Absolutely saturated models

by

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Abstract. Extensions of the theories of saturated models and models of $\mathscr{T}$ that are homogeneous universal with respect to models of $\mathscr{T}$ are obtained in NBG for structures whose universes are proper classes. The class concepts are referred to as absolutely saturated models and models of $\mathscr{T}$ that are absolutely homogeneous universal with respect to models of $\mathscr{T}$, respectively. We illustrate the theory by showing that J. H. Conway's ordered field No is (up to isomorphism) the unique absolutely saturated real-closed ordered field and the unique absolutely homogeneous universal ordered field.

0. Introduction. A model $\mathscr{A}$ with universe A of a theory $\mathscr{T}$ in a language $\mathscr{L}$ is said to be *$\varkappa$-saturated* iff for every subset X of A of cardinality $<\varkappa$, the expansion $(\mathscr{A}, a)_{a\in X}$ realizes every type $\Sigma(v)$ of the expanded language $\mathscr{L}\cup\{c_a\colon a\in X\}$ which is consistent with the theory of $(\mathscr{A}, a)_{a\in X}$ (cf. [2] for details). If $\varkappa = |A|$, $\mathscr{A}$ is said to be *saturated.* Closely related to the $\varkappa$-saturated models are the models of $\mathscr{T}$ that are *$\varkappa$-universal with respect to $\mathscr{T}$* and *$\varkappa$-homogeneous with respect to $\mathscr{T}$*, that is, the models of $\mathscr{T}$ in $\mathscr{L}$ such that every model of $\mathscr{T}$ in $\mathscr{L}$ of power $\leqslant\varkappa$ can be embedded in them, and every isomorphism between substructures of them that are models of $\mathscr{T}$ in $\mathscr{L}$ of power $<\varkappa$ can be extended to an automorphism. If $\mathscr{A}$ is such a model and $|A| = \varkappa$, then $\mathscr{A}$ is said to be *homogeneous universal with respect to T.*

In ZFC, as is well-known, it is impossible (except in rare cases (cf. [2], pp. 98–99 and Ch. 7)) to prove the existence of models that are either saturated or homogeneous universal with respect to $\mathscr{T}$. In general, the existence of such models is limited to cases where we assume instances of the GCH or the existence of inaccessible cardinals. It has been known for some time, however, that in von Neumann–Bernays–Gödel set theory with Global Choice, henceforth NBG (cf. [13], Ch. 4), one can show that On (the "cardinal" of all proper classes) is inaccessible. It is therefore natural to inquire if within this conservative extension of ZFC [6] one may obtain extensions (or partial extensions) of the following classical results for $\varkappa = \mathrm{On}$, extensions which could, for example, certainly be obtained in the stronger Morse–Kelley set theory with Global Choice (cf. [13], Ch. 4) where, unlike in NBG, ruth is generally definable in class-structures and induction can be applied freely in such structures to assertions involving global truth. (See [15], [16], [20], [18], [1] and, [11], p. 195.)

I. Morley-Vaught ([14]; [2], 5.1.5 and 5.1.13). *If* $|\mathscr{L}| \cup \omega \leqslant \gamma$, *then every complete theory* $\mathscr{T}$ *in* $\mathscr{L}$ *having an infinite model has* (*up to isomorphism*) *a unique saturated model in each inaccessible power* $\varkappa > \gamma$.

II. Jónsson-Morley-Vaught ([7]; [8]; [14] and [9], p. 153). *If* $|\mathscr{L}| \cup \omega \leqslant \gamma$, *then every Jónsson theory* $\mathscr{T}$ *in* $\mathscr{L}$ *has* (*up to isomorphism*) *a unique model that is homogeneous universal with respect to* $\mathscr{T}$ *in each inaccessible power* $\varkappa > \gamma$, whereby a Jónsson theory we mean a first-order $\forall\exists$-theory having an infinite model as well as the following properties:

(i) *Joint embedding*. For any set-models $\mathscr{B}_0, \mathscr{B}_1$ of $\mathscr{T}$ in $\mathscr{L}$ there is a set-model $\mathscr{A}$ of $\mathscr{T}$ in $\mathscr{L}$ and embeddings f_0, f_1 such that $f_0\colon \mathscr{B}_0 \to \mathscr{A}$ and $f_1\colon \mathscr{B}_1 \to \mathscr{A}$.

