On small sets in the sense of measure and category

by

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Abstract. We show assuming Martin's Axiom that there exists a set of reals of cardinality continuum whose every Borel image into the reals is Lebesgue negligible and meagre. This is an answer to a problem of E. Grzegorek. We also construct a $\gamma$-set which can be mapped onto the unit interval by a Borel function.

Assuming Martin’s Axiom, F. Galvin and A. W. Miller [6] constructed a set of reals of cardinality continuum such that every continuous image of this set into the reals is Lebesgue negligible and meagre. This was an answer to a question of Sierpiński. E. Grzegorek has posed the following question: assume Continuum Hypothesis, does there exist a set of reals of cardinality continuum such that every Borel image of it into the reals is Lebesgue negligible and meagre? (see [3]). D. H. Fremlin and J. Jasinski [3] showed that a set with this property exists if we assume Martin’s Axiom and if there exists $k < \zeta$ such that $\mathcal{P}(k)$ contains a proper uniform $\alpha$-saturated $k$-aditive ideal. But this assumption implies the negation of Continuum Hypothesis. The following theorem solves Grzegorek’s question under Martin’s Axiom.

THEOREM 0. Assume Martin’s Axiom. There exists a set $X \subseteq \mathbb{R}$ such that $|X| = \zeta$ and $f(X)$ is Lebesgue negligible and meagre for every Borel measurable function $f : X \to \mathbb{R}$.

DEFINITION 1. A topological space $X$ is a $D$-set iff for every double sequence $(J^k_n : n, k \in \omega)$ of finite Borel covers of $X$ there exists a double sequence $(J^k_n : n, k \in \omega)$ such that $\overline{J}^k_n \subseteq J^k_n$ and $|\bigcup_{n \in \omega} J^k_n| \leq 2^\zeta$ and $X \subseteq \bigcup_{k \in \omega} \bigcup_{n \in \omega} J^k_n$.

LEMMA 1. Assume Martin’s Axiom. Every subset of the reals of cardinality less than continuum is a $D$-set.

Proof. Let $X \subseteq \mathbb{R}$ be such that $|X| < \zeta$ and let $(J^k_n : k, n \in \omega)$ be a double sequence of finite Borel covers of $X$. We define a partially ordered set $(P, \leq)$ as follows:

$P = \{(f, H_0, H_1, ..., H_n) : f : m+1 \times m+1 \to \bigcup_{k \in \omega} P(J^k_n) \land f(i, j) \subseteq J^k_n \land f(i, j) \subseteq 2^\zeta \land H_i \subseteq X \land |H_i| \leq 2^\zeta, H_j \subseteq \bigcap_{i \in \omega} f((i, j), m \in \omega)\}$
and \((f, H_0, \ldots, H_m) \subseteq (f', H'_0, \ldots, H'_m)\) if and only if \(f \subset f'\) and \(H_j \subset H'_j\) for every \(0 \leq j \leq m\). It is not hard to see that \((P, \leq)\) has C.C.C. and that the sets \(G_x = \{(f, H_0, \ldots, H_m) : \exists 0 \leq j \leq m : x \in H_j\}\), \(x \in X \times \mathbb{N}\), are dense in \((P, \leq)\). So there exists a \(\mathcal{F}\)-generic filter \(\mathcal{G}\), where

\[
\mathcal{F} = \{G_x : x \in X \} \cup \{G_{ij} : i, j \in \omega\}.
\]

Let \(F = \bigcup \{f : \exists (f, H_0, \ldots, H_m) \in \mathcal{F}\}\). \(F\) is a function defined on \(\omega \times \omega\). Let \(J^n = F(n, k)\). We have \(X \subseteq \bigcup \{J^n : n \in \omega\}\.

We will use the following notation:

- \([M]^\omega = \{N \subseteq M : N \text{ is infinite}\}\)
- \([M]^* = \{N \subseteq M : N \text{ is finite}\}\)
- \([M]^\omega_1 = \{N \subseteq [\omega]^\omega : N \text{ is finite}\}\)
- \((S, M)^n = \{N \subseteq [\omega]^n : S \subseteq N \subseteq M \text{ and } S < N\}\)

where \(S\) is a finite and \(M\) an infinite subset of \(\omega\).

**Lemma 2.** Let \((U_n : n \in \omega)\) be a sequence of finite Borel covers of \([\omega]^\omega\). Then

\[
\forall S \in [\omega]^\omega \exists M \in [\omega]^\omega \exists N \in [M]^\omega \exists \{J_n : n \in \omega\} \subseteq [\omega]^\omega \text{ s.t. } J_n \subseteq J_{n+1} \subseteq \mathcal{I}^\omega \land (S, M)^n \subseteq \bigcap \{J_n : n \in \omega\}\.
\]

**Proof.** In this proof we will use the following version of the theorem of Galvin and Prikry:

For every \(J_\alpha\), a finite Borel cover of \([\omega]^\omega\), and every \(S \in [\omega]^\omega\) and \(N \in [M]^\omega\) there exist \(N \in [M]^\omega\) and \(B \subseteq J_\alpha\) such that \(N \cup B \subseteq M\).

