On small sets in the sense of measure and category

by

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Abstract. We show assuming Martin's Axiom that there exists a set of reals of cardinality continuum whose every Borel image into the reals is Lebesgue negligible and meagre. This is an answer to a problem of E. Grzegorek. We also construct a γ-set which can be mapped onto the unit interval by a Borel function.

Assuming Martin's Axiom, F. Galvin and A. W. Miller [6] constructed a set of reals of cardinality continuum such that every continuous image of this set into the reals is Lebesgue negligible and meagre. This was an answer to a question of Sierpiński. E. Grzegorek has posed the following question: assume Continuum Hypothesis, does there exist a set of reals of cardinality continuum such that every Borel image of it into the reals is Lebesgue negligible and meagre? (see [3]). D. H. Fremlin and J. Jasiński [3] showed that a set with this property exists if we assume Martin's Axiom and if there exists $k < \mathfrak{c}$ such that $P(k)$ contains a proper uniform ω_1-saturated k-additive ideal. But this assumption implies the negation of Continuum Hypothesis. The following theorem solves Grzegorek's question under Martin's Axiom.

THEOREM 0. *Assume Martin's Axiom. There exists a set* $X \subseteq \mathbf{R}$ *such that* $|X| = \mathfrak{c}$ *and* $f(X)$ *is Lebesgue negligible and meagre for every Borel measurable function* $f\colon X \to \mathbf{R}$.

DEFINITION 1. A topological space X is a *Δ-set* iff for every double sequence $(J_n^k\colon n, k \in \omega)$ of finite Borel covers of X there exists a double sequence $(\bar{J}_n^k\colon n, k \in \omega)$ such that $\bar{J}_n^k \subseteq J_n^k$ and $|\bar{J}_n^k| \leqslant 2^n$ and $X \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}_n^k$.

LEMMA 1. *Assume Martin's Axiom. Every subset of the reals of cardinality less than continuum is a Δ-set.*

Proof. Let $X \subseteq \mathbf{R}$ be such that $|X| < \mathfrak{c}$ and let $(J_n^k\colon k, n \in \omega)$ be a double sequence of finite Borel covers of X. We define a partially ordered set $(P, \leqslant)$ as follows:

$$P = \{(f, H_0, H_1, \ldots, H_m)\colon$$
$$f\colon m+1 \times m+1 \to \bigcup_{k,n} P(J_n^k) \wedge f(i,j) \subseteq J_i^j \wedge f(i,j) \leqslant 2^i \wedge H_j \subseteq X \wedge |H_j| \leqslant 2^m,$$
$$H_j \subseteq \bigcap_{i=0}^{m} f(i,j) \wedge m \in \omega\}$$

and $(f, H_0, \ldots, H_m) \leqslant (f', H'_0, \ldots, H'_m)$ iff $m \geqslant m'$ and $f \supseteq f'$ and $H_j \supseteq H'_j$ for every $0 \leqslant j \leqslant m'$. It is not hard to see that $(P, \leqslant)$ has C.C.C. and that the sets $G_x = \{(f, H_0, \ldots, H_m)\colon \exists 0 \leqslant j \leqslant m \; x \in H_j\}$ for $x \in X$ and

$$G_{ij} = \{(f, H_0, \ldots, H_m)\colon (i, j) \in \operatorname{dom} f\}$$

are dense in $(P, \leqslant)$. So there exists a $\mathscr{G}$-generic filter $\mathscr{F}$, where

$$\mathscr{G} = \{G_x\colon x \in X\} \cup \{G_{ij}\colon i, j \in \omega\}.$$

Let $F = \bigcup \{f\colon \exists (f, H_0, \ldots, H_m) \in \mathscr{F}\}$. F is a function defined on $\omega \times \omega$. Let $\bar{J}^k_n = F(n, k)$. We have $X \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}^k_n$.

We will use the following notation:

$$[M]^\omega = \{N \subseteq M\colon N \text{ is infinite}\},$$
$$[M]^{<\omega} = \{S \subseteq M\colon S \text{ is finite}\},$$
$$[M]^{*\omega} = \{N \in [\omega]^\omega\colon N - M \text{ is finite}\},$$
$$(S, M)^\omega = \{N \in [\omega]^\omega\colon S \subseteq N \subseteq S \cup M \text{ and } S < N - S\},$$

where S is a finite and M an infinite subset of ω.