(ii) *Amalgamation*. For any set-models $\mathscr{A}, \mathscr{B}_0, \mathscr{B}_1$ of $\mathscr{T}$ in $\mathscr{L}$ and embeddings f_0, f_1 where $f_0\colon \mathscr{A} \to \mathscr{B}_0$ and $f_1\colon \mathscr{A} \to \mathscr{B}_1$, there is a set-model $\mathscr{C}$ of $\mathscr{T}$ in $\mathscr{L}$ and embeddings g_0, g_1 such that $g_0\colon \mathscr{B}_0 \to \mathscr{C}$, $g_1\colon \mathscr{B}_1 \to \mathscr{C}$ and $g_0 f_0 = g_1 f_1$.

In the pages that follow, while the status of a complete class analogue of I will be left open, we will provide an affirmative answer for II, as well as an extension of I for the special case where $\mathscr{T}$ is *model complete*, (henceforth $\mathscr{T}$ is an MC-*theory*), that is, where every monomorphism between models of $\mathscr{T}$ is elementary, or equivalently, where every formula of $\mathscr{L}$ is $\mathscr{T}$-equivalent to an existential formula of $\mathscr{L}$. Since every complete $\mathscr{T}$ can be conservatively expanded to a complete $\mathscr{T}^*$ of this kind (as in [14], Theorem 3.2) we will obtain something like a class analogue of I. Since the notions of $\varkappa$-saturation and $\varkappa$-homogeneous universality grow stronger as $\varkappa$ increases, it seems appropriate to refer to the models of $\mathscr{T}$ that will concern us as *absolutely saturated* and *absolutely homogeneous universal* with respect to $\mathscr{T}$, respectively. To illustrate the class concepts, J. H. Conway's ordered field No will be identified (up to isomorphism) as the unique absolutely saturated real-closed ordered field and the unique absolutely homogeneous universal ordered field. Since the proofs of our results are extensions of familiar ideas we will only provide the plans leaving the details to the reader.

1. Main results. Since the usual definition of a sequence does not work in NBG when proper classes are involved, we follow the standard practice of understanding by a "structure" $\mathscr{A}$ where $|A| = \mathrm{On}$ and the R_α, $1 \leqslant \alpha < \mu < \mathrm{On}$, are finitary relations on A of power $\leqslant \mathrm{On}$ (which may be operations or distinguished elements treated as special relations) the class $(A \times \{0\}) \cup R$ where $R = \bigcup \{R_\alpha \times \{\alpha\}\colon 1 \leqslant \alpha < \mu\}$. Let $\mathscr{A}$ be such a structure appropriate for $\mathscr{L}$, where $|\mathscr{L}| < \mathrm{On}$, and suppose $\mathscr{A}_\beta$, $\beta < \mathrm{On}$, is a chain of substructures of $\mathscr{A}$ (obtained using Global Choice) where $|A_\beta| < \mathrm{On}$ and $\bigcup_{\beta < \mathrm{On}} \mathscr{A}_\beta = \mathscr{A}$. As the reader can readily verify, truth in $\mathscr{A}$ for $\forall\exists$-sentences of $\mathscr{L}$ *can be defined* as follows. If $\varphi = \chi$ or $\varphi = \exists x_1 \ldots \exists x_n \psi$, where χ and ψ are open, φ is true in $\mathscr{A}$ iff φ is true in some $\mathscr{A}_\beta$; and if $\varphi = \forall x_1 \ldots \forall x_m \exists y_1 \ldots \exists y_n \psi$, where ψ is open, then φ is true in $\mathscr{A}$ iff for each $a_1, \ldots, a_m \in A$,

there is a $\mathscr{A}_\beta$ containing $a_1, \ldots, a_m$ and $\exists y_1 \ldots \exists y_n \psi(a_1 \ldots a_m)$ is true in $\mathscr{A}_\beta$, where $\psi(a_1 \ldots a_m)$ indicates that each occurrence of x_i in ψ is taken to be a_i for $1 \leqslant i \leqslant m$.

