cf. Jónsson [7]). *Suppose K is a congruence distributive quasivariety of algebras and $M \subseteq K$. Then $HSP(M) \cap K_{FSI} \subseteq HSP_U(M)$.*

Proof. Let $A \in HSP(M) \cap K_{FSI}$. Then A is a homomorphic image, say, via f, of some subalgebra B of $\prod(C_i\colon i\in I)$ where $C_i\in M$ for all $i\in I$. For a subset S of I define a congruence relation Θ_S on B by $x\equiv y(\Theta_S)$ iff $\{i\in I\colon x(i)=y(i)\}\supseteq S$. Let $X=\{F\colon F$ is a filter over I and $\Theta_S\leqslant \operatorname{Ker} f$ for all $S\in F\}$. As $\{I\}\in X$, X is nonempty. Since the poset $(X,\subseteq)$ is inductive, it has maximal elements. Choose one of them and denote it by U. We may assume that $|A|\geqslant 2$ since otherwise we immediately have $A\in HSP_U(M)$. From this assumption it follows that $U\neq 2^I$. Similarly as in Jónsson [7, Corollary 3.2] one can show that U is an ultrafilter over I. Thus $B/\bigvee(\Theta_S\colon S\in U)\in ISP_U(M)$ which, by $\bigvee(\Theta_S\colon S\in U)\leqslant \operatorname{Ker} f$, yields $A\in HSP_U(M)$.

Let $V(M)$ denote the least variety containing M. From the above lemma we obtain

PROPOSITION 3.2. *Let M be a finite set of finite algebras, and let $Q(M)$ be congruence distributive. Then the interval $[Q(M), V(M)]$, in the subquasivariety lattice of $V(M)$, contains only finitely many congruence distributive quasivarieties.*

Proof. Let $K\in[Q(M), V(M)]$ be congruence distributive. Then, by Lemma 3.1, $K_{FSI}=V(M)\cap K_{FSI}\subseteq HS(M)$. The rest follows from the fact that every quasivariety is generated by its finitely subdirectly irreducible members.

The following example shows that the interval $[Q(M), V(M)]$ may contain exactly one congruence distributive quasivariety.

Let $A=(\{0,1,2\},\circ,+)$ be of type $(2,2)$ with the operations $\circ$ and $+$ satisfying $x\circ y\in\{0,1\}$ for all x, y, $x\circ y=1$ iff $x=y$, and $1+x=0+1=1$ for all x, and $x+y=0$ otherwise. Let B be a subalgebra of A generated by 0, and let C abbreviate $A/\Theta(0,1)$. Notice that A and B are the only up to isomorphism nontrivial members of $Q(A)_{FSI}$. Therefore $Q(M)_{FSI}\models\forall xyzw[((x\circ y)+(z\circ w)=x\circ x)\Leftrightarrow(x=y \text{ or } z=w)]$. Hence, by Theorem 2.3 of [4], $Q(A)$ is congruence distributive. We show that $Q(A)$ is unique in the interval $[Q(A), V(A)]$. Let $Q(A)\subseteq K\subseteq V(A)$ be congruence distributive. By Lemma 3.1, $K_{FSI}\subseteq HS(A)=I(\{A, B, C, \text{trivial algebra}\})$. When $C\notin K$, $K\subseteq Q(A)$ and hence $K=Q(A)$. Suppose $C\in K$. Notice that every operation of C takes always the same fixed value for all its arguments. The same is true for any direct power C^n of C. Therefore $\operatorname{Con} C^n$ coincides with the partition lattice on the universe of C^n. As every element in $\operatorname{Con} C^n$ is the meet of coatoms over it and for every coatom Θ in C^n we have $C^n/\Theta\cong C$, it follows that $\operatorname{Con} C^n$ coincides with $\operatorname{Con}_K C^n$. Thus K cannot satisfy any nontrivial congruence lattice identity, in particular, K is not congruence distributive, a contradiction. Thus $C\notin K$, showing that $K=Q(A)$.