LEMMA 2. *Let $(J_n\colon n \in \omega)$ be a sequence of finite Borel covers of $[\omega]^\omega$. Then*

$$\forall S \in [\omega]^{<\omega} \forall M \in [\omega]^\omega \exists N \in [M]^\omega \exists (\bar{J}_n\colon n \in \omega) \forall n \in \omega$$
$$\bar{J}_n \subseteq J_n \wedge |\bar{J}_n| \leqslant 2^n \wedge (S, N)^\omega \subseteq \bigcap_{n \in \omega} \bigcup \bar{J}_n.$$

Proof. In this proof we will use the following version of the theorem of Galvin and Prikry:

For every J, a finite Borel cover of $[\omega]^\omega$, and every $S \in [\omega]^{<\omega}$ and $M \in [\omega]^\omega$ there exist $N \in [M]^\omega$ and $B \in J$ such that $(S, N)^\omega \subseteq B$.

There exist $N^0_0 \in [M]^\omega$ and $C^0_0 \in J_0$ such that $(S, N^0_0)^\omega \subseteq C^0_0$. Let $a_1 \in N^0_0 - ((\sup S) + 1)$. Then there exist $N^1_0 \subseteq N^0_0 - (a_1 + 1)$ and $C^1_0 \in J_1$ with

$$(S \cup \{a_1\}, N^1_0)^\omega \subseteq C^1_0$$

and there exist $N^1_1 \subseteq N^1_0$ and $C^1_1 \in J_1$ with $(S, N^1_1)^\omega \subseteq C^1_1$. Of course,

$$(S, N^1_1 \cup \{a_1\})^\omega \subseteq C^1_0 \cup C^1_1.$$

Suppose that we have defined $\{a_1, a_2, \ldots, a_{n-1}\}$ and $N^{n-1}_{2^{n-1}-1}$. Let $a_n \in N^{n-1}_{2^{n-1}-1}$ and let $P_0, P_1, \ldots, P_{2^n-1}$ be all subsets of $\{a_1, a_2, \ldots, a_n\}$. Then we can find $N^n_0 \in N^{n-1}_{2^{n-1}-1} - (a_n + 1)$ and $C^n_0 \in J_n$ such that $(S \cup P_0, N^n_0)^\omega \subseteq C^n_0$. Inductively, there exist $N^n_i \subseteq N^n_{i-1}$ and $C^n_i \in J_n$ such that $(S \cup P_i, N^n_i)^\omega \subseteq C^n_i$. Observe that

$$\bigcup_{i=0}^{2^n-1} (S \cup P_i, N^n_{2^n-1})^\omega = (S, N^n_{2^n-1} \cup \{a_1, a_2, \ldots, a_n\})^\omega;$$

and so

$$(S, N^n_{2^n-1} \cup \{a_1, \ldots, a_n\})^\omega \subseteq \bigcup_{0 \leqslant i \leqslant 2^n-1} C^n_i.$$

Let $N = \{a_i\colon i \in \omega\}$ and $\bar{J}_n = \{C^n_0, C^n_1, \ldots, C^n_{2^n-1}\}$. Since, for every n,

$$(S, N)^\omega \subseteq (S, N^n_{2^n-1} \cup \{a_1, a_2, \ldots, a_n\})^\omega,$$

we have $(S, N)^\omega \subseteq \bigcap_n \bigcup \bar{J}_n$.

LEMMA 3. *Let $(J^k_n\colon n, k \in \omega)$ be a double sequence of finite Borel covers of $[\omega]^\omega$ and let $M \in [\omega]^\omega$. Then there exist $N \in [M]^\omega$ and $(\bar{J}^k_n\colon n, k \in \omega)$ such that $\bar{J}^k_n \subseteq J^k_n$ and $|\bar{J}^k_n| \leqslant 2^n$ and $[N]^{*\omega} \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}^k_n$.*

Proof. Let $\{S_k\colon k \in \omega\} = [\omega]^{<\omega}$. We first construct a decreasing sequence $(N_k\colon k \in \omega)$ and a double sequence $(\bar{J}^k_n\colon k, n \in \omega)$ such that $\bar{J}^k_n \subseteq J^k_n$, $|\bar{J}^k_n| \leqslant 2^n$ and $(S_k, N_k)^\omega \subseteq \bigcap_n \bigcup \bar{J}^k_n$ and $N_0 \subseteq M$.