Since the notion of an isomorphism between class-structures is meaningful in NBG, it is therefore now clear that if $|\mathscr{L}| < \mathrm{On}$, the concept of a model that is absolutely homogeneous universal with respect to an $\forall\exists$-theory is likewise meaningful in NBG, where the new notion is obtained from the old one by letting $\varkappa = \mathrm{On}$. A similar remark also applies to the notion of an absolutely saturated model of a MC-theory, where $|\mathscr{L}| < \mathrm{On}$, since if $\mathscr{T}$ is an MC-theory and $\mathscr{A}$ is a model of $\mathscr{T}$, the theory of $(\mathscr{A}, a)_{a \in X}$, where X is a subset of A, is also an MC-theory.

THEOREM 1. *If* $|\mathscr{L}| < \mathrm{On}$, *then every Jónsson theory* $\mathscr{T}$ *in* $\mathscr{L}$ *has* (*up to isomorphism*) *a unique model that is absolutely homogeneous universal with respect to* $\mathscr{T}$.

THEOREM 2. *If* $|\mathscr{L}| < \mathrm{On}$, *then every complete* MC-*theory* $\mathscr{T}$ *in* $\mathscr{L}$ *having an infinite model has* (*up to isomorphism*) *a unique absolutely saturated model.*

To prove Theorem 1 we require the following class analogues, for the special case of first-order theories, of Jónsson's conditions V and $\mathrm{VI}_\varkappa$ of [7] and his Lemma 2.5 of [7]. Let β be an ordinal.

(i) If $\mathscr{A}$ is the union of a chain $\mathscr{A}_\beta$, $\beta < \mathrm{On}$, of models of an $\forall\exists$-theory $\mathscr{T}$, where $|A| = \mathrm{On}$ and $|A_\beta| < \mathrm{On}$, then $\mathscr{A}$ is a model of $\mathscr{T}$.

(ii) If $\mathscr{A}$ is a class-model of an $\forall\exists$-theory $\mathscr{T}$ where $|\mathscr{L}| \cup \omega \leqslant \varkappa < \mathrm{On}$, and C is a subset of A of power $\sigma < \mathrm{On}$, then if γ is a cardinal such that $\varkappa, \sigma \leqslant \gamma < \mathrm{On}$, there is a model $\mathscr{B}$ of $\mathscr{T}$ in $\mathscr{L}$ of power γ such that $C \subseteq B \subset A$.

(iii) If $\mathscr{A}$ is a class-model of an $\forall\exists$-theory, where $|\mathscr{L}| < \mathrm{On}$, then $\mathscr{A}$ is the union of a chain $\mathscr{A}_\beta$, $\beta < \mathrm{On}$, of set-models of $\mathscr{T}$ in $\mathscr{L}$.

The proof of (i) is trivial, (iii) follows from (ii), and (ii) is established by a weakened version of Tarski's proof of the downward Löwenheim–Skolem Theorem (cf. [2], 3.1.6) in which while limiting oneself to the open formulas of $\mathscr{L}$ one otherwise mimicks the proof stopping short of showing that the inductively defined class B is the universe of an elementary submodel of $\mathscr{A}$, something which will not in general be the case unless $\mathscr{T}$ is an MC-theory. Since $\mathscr{T}$ is an $\forall\exists$-theory, however, it is easy to show that $\mathscr{A}|B$ is a model of $\mathscr{T}$. Moreover, the inductive proof that B has power γ is permissible in NBG despite the presence in its definition of assertions like '$\mathscr{A} \models \exists x \varphi[a_1 \ldots a_n]$' and '$\mathscr{A} \models \varphi[b a_1 \ldots a_n]$', where φ is open, since these assertions and the defining condition more generally can be expressed in the primitive language of NBG without class-quantifiers and the induction schema is provable in NBG for such conditions (cf. [11], p. 198).

