Imbeddings into R^n and dimension of products

by

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Dedicated to Prof. Yukihiro Kodama
on the occasion of his sixtieth birthday

Abstract. The aim of this note is to answer a question from [M–R], and to show that the examples constructed in [M–R] have several other properties.

In [M–R] D. McCullough and L. R. Rubin presented an interesting construction and proved the following result: for each $n \geqslant 2$ there exists an n-dimensional continuum X such that the space $E(X, R^{2n})$ of imbeddings from X into R^{2n} is dense in the space $C(X, R^{2n})$ of continuous maps to R^{2n}. The authors asked the following question:

Is this property related to the phenomenon of n-dimensional compacta whose products have dimension less than $2n$?

The aim of this note is to answer this question (see Theorem 1.1), and to show that this property is equivalent to some other properties (see Theorem 2.2). At the end of this note we pose a problem (1) which, taking into account the above result of D. McCullough and L. R. Rubin and the results of this note, seems to be very important. It is a more concrete form of the above problem.

1. An answer to the question. In this section we prove the following

1.1. THEOREM. *Let X be an n-dimensional compactum, $n \geqslant 1$. If $E(X, R^{2n})$ is dense in $C(X, R^{2n})$ then* $\dim X \times X < 2n$.

This theorem is a direct consequence of 1.4 and 1.5.

By D^n we denote the closed unit ball in R^n.

A mapping $f\colon X \to D^m$ is said to be *transversely trivial with respect to* a mapping $g\colon Y \to D^n$ (briefly: f and g are *transversely trivial*) if there exist two mappings $F\colon X \to D^m \times D^n$ and $G\colon Y \to D^m \times D^n$ satisfying the conditions:

(1) **Added in proof.** Affirmatively solved by S. Spież.

(i) $F(x) = (f(x), 0)$ for $x \in f^{-1}(\partial D^m)$,
(ii) $G(x) = (0, g(y))$ for $y \in g^{-1}(\partial D^n)$,
(iii) $F(X) \cap G(Y) = \emptyset$.

Here 0 denotes the origin of R^k. If f and g are not transversely trivial then they are said to be *transversely essential.* If f is transversely trivial with respect to itself then we simply say that it is transversely trivial.

For the definition and properties of essential mappings see e.g. [K].

1.2. *If $f \times g$: $X \times Y \to D^m \times D^n$ is essential then f and g are transversely essential.*

Proof. For suppose there exist mappings $F = (f_1, f_2)$ and $G = (g_1, g_2)$ as in the definition above. By (iii) and [K–L], 2.2, $f_1 \times g_2$ $X \times Y \to D^m \times D^n$ is inessential. By (i) and (ii) we have $f_1 | f^{-1}(\partial D^m) = f | f^{-1}(\partial D^m)$, and similar equality holds for g_2 and g. It follows that $f \times g$ is inessential (see e.g. [K], 1.1), a contradiction.

1.3. *If f: $X \to D^m$ and g: $Y \to D^n$ are transversely essential then there exists an $\varepsilon > 0$ such that for every mapping F: $X \to R^m \times R^n$ which is ε-close to $(f, 0)$: $X \to R^m \times R^n$, and every mapping G: $Y \to R^m \times R^n$ which is ε-close to $(0, g)$: $Y \to R^m \times R^n$ we have $F(X) \cap G(Y) \neq \emptyset$.*

Proof. Put $\varepsilon = \frac{1}{6}$. Consider two mappings $F = (f_1, f_2)$ and $G = (g_1, g_2)$ satisfying the hypothesis. Let α, β: $[0, \infty) \to [0, 1]$ be given by the formulae:

$$\alpha(t) = \begin{cases} 0 & t \leqslant \frac{1}{2}, \\ 6t-3 & \frac{1}{2} \leqslant t \leqslant \frac{2}{3}, \\ 1 & t \geqslant \frac{2}{3}, \end{cases} \qquad \beta(t) = \begin{cases} 1 & t \leqslant \frac{1}{2}, \\ -6t+4 & \frac{1}{2} \leqslant t \leqslant \frac{2}{3}, \\ 0 & t \geqslant \frac{2}{3}. \end{cases}$$

Define F': $X \to R^m \times R^n$ and G': $Y \to R^m \times R^n$ by

$$F'(x) = \Big(\big(1-\alpha(|f_1(x)|)\big) f_1(x) + \alpha(|f_1(x)|) f(x),\ \beta(|f_1(x)|) f_2(x)\Big),$$
$$G'(y) = \Big(\beta(|g_2(y)|) g_1(y),\ \big(1-\alpha(|g_2(y)|)\big) g_2(y) + \alpha(|g_2(y)|) g(y)\Big).$$