There are $N_0 \in [M]^\omega$ and $(\bar{J}^0_n\colon n \in \omega)$ such that $\bar{J}^0_n \subseteq J^0_n$ and $|\bar{J}^0_n| \leqslant 2^n$ and $(S_0, N_0)^\omega \subseteq \bigcap_n \bigcup \bar{J}^0_n$ (Lemma 2) and inductively, there exist $N_{k+1} \in [N_k]^\omega$ and $(\bar{J}^{k+1}_n\colon n \in \omega)$ with $\bar{J}^{k+1}_n \subseteq J^{k+1}_n$ and $|\bar{J}^{k+1}_n| \leqslant 2^n$ and $(S_{k+1}, N_{k+1})^\omega \subseteq \bigcap_n \bigcup \bar{J}^{k+1}_n$ (Lemma 2).

We define a sequence $(b_l\colon l \in \omega)$. Let $b_0 = \min(N_0)$ and

$$b_{l+1} = \min\bigl(N_{k_{l+1}} - (b_l + 1)\bigr)$$

where $k_{l+1} = \sup\{k \in \omega\colon S_k \subseteq b_l + 1\}$. Let $N = \{b_l\colon l \in \omega\}$. We claim that $[N]^{*\omega} \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}^k_n$. Let $L \subseteq^* N$. Then there exists b_l such that $L - (b_l + 1) \subseteq N$. There is a $k \in \omega$ for which $S_k = L \cap (b_l + 1)$. Then $(S_k, N_k)^\omega \subseteq \bigcap_n \bigcup \bar{J}^k_n$ and $\{b_m\colon m > l\} \subseteq N_{k_{l+1}} \subseteq N_k$. Thus $(S_k, L - (b_l + 1))^\omega \subseteq \bigcap_n \bigcup \bar{J}^k_n$. Since $(b_l\colon l \in \omega)$ is increasing, $L \in \bigl(S_k, L - (b_l + 1)\bigr)^\omega$.

THEOREM 1. *Assume Martin's Axiom. Then there exists a Δ-set of reals of cardinality continuum.*

Proof. From Lemma 3, using standard methods (see [2], [4], [5], [6]), we get a set $X = \{X_\alpha\colon \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ such that $|X| = \mathfrak{c}$ and for every $\alpha, \beta < \mathfrak{c}$ if $\alpha < \beta$ then $X_\beta \subseteq^* X_\alpha$ and for every double sequence $(J^k_n\colon k, n \in \omega)$ of finite Borel covers of $[\omega]^\omega$ there exist $(\bar{J}^k_n\colon k, n \in \omega)$ and $\alpha < \mathfrak{c}$ such that $\bar{J}^k_n \subseteq J^k_n$ and $|\bar{J}^k_n| \leqslant 2^n$ and $\{X_\beta\colon \beta \geqslant \alpha\} \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}^k_n$.

We will show that X is a Δ-set. For every $(I^k_n\colon k, n \in \omega)$, a double sequence of finite Borel covers of X, there exists a double sequence $(J^k_n\colon k, n \in \omega)$ of finite Borel covers of $[\omega]^\omega$ such that $I^k_n = \{B \cap X\colon B \in J^k_n\}$. There exist $\alpha < \mathfrak{c}$ and $(\bar{J}^k_n\colon k, n \in \omega)$ such that $\bar{J}^k_n \subseteq J^k_n$, $|\bar{J}^k_n| \leqslant 2^n$ and

$$\{X_\beta\colon \beta \geqslant \alpha\} \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}^{2k}_n \quad \text{and} \quad \{X_\beta\colon \beta < \alpha\} \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}^{2k+1}_n$$

(Lemma 1). Thus the double sequence $(\bar{I}_n^k\colon k, n \in \omega)$, where $\bar{I}_n^k = \{X \cap B\colon B \in \bar{J}_n^k\}$ has the properties required in the definition of a Δ-set.