Proof of Theorem 1. Let β^+ be the least cardinal $>$ the cardinal β. Beginning with a model of $\mathscr{T}$ of power $|\mathscr{L}| \cup \omega$ and employing the classic existence result of Jónsson ([8], Theorem A) one constructs a continuous chain $\mathscr{A}_\beta$, $|\mathscr{L}| \cup \omega < \beta < \mathrm{On}$, such that $\mathscr{A}_{\beta^+}$ is β^+-homogeneous and β^+-universal with respect to $\mathscr{T}$ of power 2^β, and sets $\mathscr{A}$ equal to the union of the chain. One now completes the proof along classical lines (*op. cit.*) except that instead of referring to elements of the Jónnson class one appeals to models of $\mathcal{T}$, and in place of conditions V and $\mathrm{VI}_\varkappa$ and Lemma 2.5 one appeals to (i)–(iii), respectively.

The following supplementary result, whose proof also uses (i)–(iii), will prove useful in Section 2. Like Theorem 1, it is a straightforward extension, for the special case of first-order theories, of a result due to Jónsson. (See [7], Definitions 2.2 and 2.3, [14], Corollary 2.4e, and the proofs of Theorems A and B in [8]).

COROLLARY 1. *If $\mathcal{T}$ is a Jónsson theory, where $|\mathcal{L}| <$ On, then $\mathcal{A}$ is absolutely homogeneous universal with respect to $\mathcal{T}$ iff $\mathcal{A}$ is a universally extending model of $\mathcal{T}$, i.e., for every substructure $\mathcal{B}$ of $\mathcal{A}$ and extension $\mathcal{C}$ of $\mathcal{B}$ where $\mathcal{B}$ and $\mathcal{C}$ are models of $\mathcal{T}$ in $\mathcal{L}$ of power $<$ On, there is a substructure $\mathcal{C}'$ of $\mathcal{A}$ that is a model of $\mathcal{T}$ in $\mathcal{L}$ and an isomorphism from $\mathcal{C}$ onto $\mathcal{C}'$ that is an extension of the identity map on $\mathcal{B}$.*

Proof of Theorem 2. Since the Elementary Chain Theorem continues to hold for "long" chains of set-models in NBG (cf. [4], pp. 253–254), existence can be proved classically (*op. cit.*). Uniqueness cannot be established classically, however, since to do so would require working with class-models having a proper class of distinguished elements (cf. [2], 5.1.11) and clearly such entities cannot be defined using the aforementioned definition of truth. The following more cumbersome approach is available, however. One first proves

LEMMA 1. *If $|\mathcal{L}| <$ On and $\mathcal{A}$ is an absolutely saturated model of an* MC*-theory, then $\mathcal{A}$ is the union of a continuous elementary chain $\mathcal{A}_\beta$, $|\mathcal{L}| \cup \omega < \beta <$ On, where $\mathcal{A}_{\beta^+}$ is β^+-saturated of power 2^β.*

To establish this one begins by showing

(iv) Suppose $|\mathcal{L}| \cup \omega \leqslant \alpha <$ On, $\omega \leqslant |A| \leqslant 2^\alpha$ and $\mathcal{A}$ is an elementary submodel of an absolutely saturated model $\mathcal{A}'$ of an MC-theory. Then there exists an elementary substructure $\mathcal{B}$ of $\mathcal{A}'$ that is an elementary extension of $\mathcal{A}$ of power 2^α such that for every $X \subseteq A$ of power α, $(\mathcal{B}, a)_{a\in X}$ realizes each type $\Sigma(v)$ of $(\mathcal{A}, a)_{a\in X}$.

To obtain an appropriate $\mathcal{B}$ one proceeds as in the proof of 5.1.3 of [2] except that the elements required for the extension come from A'–A. That $\mathcal{B}$ is an elementary substructure of $\mathcal{A}'$ follows from the fact that $\mathcal{T}$ is an MC-theory. One next proves

(v) If $|\mathcal{L}| \cup \omega \leqslant \alpha <$ On, $\omega \leqslant |B| \leqslant 2^\alpha$ and $\mathcal{B}$ is an elementary substructure of an absolutely saturated model $\mathcal{A}$ of an MC-theory $\mathcal{T}$, there is an α^+-saturated elementary extension $\mathcal{B}'$ of $\mathcal{B}$ of power 2^α that is an elementary substructure of $\mathcal{A}$.

The proof of (v) proceeds like the proof of 5.1.4 of [2] except that instead of appealing to 5.1.3 of [2] one appeals to (iv). That $\mathcal{B}'$ is an elementary substructure follows as above.