Since $\operatorname{im} F' \cup \operatorname{im} G' \subset D^m \times D^n$, $F'(x) = (f(x), 0)$ for $f(x) \in \partial D^m$ and $G'(y) = (0, g(y))$ for $g(y) \in \partial D^n$, by our hypothesis there is a point $(x, y) \in X \times Y$ such that $F'(x) = G'(y)$. Then we have

$$\big|\big(1-\alpha(|f_1(x)|)\big) f_1(x) + \alpha(|f_1(x)|) f(x)\big| \leqslant |g_1(y)| \leqslant \tfrac{1}{6}.$$

It follows that $|f_1(x)| \leqslant \frac{1}{2}$ since $|f_1(x) - f(x)| < \frac{1}{3}$. Similarly $|g_2(y)| \leqslant \frac{1}{2}$, and we conclude that $F'(x) = F(x)$ and $G'(y) = G(y)$. This completes the proof.

1.4. *If X is a compactum such that $E(X, R^{2n})$ is dense in $C(X, R^{2n})$ then any two mappings f_1: $X_1 \to D^n$ and f_2: $X_2 \to D^n$, where X_1 and X_2 are disjoint closed subsets of X, are transversely trivial. In particular, $f_1 \times f_2$: $X_1 \times X_2 \to D^n \times D^n$ is inessential.*

Proof. Suppose f_1 and f_2 are transversely essential. Let $\varepsilon > 0$ satisfy 1.3 for $f = f_1$ and $g = f_2$. Let h: $X \to R^n \times R^n$ be a mapping whose restriction to X_1 is $(f_1, 0)$ and restriction to X_2 is $(0, f_2)$. By our hypothesis there exists an imbedding H: $X \to R^n \times R^n$ which is an ε-approximation to h. Then $F = H | X_1$ and $G = H | X_2$ are ε-approximations to $(f_1, 0)$ and $(0, f_2)$ respectively, with disjoint images. This contradicts 1.3 and proves the first assertion. The second follows now from 1.2.

1.5. *For an n-dimensional compactum X the following are equivalent:*
(1) $\dim X \times X < 2n$,
(2) *for every mapping f: $X \to D^n$ its square $f \times f$: $X \times X \to D^n \times D^n$ is inessential,*
(3) *for any two mappings f_1: $X_1 \to D^n$ and f_2: $X_2 \to D^n$, where X_1 and X_2 are disjoint closed subsets of X, $f_1 \times f_2$: $X_1 \times X_2 \to D^n \times D^n$ is inessential.*

Proof. (1)$\Rightarrow$(2). This is well known. (2)$\Rightarrow$(3). Suppose $f_1 \times f_2$ is essential. Let f: $X \to D^n$ be any extension of the mapping $X_1 \cup X_2 \to D^n$ given by f_1 and f_2. Then $f \times f$ is essential since its restriction to $X_1 \times X_2 \subset X \times X$ (equal to $f_1 \times f_2$) is essential. This contradicts (2).

(3)$\Rightarrow$(1). Suppose $\dim X \times X = 2n$. Then there exists an $\varepsilon_0 > 0$ such that for every ε_0-mapping F: $X \times X \to Y$ we have

(a) $$\dim F(X \times X) \geqslant 2n.$$

There exists an $\eta > 0$ such that for every η-mapping f: $X \to P$ (P is to be fixed from the next sequence on) we have

(b) $$f \times f\colon X \times X \to P \times P \text{ is an } \varepsilon_0\text{-mapping}.$$

There exists an n-dimensional compact polyhedron P and an η-mapping f: $X \to P$. There exists an $\varepsilon_1 > 0$ such that

(c) $$(A \subset P \times P) \,\&\, (\operatorname{diam} A < \varepsilon_1) \Rightarrow \operatorname{diam}(f \times f)^{-1}(A) < \varepsilon_0.$$

Let K be a finite simplicial complex such that $|K| = P$, and let $K \times K$ denote the cell complex composed of all cells $s_1 \times s_2 \subset P \times P$, where $s_1, s_2 \in K$. We can choose K so fine that

(d) $$\operatorname{diam}(\operatorname{st}(\sigma, K \times K)) < \varepsilon_1 \quad \text{for each } \sigma \in K \times K.$$

Now we are going to show that there exist two n-simplices $s_1, s_2 \in K$ such that

(e) $$f \times f\colon (f \times f)^{-1}(s_1 \times s_2) \to s_1 \times s_2 \text{ is essential}.$$

Suppose not. Then there exists a mapping F: $X \times X \to |(K \times K)^{2n-1}|$ such that for each $z \in X \times X$

(f) both $F(z)$ and $(f \times f)(z)$ belong to one cell of $K \times K$.