THEOREM 2. (a) *Every Borel image of a Δ-set is a Δ-set.*

(b) *Every Δ-set of reals is Lebesgue negligible and meagre.*

Proof. (a) Evident.

(b) Let $X \subseteq R$ be a Δ-set. We may assume that $X \subseteq (0, 1)$. Let

$$J_n^k = \{[i\cdot 2^{-2^n}, (i+1)\cdot 2^{-2^n})\colon 0 \leqslant i \leqslant 2^{2^n}-1\}.$$

Then there exists $(\bar{J}_n^k\colon k, n \in \omega)$ such that $\bar{J}_n^k \subseteq J_n^k$, $|\bar{J}_n^k| \leqslant 2^n$ and $X \subseteq \bigcup_k \bigcap_n \bigcup \bar{J}_n^k$. Observe that $m(\bigcup \bar{J}_n^k) \leqslant 2^{-n}$ (m denoting Lebesgue measure), so that $m(\bigcup_k \bigcap_n \bigcup \bar{J}_n^k) = 0$. Let $(a, b) \subseteq (0, 1)$ and let $n \in \omega$ be such that $2\cdot 2^{-n} \leqslant b-a$. Then there exists i such that $0 \leqslant i \leqslant 2^{2^n}-1$ and $[i\cdot 2^{-2^n}, (i+1)\cdot 2^{-2^n}) \cap \bigcup \bar{J}_n^k = \emptyset$ and $[i\cdot 2^{-2^n}, (i+1)\cdot 2^{-2^n}) \subseteq (a, b)$. Hence, for every $k \in \omega$ the set $\bigcap_n \bigcup \bar{J}_n^k$ is nowhere dense.

Theorem 0 is a direct consequence of Theorems 1 and 2.

Remarks. Every Δ-set is a C'-set.

A. W. Miller has proved the following theorem: It is consistent with ZFC that for every set of reals of cardinality continuum there is a continuous map from that set onto $[0, 1]$. It follows that we cannot prove Theorem 0 in ZFC.

Recently I have learnt that S. Todorčević solved Grzegorek's problem independently. In 1981 he showed that under MA there exists a set of reals of cardinality continuum such that every Borel image of it is a γ-set. This implies Theorem 0. This result is not published.

We can take any positive sequence (m_n) with $m_n \to +\infty$ in place of (2^n) in the definition of a Δ-set.

Let $(J_n^k\colon k, n \in \omega)$ be a sequence of finite Borel covers X. Let $n_0 = 0$ and let (n_l) be strictly increasing and such that $\forall n \geqslant n_l\ m_n \geqslant 2^l$. Let

$$J^{*k}_n = \{A_0 \cap A_1 \cap \ldots \cap A_{n_l}\colon A_j \in J_j^k\}$$

and let $\bar{J}^{*k}_l \subseteq J^{*k}_l$, $|\bar{J}^{*k}_l| \leqslant 2^l$. For $n \in \omega$ let

$$\bar{J}_n^k = \{A_n\colon A_0 \cap A_1 \cap \ldots \cap A_n \cap \ldots \cap A_{n_{l_0}} \in \bar{J}^{*k}_{l_0}\}$$

where $l_0 = \min\{l\colon n_l > n\}$. Since $n \geqslant n_{l_0-1}$, $m_n \geqslant 2^{2l_0-1}$, we see that $|\bar{J}_n^k| \leqslant m_n$. Observe that $\bigcup_k \bigcap_l \bigcup \bar{J}^{*k}_l \subseteq \bigcup_k \bigcap_l \bigcup \bar{J}_l^k$.

A family $J \subseteq P(X)$ is an *ω-cover* if for every finite set $F \subseteq X$ there exists $B \in J$ such that $F \subseteq B$.

A topological space X is a *γ-set* if for every J, an open ω-cover of X, there exists a family $\{D_n\colon n \in \omega\} \subseteq J$ such that $X \subseteq \bigcup_m \bigcap_{n \geqslant m} D_n$.