There exist \(N_0^\omega \in [M]^\omega\) and \(C_0 \in J_0\) such that \((S, N_0^\omega) \subseteq C_0\). Let \(a_1 \in N_0^{\omega-\omega} = \{(\text{sup } S) + 1\}\). Then there exist \(N^\omega_1 = N_0^{\omega-\omega} \cup \{a_1\} \subseteq C_0\) with

\[
(S, N_1^\omega) \subseteq C_0\text{ and } (S, N_1^\omega) \subseteq C_0 \subseteq C_1.
\]

and there exist \(N_1^\omega \subseteq N_0^\omega \) and \(C_1 \subseteq J_1\) with \((S, N_1^\omega) \subseteq C_1\). Of course,

\[
(S, N_1^\omega) \subseteq C_0 \cup C_1.
\]

Suppose that we have defined \((a_1, a_2, \ldots, a_{n-1})\) and \(N_0^{\omega-1-\omega}\). Let \(a_n \in N_0^{\omega-1-\omega}\) and let \(P_0, P_1, \ldots, P_{n-1}\) be all subsets of \((a_1, a_2, \ldots, a_{n-1})\). Then we can find \(N_{n}^\omega \in [P_0]^{\omega-\omega} \cup \{a_1\}\) and \(C_n \subseteq J_n\) such that \((S \cup P_0, N_{n}^\omega) \subseteq C_n\). Inductively, there exist \(N_{n+1}^\omega \subseteq N_{n}^\omega\) and \(C_{n+1} \subseteq J_{n+1}\) such that \((S \cup P_n, N_{n+1}^\omega) \subseteq C_{n+1}\). Observe that

\[
\bigcup_{n=0}^{\omega-1} (S \cup P_n, N_{n+1}^\omega) = (S, N_{\omega-1}^\omega) \cup (a_1, a_2, \ldots, a_{n})^\omega
\]

and so

\[
(S, N_{\omega-1}^\omega) \cup (a_1, a_2, \ldots, a_{n})^\omega \subseteq \bigcup_{n=0}^{\omega-1} C_n.
\]

Let \(N = \{a_i : i \in \omega\}\) and \(J_n = (C_0, C_1, \ldots, C_{n-1})\). Since, for every \(n\),

\[
(S, N)^n \subseteq (S, N_{\omega-1}^\omega) \cup (a_1, a_2, \ldots, a_{n})^\omega
\]

we have \((S, N)^\omega \subseteq \bigcup_{n=0}^{\omega} J_n\).

**Lemma 3.** Let \((J_\alpha : \alpha < \omega_1)\) be a double sequence of finite Borel covers of \([\omega]^\omega\) and \(M \in [\omega]^\omega\). Then there exist \(N_0 \in [M]^\omega\) and \((J_\alpha : \alpha < \omega_1)\) such that \(J_\alpha \subseteq J_\beta \subseteq \mathcal{I}^\omega \) and \((S, N_0)^\omega \subseteq \bigcup_{n=0}^{\omega} J_n\).

**Proof.** Let \((S, N_0) = [\omega]^\omega\). We first construct a decreasing sequence \((N_k : k \in \omega)\) and a double sequence \((J_\alpha : \alpha < \omega_1, \alpha < \omega_1)\) such that \(J_\alpha \subseteq J_\beta \subseteq \mathcal{I}^\omega \) and \((S, N_k)^\omega \subseteq \bigcup_{n=0}^{\omega} J_n\).

There exist \(N_0 \in [M]^\omega\) and \((J_\alpha : \alpha < \omega_1)\) such that \(J_\alpha \subseteq J_\beta \subseteq \mathcal{I}^\omega \) and \((S, N_0)^\omega \subseteq \bigcup_{n=0}^{\omega} J_n\) (Lemma 2) and inductively, there exist \(N_\omega \in [M]^\omega\) and \((J_\alpha : \alpha < \omega_1)\) with \(J_\alpha \subseteq J_\beta \subseteq \mathcal{I}^\omega \) and \((S, N_\omega)^\omega \subseteq \bigcup_{n=0}^{\omega} J_n\) (Lemma 2).

We define a sequence \((b_i : i \in \omega)\). Let \(b_0 = \min(N_0)\) and

\[
b_{i+1} = \min(N_{i+1} - (b_i + 1))
\]

where \(k_{i+1} = \sup\{k : k \in \omega : S_k \subseteq b_{i+1}\}\). Let \(N = \{b_i : i \in \omega\}\). We claim that \((N)^\omega \subseteq \bigcup_{n=0}^{\omega} J_n\). Let \(L \subseteq N\). Then there exists \(b_i\) such that \(L \subseteq (b_i + 1) \subseteq N\).

There is a \(k \in \omega\) for which \(S_k = L \cap (b_i + 1)\). Then \((S_k, N_k)^\omega \subseteq \bigcup_{n=0}^{\omega} J_n\).