It is worth noticing that $V(M)$ does not satisfy any nontrivial congruence lattice identity though the quasivariety $Q(A)$ is congruence distributive.

PROPOSITION 3.3. *Every finitely generated and congruence distributive quasivariety K of algebras contains only finitely many congruence distributive quasivarieties and they all form a distributive sublattice in the subquasivariety lattice of K.*

Proof. Let $K=Q(M)$ where M is a finite set of finite algebras. By Corollary 2.4 and Lemma 3.1, $L_{FSI}\subseteq HS(M)$ for every congruence distributive quasivariety L contained in K. Hence K contains only finitely many congruence distributive quasivarieties, and all of them are finitely generated. As every finitely generated quasivariety of algebras contains no infinite finitely subdirectly irreducible member, to complete the proof it suffices to show that for all finite $A\in K$ and congruence distributive quasivarieties $L, M\subseteq K$, $A\in(L+M)_{FSI}$ implies $A\in L_{FSI}\cup M_{FSI}$, and $A\in(L\cap M)_{FSI}$ implies $A\in L_{FSI}\cap M_{FSI}$. The first implication is obvious since $L+M=ISP(L\cup M)$. To prove the second, let us assume that $A\in(L\cap M)_{FSI}$ and $A\notin L_{FSI}$ or $A\notin M_{FSI}$. As every member of $L_{FSI}\cup M_{FSI}$ is finitely subdirectly irreducible in the absolute sense, $A\in L_{FSI}\cup M_{FSI}$ and $A\in L\cap M$ imply $A\in L_{FSI}\cap M_{FSI}$. So, we may assume that $A\notin L_{FSI}$ and $A\notin M_{FSI}$. Then, by Lemma 2.2, $M_A^L=M_A^M$ where M_A^L and M_A^M denote the sets of all minimal finitely meet irreducible elements of $\operatorname{Con}_L A$ and $\operatorname{Con}_M A$, respectively. Hence $A\notin(L\cap M)_{FSI}$, a contradiction.

4. Finite axiomatizability. The proof of our main result of this section will refer to the following theorem the proof of which can be easily found by a suitable modification of the proof of Theorem 4.1 from Blok and Pigozzi [3].

THEOREM 4.1 (cf. Blok and Pigozzi [3, Theorem 4.1]). *Let K be a congruence distributive quasivariety of algebras of finite type, and let $K=\operatorname{Mod}\operatorname{Id}(K)\cup\Gamma$ for some finite set Γ of quasiidentities. If K_{FSI} is finitely axiomatizable then so is K.*

In the above theorem $\operatorname{Id}(K)$ denotes the set of all identities valid in K.

LEMMA 4.2. *Let M be a locally finite quasivariety of algebras of finite type. Then for a quasivariety K contained in M the following conditions are equivalent:*

(i) *K is not finitely axiomatizable relative to M;*

(ii) *There exists an infinite sequence $A_0, A_1, A_2, \ldots$ of finite algebras of M satisfying:*

(1) $$|A_i|<|A_{i+1}| \quad \text{for all } i;$$

(2) $$A_i\notin K \quad \text{for all } i;$$

(3) *Every proper subalgebra of every A_i belongs to K.*

Proof. Denote by Γ_n the set of all at most n-variable quasiidentities valid in K, and by $\operatorname{Mod}_M\Gamma_n$ the class of all members of M validating Γ_n. Evidently, (a) $K=\bigcap(\operatorname{Mod}_M\Gamma_n\colon n<\omega)$ and (b) $\operatorname{Mod}_M\Gamma_{n+1}\subseteq\operatorname{Mod}_M\Gamma_n$ for all n. As the type of M is finite, by (i) we also have (c) $K\neq\operatorname{Mod}_M\Gamma_n$ for all n. Now, using (a), (b), (c) and local finiteness of M one can easily find an infinite sequence of finite algebras of M satisfying (1), (2) and (3) of (ii). Thus (i) implies (ii). To prove the remaining part assume (ii) and suppose that (i) fails. Then $K=\operatorname{Mod}_M\Gamma$ for some finite set Γ of quasiidentities. Let $n=\max\{|\operatorname{var}\varphi|\colon \varphi\in\Gamma\}$ where $\operatorname{var}\varphi$ denotes the set of all individual variables occurring in φ. By (2) and (3) of (ii), every A_i is n-generated. Hence, as M is locally finite, the sequence $|A_i|$, $i<\omega$, must be bounded which contradicts (1) of (ii).