Now we construct an example of a γ-set which can be mapped onto $[0, 1]$ by a Borel function. Observe that every continuous image of a γ-set is Lebesgue negligible and meagre.

THEOREM 3. *Assume Martin's Axiom. Then there exists a set $X \subseteq \mathbf{R}$ with the properties:*

(1) *There exists a countable set D such that $X \cup D$ is a γ-set.*

(2) *$X+F$ is Lebesgue negligible (meagre) if F is Lebesgue negligible (meagre).*

(3) *There exists a continuous function $f\colon X \xrightarrow{\text{onto}} [0, 1]$.*

Define a function $g\colon [\omega]^\omega \to 2^\omega$ by

$$g(A) = \varphi \quad \text{iff} \quad \varphi(n) = \begin{cases} 0 & \text{if the } n\text{th term of } A \text{ is even},\\ 1 & \text{otherwise}. \end{cases}$$

Let $E = \{2n\colon n \in \omega\}$.

Fact 1. *g is continuous.*

Fact 2. *For every $A \in [\omega]^\omega$ with $|A \cap E| = |A \cap (\omega - E)| = \omega$ we have $g([A]^\omega) = 2^\omega$.*

LEMMA 4. *Let J be an open ω-cover of $[\omega]^\omega$ in $P(\omega)$ and let $M \in [\omega]^\omega$ be such that $|M \cap E| = |M \cap (\omega - E)| = \omega$. Then there exists a family $\{D_n\colon n \in \omega\} \subseteq J$ and $N \in [M]^\omega$ such that $[N]^{*\omega} \subseteq \bigcup_m \bigcap_{n \geqslant m} D_n$ and $|N \cap E| = |N \cap (\omega - E)| = \omega$.*

The proof of Lemma 4 is similar to the proof of Lemma 1.2 in [6].

Proof of Theorem 3. Let $\{J_\alpha\colon \alpha < \mathfrak{c}\}$ be the family of all open ω-covers of $[\omega]^{<\omega}$ in $P(\omega)$ and let $\{h_\alpha\colon \alpha < \mathfrak{c}\} = \{h \in 2^\omega\colon |h^{-1}(1)| = |h^{-1}(0)| = \omega\}$ and $\omega^\omega = \{m_\alpha\colon \alpha < \mathfrak{c}\}$.

Using Lemma 4 and methods of [5] and [6] we construct a set $Y = \{Y_\alpha\colon \alpha < \mathfrak{c}\}$ such that

(a) $Y \subseteq [\omega]^\omega$ and $|Y| = \mathfrak{c}$,

(b) $\forall \alpha, \beta\ \alpha < \beta \Rightarrow Y_\beta \subseteq^* Y_\alpha$,

(c) For every α, if J_α is an open ω-cover of $Y \cup [\omega]^{<\omega}$ then there exists a family $\{D_n\colon n \in \omega\} \subseteq J_\alpha$ such that $Y \cup [\omega]^{<\omega} = \bigcup_m \bigcap_{n > m} D_n$,

(d) $\forall \alpha \forall n$ the nth term in Y_α is greater than $m_\alpha(n)$,

(e) $\forall \alpha\ g(Y_\alpha) = h_\alpha$.

Let i be the standard homeomorphism from $P(\omega)$ onto the Cantor set on the real line and let j be a continuous function from $\{h \in 2^\omega\colon h^{-1}(1) = |h^{-1}(0)| = \omega\}$ onto $[0, 1]$. Define $X = i(Y)$, $D = i([\omega]^{<\omega})$ and $f = j \circ g \circ i^{-1} \wedge X$. The sets X and D and the function f have properties as required (see [5], [6]).

COROLLARY 1. *Assume Martin's Axiom. Then there exists a γ-set which can be mapped onto $[0, 1]$ by a Borel function.*

Proof. The function f from Theorem 3 can be extended to a Borel function on $X \cup D$.

The author wishes to express his thanks to D. H. Fremlin, E. Grzegorek, J. Jasiński and J. Tryba for fruitful discussions.

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Received 6 April 1987;
in revised form 14 July 1987

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