**Theorem 1.** Assume Martin's Axiom. Then there exists a \(\Delta\)-set of reals of cardinality \(\omega_1\).

**Proof.** From Lemma 3, using standard methods (see [2], [4], [5], [6]), we get a set \(X = \{X_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega\) such that \(X_\alpha \equiv \alpha\) and for every \(\alpha, \beta < \omega_1\) if \(\alpha < \beta\) then \(X_\alpha \subseteq X_\beta\) and for every double sequence \((J_\alpha : k, n \in \omega)\) of finite Borel covers of \([\omega]^\omega\) there exist \(J_\beta \supseteq J_\alpha\) and \(a < \alpha\) such that \(J_\beta \subseteq J_\alpha\) and \(J_\beta \subseteq \mathcal{I}^\omega\) and \((X_\beta : \beta < \alpha) \subseteq \bigcup_{n=0}^{\omega} J_n\).

We will show that \(X\) is a \(\Delta\)-set. For every \((J_\alpha : k, n \in \omega)\), a double sequence of finite Borel covers of \(X\), there exists a double sequence \((J_\beta : k, n \in \omega)\) of finite Borel covers of \([\omega]^\omega\) such that \(J_\beta = \{X_\beta : X_\beta \cap X \subseteq J_\beta\}\). There exist \(a < \alpha\) and \((J_\alpha : k, n \in \omega)\) such that \(J_\beta \subseteq J_\alpha\) and \(\alpha < \beta\) and

\[
(X_\beta : \beta > a) \subseteq \bigcup_{n=0}^{\omega} J_n
\]

and

\[
(X_\beta : \beta < a) \subseteq \bigcup_{n=0}^{\omega} J_n + 1
\]
by a Borel function. Observe that every continuous image of a $\gamma$-set is Lebesgue negligible and meagre.

**Theorem 3.** Assume Martin’s Axiom. Then there exists a set $X \subseteq R$ with the properties:

1. There exists a countable set $D$ such that $X \cup D$ is a $\gamma$-set.
2. $X \cup F$ is Lebesgue negligible (meagre) if $F$ is Lebesgue negligible (meagre).
3. There exists a continuous function $f: X^{\omega_1} \to [0, 1]$.

Define a function $g: [\omega]^{\omega} \to 2^\omega$ by

$$
g(A) = \varphi \quad \text{iff} \quad \varphi(n) = \begin{cases} 0 & \text{if the } n\text{th term of } A \text{ is even}, \\ 1 & \text{otherwise}. \end{cases}
$$

Let $E = \{2n: n \in \omega\}$.

**Fact 1.** $g$ is continuous.

**Fact 2.** For every $A \in [\omega]^\omega$ with $|A \cap E| = |A \cap (\omega - E)| = \omega$ we have $g(|A|) = 2^\omega$.

**Lemma 4.** Let $J$ be an open $\omega$-cover of $[\omega]^\omega$ in $P(\omega)$ and let $M \in [\omega]^\omega$ be such that $|M \cap E| = |M \cap (\omega - E)| = \omega$. Then there exists a family $\{D_n: n \in \omega\} \subseteq J$ and $N \in [M]^\omega$ such that $|N|^\omega \subseteq \bigcup_{n \in \omega} D_n$ and $N \cap E = |N \cap (\omega - E)| = \omega$.

The proof of Lemma 4 is similar to the proof of Lemma 1.2 in [6].

**Proof of Theorem 3.** Let $\langle \mathcal{A}_n: n < \omega \rangle$ be the family of all open $\omega$-covers of $[\omega]^\omega$ in $P(\omega)$ and let $\langle h_n: n < \omega \rangle = \{h: h \in \omega^\omega\}$.

Using Lemma 4 and methods of [5] and [6] we construct a set $Y = \{Y_n: n < \omega\}$ such that

1. $Y \subseteq [\omega]^\omega$ and $|Y| = \omega$.
2. $\forall x, b \in [\omega]^\omega, Y_b \subseteq \mathcal{A}_x$.
3. For every $n$, if $\mathcal{A}_n$ is an open $\omega$-cover of $Y \cup [\omega]^\omega$ then there exists a family $\{D_n: n \in \omega\} \subseteq \mathcal{A}_n$ such that $Y \cup [\omega]^\omega \subseteq \bigcup_{n \in \omega} D_n$.
4. $\forall x \in Y$, the $n$th term in $Y_x$ is greater than $m_n(x)$.
5. $\forall x \in Y$, $g(Y_x) = h_n$.

Let $I$ be the standard homeomorphism from $P(\omega)$ onto the Cantor set on the real line and let $f$ be a continuous function from $[h \in \omega^\omega]: h^{-1}(0) = \omega$ onto $[0, 1]$. Define $X = (I(Y), D = (i(\omega)^\times)^\times)$ and $f = f + g$ on $X$. The sets $X$ and $D$ and the function $f$ have properties as required (see [5], [6]).

**Corollary 1.** Assume Martin’s Axiom. Then there exists a $\gamma$-set which can be mapped onto $[0, 1]$ by a Borel function.

**Proof.** The function $f$ from Theorem 3 can be extended to a Borel function on $X \cup D$.

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References


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