For a quasivariety K of algebras and a finite $A\in K$, let M_A^K denote the set of all minimal finitely meet irreducible elements of $\operatorname{Con}_K A$. We have

LEMMA 4.3. *Let $\mathbf{K}$ and $\mathbf{L}$ be congruence distributive quasivarieties, and let $\mathbf{L} \subseteq \mathbf{K}$. Then for a finite algebra $A \in \mathbf{K}$, $A \in \mathbf{L}$ iff $A/\Theta \in \mathbf{L}$ for every $\Theta \in M_A^{\mathbf{K}}$.*

Proof. $\Rightarrow$: Let $A \in \mathbf{L}$. By Theorem 2.3, there exists a sequence $\Theta_0, \ldots, \Theta_{n-1}$ of finitely meet irreducible elements of $\mathrm{Con}\, A$ with $\bigwedge (\Theta_i\colon i<n) = \omega_A$ and $A/\Theta_i \in \mathbf{L}$ for all $i<n$. Let $\Theta \in M_A^{\mathbf{K}}$. Then $\bigwedge (\Theta_i\colon i<n) \leqslant \Theta$ and, by Lemma 2.2, $\Theta_k \leqslant \Theta$ for some $k<n$. But Θ_k is a finitely meet irreducible element of $\mathrm{Con}_{\mathbf{K}} A$. Hence, by minimality of Θ, $\Theta_k = \Theta$. Thus $A/\Theta \in \mathbf{L}$ for all $\Theta \in M_A^{\mathbf{K}}$. $\Leftarrow$: This part is obvious since $\bigwedge (\Theta\colon \Theta \in M_A^{\mathbf{K}}) = \omega_A$.

For an algebra A and congruence relations Θ, ψ on A, let $A(\Theta, \psi)$ denote the set $\{([a]\Theta, [a]\psi)\colon a \in A\}$. Evidently, $A(\Theta, \psi)$ forms a subalgebra of $A/\Theta \times A/\psi$ and, moreover, the map $a \mapsto ([a]\Theta, [a]\psi)$ establishes an isomorphism between A and $A(\Theta, \psi)$ whenever $\Theta \wedge \psi = \omega_A$.

LEMMA 4.4. *Let A be a finite algebra belonging to a congruence distributive quasivariety $\mathbf{K}$, Θ be an element of $M_A^{\mathbf{K}}$, $\{b_0, \ldots, b_{k-1}\}$ be a fixed selector of $\{[a]\Theta\colon a \in A\}$, (c, d) be a fixed element of $\psi \setminus \Theta$ where ψ abbreviates*

$$\bigwedge (\sigma \in M_A^{\mathbf{K}}\colon \sigma \neq \Theta),$$

and let B be a subalgebra of A/ψ generated by the set $\{[b_i]\psi\colon i<k\} \cup \{[c]\psi\}$. Then the following conditions are fulfilled:

(i) *The set $\{([a]\psi, [a]\Theta)\colon a \in A$ and $[a]\psi \in B\}$ forms a subalgebra C of $A(\psi, \Theta)$;*

(ii) *If B is a proper subalgebra of A/ψ then C is a proper subalgebra of $A(\psi, \Theta)$;*

(iii) *$C/\pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C \cong A/\Theta$;*

(iv) *$\pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C \in M_C^{\mathbf{K}}$*

where π_2 is the projection of $A/\psi \times A/\Theta$ onto A/Θ.

Proof. (i): This condition is obvious.