Note that F is an ε_0-mapping. Indeed, given $y \in \operatorname{im} F$ there is $\sigma \in K \times K$ containing y and by (f) we have $F^{-1}(y) \subset (f \times f)^{-1}(\operatorname{st}(\sigma, K \times K))$. The assertion follows now from (d) and (c). But this is impossible: $\dim(\operatorname{im} F) < 2n$, contrary to (a). This proves (e).

Now take two n-simplices $\tau_1 \subset s_1$ and $\tau_2 \subset s_2$ so that $\tau_1 \cap \tau_2 = \emptyset$ and let $X_1 = f^{-1}(\tau_1)$, $X_2 = f^{-1}(\tau_2)$. Then X_1 and X_2 are disjoint closed subsets of X. Let $f_1\colon X_1 \to \tau_1$ and $f_2\colon X_2 \to \tau_2$ be the corresponding restrictions of f. By (e) and [K], 1.9, $f_1 \times f_2\colon X_1 \times X_2 \to \tau_1 \times \tau_2$ is essential, a contradiction.

2. Some equivalent properties. Theorem 2.2 shows that some seemingly different properties of compacta are equivalent. Its proof relies heavilly on the following lemma.

2.1. *Let X be an n-dimensional compactum. Assume any two mappings $\varphi\colon A \to \boldsymbol{D}^n$ and $\psi\colon B \to \boldsymbol{D}^n$, where A and B are disjoint closed subsets of X, are transversely trivial. Then $E(X\times\{0,1\}, \boldsymbol{R}^{2n})$ is dense in $C(X\times\{0,1\}, \boldsymbol{R}^{2n})$.*

Proof. It suffices to show that the set of ε-mappings $C_\varepsilon(X\times\{0,1\}, \boldsymbol{R}^{2n})$ is dense in $C(X\times\{0,1\}, \boldsymbol{R}^{2n})$, for every $\varepsilon > 0$. So, let $\varepsilon > 0$, $f\colon X\times\{0,1\} \to \boldsymbol{R}^{2n}$, and $\delta > 0$ be given. It remains to construct a mapping h such that

$$(1)\qquad h \in C_\varepsilon(X\times\{0,1\}, \boldsymbol{R}^{2n}) \quad\text{and}\quad d(h,f) < \delta .$$

In this proof we assume that X is equipped with a metric ϱ bounded by 1, and the metric on $X\times\{0,1\}$ is given by

$$d((x,i),(y,j)) = \begin{cases} \varrho(x,y) & \text{for } i = j,\\ 1 & \text{for } i \neq j.\end{cases}$$

We claim that there exists a mapping $p\colon X \to P$, where P is a compact polyhedron with $\dim P \leqslant n$, and a mapping $g\colon P\times\{0,1\} \to \boldsymbol{R}^{2n}$ such that

$$(2)\qquad p \text{ is an } \varepsilon\text{-mapping} \quad\text{and}\quad d(g\circ(p\times \mathrm{id}), f) < \delta/2 .$$

Indeed, let $X = \varprojlim\{X_1 \xleftarrow{p_{12}} X_2 \xleftarrow{p_{23}} \ldots\}$, where X_k, $k \geqslant 1$, is a compact polyhedron with $\dim X_k \leqslant n$. Then there exist $r \geqslant 1$ and a mapping $g'\colon X_r\times\{0,1\} \to \boldsymbol{R}^{2n}$ such that $d(g'\circ(p_r\times\mathrm{id}), f) < \delta/2$, where p_r stands for the projection $X \to X_r$. Let $s \geqslant r$ be so large that $p_s\colon X \to X_s$ is an ε-mapping. Setting $P = X_s$, $p = p_s$ and $g = g'\circ(p_{rs}\times\mathrm{id})$ one easily sees that (2) is satisfied. This proves the claim.