It should be emphasized that the above proof proceeds by transfinite induction along an inductively defined class where assertions of global truth play a prominent role. We leave it to the reader to verify (using well-known facts about the definability of the relevant set-theoretic and model-theoretic notions in terms of formulas of Levy's Hierarchy (cf. [12], [4], pp. 76–98, and [19])) that the induction is warranted by the following principle due to Takahashi [18]: If $\varphi(x, X)$ is a formula of NBG (in which class variables and special classes may occur), then the least class (or set) X satisfying ($x \in X$ iff $\varphi(x, X)$) exists, if the variable X occurs only in positive parts of φ in the form $y \in X$ or $Y \in X$ and NBG $\vdash \varphi(x, X) \equiv \exists x\psi$, where ψ contains no class-quantifiers and each set-quantifier has the form $\exists x[x \in y \wedge \ldots]$, $\forall x[x \in y \to \ldots]$, $\exists x[x \subseteq y \wedge \ldots]$ or $\forall x[x \subseteq y \to \ldots]$. Moreover, to prove $X \subseteq P$ it suffices to prove $x \in P$ under the assumption $\varphi(x, P)$ or the assumption $x \in X \wedge \varphi(x, X \cap P)$.

Now let $\mathcal{A}$ be an absolutely saturated model of a complete MC-theory $\mathcal{T}$. Since MC-theories are $\forall\exists$-theories we can appeal to (ii) to obtain an elementary submodel of $\mathcal{A}$ of power $|\mathcal{L}| \cup \omega$. By invoking (v), in place of Lemma 5.1.4 of [2], appealing to (ii) after each such application to obtain an elementary extension of equal power containing the first remaining element in a well-ordering of A, and otherwise mimicking the classical existence proof of saturated models of inaccessible powers (*op. cit.*), one proves Lemma 1. Moreover, by now invoking this lemma one can decompose two absolutely saturated models $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{T}$ into continuous elementary chains $\mathcal{A}_\beta$, $\mathcal{A}'_\beta$, $|\mathcal{L}| \cup \omega < \beta <$ On, where A_{β^+}, $\mathcal{A}'_{\beta^+}$ are β^+-saturated models of $\mathcal{T}$ of power 2^β. Since all models of $\mathcal{T}$ are elementary equivalent, one can prove uniqueness (and thereby complete the proof of Theorem 2) by means of a routine (local) back and forth argument using the fact that β^+-saturated models are β^+*-universal* and β^+*-homogeneous* in the (elementary) sense of Keisler ([2], 5.1.14).

Using similar techniques one can also extend the classic equivalence referred to above to obtain:

COROLLARY 2. *If $\mathcal{T}$ is a complete* MC*-theory having an infinite model, where $|\mathcal{L}| <$ On, then $\mathcal{A}$ is an absolutely saturated model of $\mathcal{T}$ iff $\mathcal{A}$ is an* On*-universal and* On*-homogeneous model of $\mathcal{T}$, i.e., every elementary equivalent model can be elementary embedded in $\mathcal{A}$, and either* (i) *given any two elementary equivalent submodels of $\mathcal{A}$ of power $<$* On *and an isomorphism between them, the isomorphism can be extended to an automorphism of $\mathcal{A}$, or* (ii) *(which is equivalent to* (i) *given the first condition) for all $\xi <$* On*, $a \in {}^\xi A$, $b \in {}^\xi A$ and $c \in A$, if $(\mathcal{A}, a_\eta)_{\eta<\xi} \equiv (\mathcal{A}, b_\eta)_{\eta<\xi}$, then there is a $d \in A$ such that $(\mathcal{A}, a_\eta, c)_{\eta<\xi} \equiv (\mathcal{A}, b_\eta, d)_{\eta<\xi}$.*

In virtue of our introductory remarks we also have:

COROLLARY 3. *If $|\mathcal{L}| <$ On, then every complete theory $\mathcal{T}$ in $\mathcal{L}$ with an infinite model has a conservative expansion to a theory $\mathcal{T}^*$ that has (up to isomorphism) a unique absolutely saturated model.*