(ii): Assume that B is a proper subalgebra of A/ψ. Then $[a]\psi \in A/\psi \setminus B$ for some $a \in A$. Obviously, $([a]\psi, [a]\Theta) \notin C$ and $([a]\psi, [a]\Theta) \in A(\psi, \Theta)$. Thus C is a proper subalgebra of $A(\psi, \Theta)$.

(iii): In view (i) it suffices to prove that $\mathrm{Im}\,\pi_2 \upharpoonright C = A/\Theta$. Evidently, $\mathrm{Im}\,\pi_2 \upharpoonright C \subseteq A/\Theta$. Let $[a]\Theta \in A/\Theta$. Then $[a]\Theta = [b_i]\Theta$ for some $i<k$. Since $[b_i]\psi \in B$, $([b_i]\psi, [a]\Theta) = ([b_i]\psi, [b_i]\Theta) \in C$. Hence, as $\pi_2([b_i]\psi, [a]\Theta) = [a]\Theta$, we obtain $A/\Theta \subseteq \mathrm{Im}\,\pi_2 \upharpoonright C$. Thus $\mathrm{Im}\,\pi_2 \upharpoonright C = A/\Theta$.

(iv): By (i) just proved, C belongs to $\mathbf{K}$ because A/ψ and A/Θ are members of $\mathbf{K}$. Moreover, as Θ is finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} A$ and $C/\pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C \cong A/\Theta$ (see (iii)), it follows that $\pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C$ is a finitely meet irreducible element of $\mathrm{Con}_{\mathbf{K}} C$. We claim $\pi_1^{-1}(\omega_{A/\psi}) \upharpoonright C \nleqslant \pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C$ where π_1 is the projection of $A/\psi \times A/\Theta$ onto A/ψ. As $[c]\psi \in B$ and $[c]\psi = [d]\psi$, the pairs $([c]\psi, [c]\Theta)$ and $([d]\psi, [d]\Theta)$ belong to C. Evidently, $([c]\psi, [c]\Theta) \equiv ([d]\psi, [d]\Theta)(\pi_1^{-1}(\omega_{A/\Psi}) \upharpoonright C)$ and, by

$(c, d) \notin \Theta$, $([c]\psi, [c]\Theta) \not\equiv ([d]\psi, [d]\Theta)(\pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C)$. Thus

$$\pi_1^{-1}(\omega_{A/\Psi}) \upharpoonright C \nleqslant \pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C,$$

proving the claim.

Let Σ be a finitely meet irreducible element of $\mathrm{Con}_{\mathbf{K}} C$ with $\Sigma \leqslant \pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C$. Since $\pi_1^{-1}(\omega_{A/\Psi}) \upharpoonright C \wedge \pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C = \omega_C$, $\pi_1^{-1}(\omega_{A/\Psi}) \upharpoonright C \wedge \pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C \leqslant \Sigma$ which, by Lemma 2.2 and the above claim, yields $\pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C \leqslant \Sigma$. Thus $\Sigma = \pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C$, completing the proof that $\pi_2^{-1}(\omega_{A/\Theta}) \upharpoonright C$ belongs to $M_A^{\mathbf{K}}$.

We are ready now to prove the following

THEOREM 4.5. *For a finitely generated and congruence distributive quasivariety $\mathbf{K}$ of algebras of finite type the following conditions are equivalent:*

(i) *$\mathbf{K}$ is finitely axiomatizable;*

(ii) *$\mathrm{Mod}\,\mathrm{Id}(\mathbf{K}) \cup \Gamma$ is congruence distributive for some finite set Γ of quasiidentities valid in $\mathbf{K}$;*

(iii) *$\mathrm{Mod}\,\Sigma \cup \Gamma$ is congruence distributive for some set Σ of identities and some finite set Γ of quasiidentities such that $\mathbf{K} \subseteq \mathrm{Mod}\,\Sigma \cup \Gamma$.*