Since p is an ε-mapping there exists a finite simplicial complex K such that $|K| = P$ and

$$(3)\qquad s \in K \Rightarrow \operatorname{diam} p^{-1}(s) < \varepsilon .$$

Applying the general position theory we may further assume that

(4) g maps vertices of $K\times\{0,1\}$ to a maximally independent subset of $\boldsymbol{R}^{2n}$, and g is linear on each simplex of $K\times\{0,1\}$,

(5) $g(S(g)) = \{z_1, \ldots, z_t\}$ and each set $g^{-1}(z_i)$ consists of two distinct points (a_i, i') and (b_i, i'')

(if $S(g) = \emptyset$ then $h = g\circ(p\times\mathrm{id})$ satisfies (1) and the proof is complete),

$$(6)\qquad q|S(g)\colon S(g) \to P \text{ is an injection}.$$

Here $S(g) = \{x \in P\times\{0,1\}\colon (x) \neq g^{-1}g(x)\}$ and $q\colon P\times\{0,1\} \to P$ is the projection.

For $x \in P$ let $s(x)$ denote the simplex of K which contains x in its interior. Then $s(a_i)$ and $s(b_i)$ are n-dimensional by (4). By (5) and (6), for each $i \leqslant t$, there exist imbeddings

$$u_i\colon (\boldsymbol{D}^n, 0) \to (s(a_i), a_i) \qquad v_i\colon (\boldsymbol{D}^n, 0) \to (s(b_i), b_i)$$

having pairwise disjoint images. Let $A_i' = \operatorname{im} u_i$ and let $B_i' = \operatorname{im} v_i$. Moreover, we can choose the imbeddings in such a way that there exist imbeddings

$$w_i\colon \boldsymbol{D}^n\times\boldsymbol{D}^n \to \boldsymbol{R}^{2n},\quad i \leqslant t,$$

with disjoint images and satisfying the conditions:

$$(7)\qquad \operatorname{diam}(\operatorname{im} w_i) < \delta/2 ,$$

$$(8)\qquad w_i(x,0) = g(u_i(x), i'),\qquad w_i(0,x) = g(v_i(x), i''),$$

$$(9)\qquad g^{-1}(\operatorname{im} w_i) = A_i'\times(i') \cup B_i'\times(i'') .$$

The sets $A_i = p^{-1}(A_i')$, $B_i = p^{-1}(B_i')$, $i \leqslant t$, are pairwise disjoint and closed in X. By our hypothesis the mappings $\varphi_i\colon A_i \to \boldsymbol{D}^n$, $x \to u_i^{-1}p(x)$, and $\psi_i\colon B_i \to \boldsymbol{D}^n$, $x \to v_i^{-1}p(x)$, are transversely trivial. It follows that there exist mappings

$$F_i\colon A_i \to \boldsymbol{D}^n\times\boldsymbol{D}^n,\qquad G_i\colon B_i \to \boldsymbol{D}^n\times\boldsymbol{D}^n$$

such that

$$(10)\qquad F_i(x) = (u_i^{-1}p(x), 0) \quad\text{if}\quad p(x) \in \partial A_i',$$

$$(11)\qquad G_i(x) = (0, v_i^{-1}p(x)) \quad\text{if}\quad p(x) \in \partial B_i',$$

$$(12)\qquad F_i(A_i) \cap G_i(B_i) = \emptyset .$$

Here ∂ stands for the manifold boundary of the topological cells. For a cell A by $\operatorname{int} A$ we denote its manifold interior. Note that $\operatorname{int} A_i'$ and $\operatorname{int} B_i'$ are open in P since they are n-cells lying in n-simplices of P and $\dim P = n$. It follows that the set

$$Y = X\times\{0,1\} \setminus \bigcup_{i=1}^{t} [p^{-1}(\operatorname{int} A_i')\times(i') \cup p^{-1}(\operatorname{int} B_i')\times(i'')]$$

is closed in $X\times\{0,1\}$.

Now we are ready to define the mapping $h\colon X\times\{0,1\} \to \boldsymbol{R}^{2n}$.

We define h as follows:

$$h(x,j) = \begin{cases} g\circ(p\times\mathrm{id})(x,j) & \text{for } (x,j) \in Y,\\ w_iF_i(x) & \text{for } (x,j) \in A_i\times(i'),\\ w_iG_i(x) & \text{for } (x,j) \in B_i\times(i''),\end{cases}$$