2. Conway's ordered field No. Let $\mathcal{T}$ be a complete MC-theory with an infinite model, where $|\mathcal{L}| <$ On. Since complete MC-theories with infinite models are also Jónsson theories (cf. [17], 4.2.2 and [10], 3.5), it is easy to see that the four types of models of $\mathcal{T}$ whose existence and uniqueness is noted in Theorems 1–2 and Corollaries 1–2 coincide. Now suppose that $\mathcal{T}'$ is also a theory in $\mathcal{L}$, $\mathcal{T}' \subset \mathcal{T}$ and every set-model $\mathcal{A}'$, of $\mathcal{T}'$ has an extension to a set-model $\mathcal{A}$ of $\mathcal{T}$, such that each monomorphism from $\mathcal{A}'$ into a model $\mathcal{B}$ of $\mathcal{T}$ has a unique extension to a monomorphism from $\mathscr{A}$ into $\mathscr{B}$. Clearly, since $\mathscr{T}$ is a Jónsson theory in $\mathscr{L}$ so is $\mathscr{T}'$. Moreover, it is not difficult to see that the absolutely homogeneous universal models of $\mathscr{T}'$ (and hence the universally extending models of $\mathscr{T}'$) coincide with those of $\mathscr{T}$. The following pairs of theories $(\mathscr{T}, \mathscr{T}')$ are well-known to satisfy the above hypotheses.

$\mathscr{T}$ = real-closed ordered fields
= non-trivial divisible ordered abelian groups
= open densely ordered classes

$\mathscr{T}'$ = ordered fields
= non-trivial ordered abelian groups
= ordered classes

In [5] the author extended a result of Conway ([3], Theorem 28) by showing that No considered as an ordered field (resp. an ordered abelian group; resp. an ordered class) is a universally extending ordered field (resp. ordered abelian group; resp. ordered class). It therefore follows from the above that we now have six characterizations of No considered as an ordered field, as an ordered group and as an ordered class.

Addendum. Following the completion of this paper Professor H. J. Keisler kindly informed the author that since for each $n > 0$ satisfaction can be defined for formulas which have at most n alternations of blocks of universal and existential quantifiers, Theorem 2 can be extended to include complete theories where for some $n > 0$ every formula is $\mathscr{T}$-equivalent to such a formula. More importantly, he noted that by restricting the notion of an absolutely saturated model one may obtain an even stronger result. Specifically, let $|\mathscr{L}| < \mathrm{On}$, A be a class and $\mathscr{L}(A)$ be the language obtained from $\mathscr{L}$ by adding a constant symbol c_a for each $a \in A$. The syntax of $\mathscr{L}(A)$ can be formalized in NBG in the usual way, with variables v_n, $n \in \omega$. There are formulas ATOMIC$(\mathscr{L}, A, x)$ and FORM$(\mathscr{L}, A, x)$ of NBG saying that x is an atomic formula, or formula, of $\mathscr{L}(A)$ with at most one free variable v_0. Moreover, there is a formula HOLD$(\mathscr{L}, A, \mathscr{M}, x, y)$ saying that $\mathscr{M}$ is a class-structure appropriate for $\mathscr{L}$ with universe A, $y \in A$, ATOMIC$(\mathscr{L}, A, x)$, and x is satisfied by y in $\mathscr{M}$. Then a class-structure $\mathscr{M} = (A, R)$ for $\mathscr{L}$ is said to be On-*$saturated$ if and only if it satisfies the formula $\exists S[\psi$ and $\theta]$ of NBG where the formulas ψ and θ are as follows.

ψ: $\forall x \forall y \forall t \forall u \forall n \in \omega$

[if $S(x, y)$ then [FORM$(\mathscr{L}, A, x)$ and $y \in A$]] and
[if ATOMIC$(\mathscr{L}, A, x)$ than [$S(x, y)$ iff HOLD$(\mathscr{L}, A, \mathscr{M}, x, y)$]] and
[if x is $t \wedge u$ then [$S(x, y)$ iff [$S(t, y)$ and $S(u, y)$]]] and
[if x is $-t$ then [$S(x, y)$ iff not $S(t, y)$]] and
[if x is $\forall v_n t$ then [$S(x, y)$ iff $\forall z \in A$ $S(t(v_n|c_z), y)$]].