Proof. That (i) implies (ii) and (ii) implies (iii) is obvious. To prove that (iii) implies (i), let $\mathbf{K} = Q(\mathbf{M})$ where $\mathbf{M}$ is a finite set of finite algebras. Let $m = \max\{|A|\colon A \in \mathbf{M}\}$ and $\mathbf{M} = Q(\{A \in \mathrm{Mod}\,\Sigma \cup \Gamma\colon |A| \leqslant m\})$. First, applying Theorem 4.1 we show that $\mathbf{M}$ is finitely axiomatizable. Let $B \in \mathbf{M}_{FSI}$. Then $B \in IS(\{A \in \mathrm{Mod}\,\Sigma \cup \Gamma\colon |A| \leqslant m\})$ and hence $|B| \leqslant m$. Therefore $B \in (\mathrm{Mod}\,\Sigma \cup \Gamma)_{FSI}$. Thus $\mathbf{M}_{FSI} \subseteq (\mathrm{Mod}\,\Sigma \cup \Gamma)_{FSI}$ which, by Theorem 2.3 and Corollary 2.4, yields that $\mathbf{M}$ is congruence distributive. Let $B \in \mathrm{Mod}\,\mathrm{Id}(\mathbf{M}) \cup \Gamma_{FSI}$. Then $B \in HSP(\{A \in \mathrm{Mod}\,\Sigma \cup \Gamma\colon |A| \leqslant m\}) \cap (\mathrm{Mod}\,\Sigma \cup \Gamma)_{FSI}$ and hence, by Lemma 3.1, $|B| \leqslant m$. So, $\mathrm{Mod}\,\mathrm{Id}(\mathbf{M}) \cup \Gamma \subseteq \mathbf{M}$. Evidently, $\mathbf{M} \subseteq \mathrm{Mod}\,\mathrm{Id}(\mathbf{M}) \cup \Gamma$. Thus $\mathbf{M} = \mathrm{Mod}\,\mathrm{Id}(\mathbf{M}) \cup \Gamma$. As $\mathbf{M}$ is finitely generated, the class $\mathbf{M}_{FSI}$ is finitely axiomatizable. Hence, by Theorem 4.1, $\mathbf{M}$ is finitely axiomatizable. So, to complete the proof it suffices to show that $\mathbf{K}$ is finitely axiomatizable relative to $\mathbf{M}$. Suppose otherwise. Then, by Lemma 4.2, there exists an infinite sequence $A_0, A_1, A_2, \ldots$ of finite algebras of $\mathbf{M}$ with properties (1), (2) and (3) of Lemma 4.2. By (2) and Lemma 4.3, for each $i<\omega$ there exists $\Theta_i \in M_{A_i}^{\mathbf{M}}$ such that $A_i/\Theta_i \notin \mathbf{K}$. As each A_i/Θ_i belongs to $\mathbf{M}_{FSI}$ and $\mathbf{M}_{FSI}$ contains only finitely many non-isomorphic members, we conclude that there exist only finitely many non-isomorphic algebras among A_i/Θ_i's. This allows us to assume that all A_i/Θ_i's are isomorphic. Let ψ_i abbreviate $\bigwedge (\sigma \in M_{A_i}^{\mathbf{M}}\colon \sigma \neq \Theta_i)$. Since each A_i is isomorphic to a subdirect product of A_i/ψ_i and A_i/Θ_i, then by (1) we can assume that our sequence also satisfies (4) $|A_i/\psi_i| < |A_{i+1}/\psi_{i+1}|$ for all $i<\omega$. Let $k = |A_0/\Theta_0|$ and let s be a smallest natural number such that every $(k+1)$-generated algebra of $\mathbf{M}$ is of cardinality $\leqslant s$. By (4), $s+1 \leqslant |A_n/\psi_n|$ for some n. Let $\{b_0, \ldots, b_{k-1}\}$ be a fixed selector of $\{[a]\Theta_n\colon a \in A_n\}$; recall that all A_i/Θ_i's are isomorphic. Since $\psi_n \setminus \Theta_n \neq \emptyset$, we can choose an element, say, (c, d), of $\psi_n \setminus \Theta_n$. Let B_n be a subalgebra of A_n/ψ_n