where $j = 0, 1$ and $i = 1, \ldots, t$. By (10), (11) and (8), h is well defined. Hence h is continuous as a combination of mappings on closed sets. Note that by (9) and (7), h is $(\delta/2)$-close to $g \circ (p \times \mathrm{id})$. Hence by (2) we have $d(h, f) < \delta$. Now we shall show that h is an ε-mapping. So let $z \in \mathrm{im}\, h$. First consider the case where $z \notin \bigcup \mathrm{im}\, w_i$. Then $h^{-1}(z) \subset Y$, and therefore $(z) = g((p \times \mathrm{id})(h^{-1}(z)))$ and $(p \times \mathrm{id})(h^{-1}(z)) \subset P \times \{0, 1\} \setminus S(g)$. Since g is injective on the latter set it follows that $(p \times \mathrm{id})(h^{-1}(z))$ is a one-point set $((c, j))$. Hence $h^{-1}(z) \subset p^{-1}(c) \times (j)$ and by (2) we infer that $\mathrm{diam}\, h^{-1}(z) < \varepsilon$. Now consider the case where $z \in \mathrm{im}\, w_i$, for some $i \leqslant t$. Since the images $\mathrm{im}\, w_j$ are pairwise disjoint, by (9) it follows that $h^{-1}(z) \subset A_i \times (i') \cup B_i \times (i'')$. From the formula defining h and (12) we infer that either $h^{-1}(z) \subset A_i \times (i')$ or $h^{-1}(z) \subset B_i \times (i'')$. It follows that $(p \times \mathrm{id})(h^{-1}(z))$ is a subset of a simplex of $K \times \{0,1\}$ and by (3) we conclude that $\mathrm{diam}\, h^{-1}(z) < \varepsilon$. Hence in either case $\mathrm{diam}\, h^{-1}(z) < \varepsilon$. This proves that h satisfies (1) and the proof is complete.

Two mappings $f, g\colon X \to Y$ are said to have the *disjoint images property* if for every $\varepsilon > 0$ there exist mappings $f', g'\colon X \to Y$ with disjoint images and such that $d(f, f') < \varepsilon$ and $d(g, g') < \varepsilon$.

2.2. THEOREM. *Let $n \geqslant 1$. For an n-dimensional compactum X the following are equivalent*:

(1) *any two mappings $f, g\colon X \to \boldsymbol{R}^{2n}$ have the disjoint images property,*

(2) *any two mappings $f, g\colon X \to \boldsymbol{D}^n$ are transversely trivial,*

(3) *every mapping $f\colon X \to \boldsymbol{D}^n$ is transversely trivial,*

(4) *any two mappings $f_1\colon X_1 \to \boldsymbol{D}^n$ and $f_2\colon X_2 \to \boldsymbol{D}^n$, where X_1, X_2 are disjoint closed subsets of X, are transversely trivial,*

(5) *$E(X \times \{0, 1\}, \boldsymbol{R}^{2n})$ is dense in $C(X \times \{0, 1\}, \boldsymbol{R}^{2n})$,*

(6) *$E(X \times \boldsymbol{C}, \boldsymbol{R}^{2n})$ is dense in $C(X \times \boldsymbol{C}, \boldsymbol{R}^{2n})$, where $\boldsymbol{C}$ is the Cantor set,*

(7) *$E(X, \boldsymbol{R}^{2n})$ is dense in $C(X, \boldsymbol{R}^{2n})$.*

Proof. (1)⇒(2). This follows from 1.3. (2)⇒(3). Trivial. (3)⇒(4). Let $f\colon X \to \boldsymbol{D}^n$ be an extension of the mapping $X_1 \cup X_2 \to \boldsymbol{D}^n$ defined by f_1 and f_2. By the hypothesis there exist mappings $F, G\colon X \to \boldsymbol{D}^n \times \boldsymbol{D}^n$ satisfying the definition of transverse triviality for f. Then the mappings $F_1 = G|X_1$ and $F_2 = G|X_2$ satisfy the definition of transverse triviality for f_1 and f_2. (4)⇒(5). This is lemma 2.1. (5)⇒(6). It follows from (5) that, for each $k \geqslant 1$, the set $E(X \times \{0, \ldots, k\}, \boldsymbol{R}^{2n})$ is dense in $C(X \times \{0, \ldots, k\}, \boldsymbol{R}^{2n})$. Then (6) is a direct consequence of this fact. (6)⇒(7). Trivial. (7)⇒(1). By 1.4 we have (7)⇒(4). Since (4)⇒(5) and (5)⇒(1), the conclusion follows. This completes the proof.

PROBLEM (2). Let X be an n-dimensional compactum, $n \geqslant 2$, such that $\dim X \times X < 2n$. Is it true that every mapping $f\colon X \to \boldsymbol{D}^n$ is transversely trivial?

(2) See footnote (1) on page 247.

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