θ. For every set $B \subseteq A$, and every set F of formulas of $\mathscr{L}(B)$ with only v_0 free, if $\forall$ finite $G \subseteq F$ $\exists y \in A$ $\forall x \in G$ $S(x, y)$, then $\exists y \in A$ $\forall x \in F$ $S(x, y)$.

Note. ψ may be paraphrased by saying that S is a class of pairs consisting of formulas with v_0 free and elements of $\mathscr{M}$ which satisfies Tarski's inductive definition of satisfaction. This idea goes back at least to Montague and Vaught's "Natural Models of Set Theory", Fund. Math. 47 (1959), pp. 219–242. In general, it cannot be proved that $\exists S$ ψ holds for every class-structure $\mathscr{M}$. However, one may now prove in NBG (without the detour of Lemma 1)

THEOREM 3. *If* $|\mathscr{L}| < \mathrm{On}$, *then every complete* $\mathscr{T}$ *in* $\mathscr{L}$ *has (up to isomorphism) a unique* On*-*saturated model.*

Sketch of proof. Form an elementary chain $\mathscr{M}_\alpha$ of ω_α-saturated set models of $\mathscr{T}$ where α ranges over On, and then prove that the union of the chain is On*-saturated by taking S to be the relation $\exists \alpha$ (FORM$(\mathscr{L}, A_\alpha, x)$ and $y \in A_\alpha$ and y satisfies x in $\mathscr{M}_\alpha$). Uniqueness is established by a back and forth argument.

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Finitely generated congruence distributive quasivarieties of algebras *

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Abstract. We consider congruence distributive quasivarieties of algebras with a special emphasis on those which are finitely generated. There are two main results in the paper. The first states that within a congruence distributive quasivariety of algebras every finitely subdirectly irreducible algebra is finitely subdirectly irreducible in the absolute sense. The second provides necessary and sufficient conditions for a finitely generated and congruence distributive quasivariety of algebras of finite type to be finitely axiomatizable. In particular, applying these results we partially answer a question posed in Tumanov [9].

1. Introduction. A universal sentence whose matrix is of the form $r_0 = s_0 \& \ldots \ldots \& r_k = s_k \to r = s$ is called a *quasiidentity*. A class K of similar algebras is said to be a *quasivariety* if $K = \mathrm{Mod}\Sigma$ for some set Σ of quasiidentities. Equivalently (see [8]), K is a *quasivariety* if K is closed under isomorphisms (I), subalgebras (S), direct products (P) and ultraproducts (P_U). If M is a class of similar algebras then, by a result of [5], $ISP\dot{P}_U(M)$ is the least quasivariety containing M. Sometimes we shall write $Q(M)$ instead of $ISPP_U(M)$. A quasivariety K is said to be *finitely generated* if $K = Q(M)$ for some finite set M of finite algebras, and K is said to be *finitely axiomatizable* if $K = \mathrm{Mod}\Sigma$ for some finite set Σ of quasiidentities. For an algebra A, by $\mathrm{Con}\,A$ we denote the lattice of congruence relations on A. If K is a quasivariety and $A \in K$ then we put $\mathrm{Con}_K A = \{\Theta \in \mathrm{Con}\,A\colon A/\Theta \in K\}$. Since the set $\mathrm{Con}_K A$ is closed under arbitrary meets (in $\mathrm{Con}\,A$), it forms a complete lattice. We say that a quasivariety K is *congruence distributive* if for every $A \in K$ the lattice $\mathrm{Con}_K A$ is distributive. For varieties of algebras (= equational classes) this notion was intensively studied in the literature and many interesting results for it have been obtained. One of them, due to Baker [1], states that every finitely generated and congruence distributive *variety* of algebras of finite type is finitely axiomatizable. Our intention is to extend this result into quasivarieties. We prove (Theorem 4.5) that a finitely generated and congruence distributive quasivariety K of algebras of finite type is finitely axiomatizable iff $\mathrm{Mod}\Sigma \cup \Gamma$ is congruence distributive for some set Σ of identities and a finite set Γ of quasiidentities such that $K \subseteq \mathrm{Mod}\Sigma \cup \Gamma$.

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