generated by the set $\{[b_i]\psi_n\colon i<k\}\cup\{[c]\psi_n\}$, and let C_n denote the subalgebra of $A(\psi_n, \Theta_n)$ constructed as in Lemma 4.4. Since $|B_n|\leqslant s$, $|B_n|<|A_n/\psi_n|$ which means that B_n is a proper subalgebra of A_n/ψ_n. Hence, by Lemma 4.4(ii), C_n is a proper subalgebra of $A_n(\psi_n, \Theta_n)$ which in turn, by (3) and $A_n\cong A_n(\psi_n, \Theta_n)$, implies $C_n\in \mathbf{K}$. On the other hand, the fact $A_n/\Theta_n\notin \mathbf{K}$ and Lemmas 4.4(iii), 4.4(iv) and 4.3 yield $C_n\notin \mathbf{K}$, a contradiction.

Remark. In the second part of the above proof we have focused on showing that $\mathbf{K}$ is finitely axiomatizable relative to $\mathbf{M}$. Of course, if the interval $[\mathbf{K}, \mathbf{M}]$ in the subquasivariety lattice of $\mathbf{M}$ were finite then we could at once conclude the proof. However, in general it is not true as indicates the following example. Let $\mathbf{K} = Q(H(P))$ and $\mathbf{M} = V(H(P))$ where $H(P)$ is the Heyting algebra of all increasing subsets of the poset P of Figure 3. By Corollary 2.4, $\mathbf{K}$ is congruence distributive. Applying a method used in [6] one can show that the interval $[\mathbf{K}, \mathbf{M}]$ contains uncountable many quasivarieties.

P:

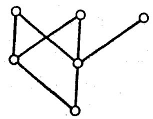

Fig. 3

Directly from Theorem 4.5 we obtain

Corollary 4.6. *Suppose $\mathbf{K}$ is a finitely generated congruence distributive quasivariety of algebras of finite type and $V(\mathbf{K})$ is also congruence distributive. Then $\mathbf{K}$ is finitely axiomatizable.*

In [9], V. I. Tumanov asked whether for a finite (modular) lattice A the following conditions are equivalent:

(i) $Q(A)$ is finitely axiomatizable

(ii) $Q(A)$ is a variety.

An answer to this question can be provided by the lattice L of Figure 2. Indeed, $Q(L)$ is congruence distributive and does not coincide with $V(L)$ (see Section 2) and, by Corollary 4.6, $Q(L)$ is finitely axiomatizable. In view of Proposition 2.5 the modular case of Tumanov's question cannot be answered by the methods of the paper.

As another corollary we have

Corollary 4.7 (see Blok and Pigozzi [2]). *Let $\mathbf{K}$ be a congruence distributive variety of algebras of finite type with $\mathbf{K}_{FSI}$ being a universal class. Then every finite set of finite members of $\mathbf{K}_{FSI}$ generates a finitely axiomatizable quasivariety.*

Proof. By Corollary 2.4, $Q(M)$ is congruence distributive. Hence, by Theorem 4.5, $Q(M)$ is finitely axiomatizable.

Concluding we want to mention that condition (ii) of Theorem 4.5 is too restrictive in applications. This is explained by the example succeeding Proposition 3.2. Condition (iii) of Theorem 4.5 seems to be more promising. Of course, the algebra A from the above mentioned example generates a finitely axiomatizable quasivariety. This follows from Corollary 3.5 of [4] and can be also proved by realizing condition (iii) of Theorem 4.5.

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Added in proof. Professor Don Pigozzi in his recent paper *Finite basis theorems for relatively congruence-distributive quasivarieties* (Trans. Amer. Math. Soc. 310 (2) (1988), 499–533) has proved by extending some ideas of [3] and [4] that every finitely generated and congruence distributive quasivariety of algebras of finite type is finitely axiomatizable. This result is evidently stronger than that established in our Corollary 4.6. It also implies that each of the conditions of Theorem 4.5 is true.

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INSTYTUT MATEMATYKI UMK
ul. Chopina 12/18
87-100 Toruń, Poland

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