Some m-dimensional compacta admitting a dense set of imbeddings into R^{2m}

by

Darryl McCullough * and Leonard R. Rubin (Norman, Okla.)

Abstract. For each $m \geq 2$, an m-dimensional compact metric space is constructed for which every map into $2m$-dimensional Euclidean space can be approximated by imbeddings.

In [K–L], J. Krasinkiewicz and K. Lorentz explored the notion of a membrane of a map, which generalizes the concept of a compactum essentially imbedded in I^n introduced in [M–R]. Having discovered an error in the proof of our Lemma 2.7 in [M–R], they gave interesting examples of disjoint essential compacta in I^n disproving the conclusion of Lemma 3.6. This left Theorem 3.7 of [M–R] in doubt, as its proof relied heavily on Lemma 3.6.

Theorem 3.7 purported to characterize the dimension of compacta in the following way:

(*) Let X be a (metric) compactum; then $\dim(X) < m$ if and only if the set of imbeddings is dense in the space of continuous maps from X to $\boldsymbol{R}^{2m}$.

It turns out that Lemma 3.6 of [M–R] is true in the case $m = 1$. This is actually the content of Theorem 3.1 of [M–R], which is reproved as Theorem 3.1 of [K–L]. It follows that the statement (*) is true for $m = 1$.

On the other hand, (*) is false when $m \geq 2$; in fact, we shall prove in this paper the following result:

THEOREM. *For each* $m \geq 2$, *there exists a compactum* X *with* $\dim(X) = m$ *such that every map from* X *to* $\boldsymbol{R}^{2m}$ *can be arbitrarily closely approximated by imbeddings.*

In terms of mapping spaces, this says that the space $\mathscr{E}(X, \boldsymbol{R}^{2m})$ of imbeddings from X to $\boldsymbol{R}^{2m}$ is dense in the space $\mathscr{C}(X, \boldsymbol{R}^{2m})$ of continuous maps to $\boldsymbol{R}^{2m}$. In this paper, we will actually prove that the space of imbeddings to the $2m$-cube I^{2m} is dense in the set of maps to I^{2m}; clearly this is sufficient to prove the Theorem.

---

* Supported in part by NSF Grant DMS-8701666.

1980 *Mathematics Subject Classification*: Primary 54F45; Secondary 54C25, 54E45.

Each space X is constructed as the inverse limit of a sequence of finite polyhedra. Crucial to the construction are certain pairs of disjoint essential compacta M_n and L_n in the $2m$-dimensional ball $D^m \times D^m$. The pairs, constructed and discussed in § 2, are a straightforward generalization of the examples given in [K–L].

In § 1 we present the facts about inverse limits needed for the construction; in particular, Proposition 1.7 shows that the inverse limit of an inverse system (indexed by the natural numbers) of polyhedra will satisfy the conclusion of the Theorem provided that each map of a polyhedron in the system can be approximated, in an appropriate sense, by an imbedding of a polyhedron "farther out" in the system. In § 2, we present the generalization of the examples from [K–L] needed for the construction, in § 3, of an inverse system satisfying the hypothesis of Proposition 1.7. Section 2 also contains some other lemmas needed in that construction.

The existence of the spaces we construct here raises many questions. For example, what is the smallest n such that there is an m-dimensional compactum whose imbeddings into R^n are dense in the maps to R^n? Can the property in the Theorem be characterized topologically? Is this property related to the phenomenon of m-dimensional compacta whose products have dimension less than $2m$, whose construction [B] is somewhat similar? We hope that these and related questions can now be approached.

1. Inverse limits. Let $\underline{X} = \{X_k, \varrho_{k,j}\}$ be an inverse sequence of compact polyhedra X_k and (continuous) maps $\varrho_{k,j}\colon X_j \to X_k$. Assume that each X_k has a fixed metric $d = d_k$. Let $X = \lim \underline{X}$; then X is a metrizable compactum and we provide it with a metric $d = d_\infty$. Let $\varrho_k\colon X \to X_k$ denote the projection.

PROPOSITION 1.1. *If for all $\varepsilon > 0$, the set of ε-maps from X to I^N is dense in $\mathscr{C}(X, I^N)$, then $\mathscr{E}(X, I^N)$ is a dense G_δ in $\mathscr{C}(X, I^N)$.*

This proposition follows from the content of Chapter V, § 3 of [H–W]. The next is 27.9 of [N].

PROPOSITION 1.2. *If $\dim(X_k) \leqslant m$ for all k, then $\dim(X) \leqslant m$.*

PROPOSITION 1.3. *For every $\varepsilon > 0$ and for each compact* ANR *P and map $h\colon X \to P$, there exists k and a map $f\colon X_k \to P$ such that $d(f \circ \varrho_k, h) < \varepsilon$.*

For Proposition 1.3, see (CR1) of Theorem 8 in Chapter I, § 5.2 of [M–S].

PROPOSITION 1.4. *Suppose that for each k, $S^{m-1} \subseteq X_k$. Suppose further that for $j \geqslant k$, $\varrho_{k,j}|S^{m-1} = \mathrm{id}\colon S^{m-1} \to S^{m-1}$ and that X_k does not retract to S^{m-1}. Then $\dim(X) \geqslant m$.*

Proof. Clearly the sequence $\underline{S} = \{S^{m-1}, \varrho_{k,j}|S^{m-1}\colon S^{m-1} \to S^{m-1}\}$ is an inverse sequence whose limit S^{m-1} is a closed subspace of X. We need only show that $\mathrm{id}\colon S^{m-1} \to S^{m-1}$ does not extend to a map $h\colon X \to S^{m-1}$. Suppose, to the contrary, that such a map h does exist. Choose $\varepsilon > 0$ such that if $f\colon S^{m-1} \to S^{m-1}$ is any map and $d(f, h) < \varepsilon$, then $f \simeq h$. Apply Proposition 1.3 to get an index k and a map $g\colon X_k \to S^{m-1}$ with $d(g \circ \varrho_k, h) < \varepsilon$. Since all $\varrho_{k,j}|S^{m-1}$ are the identity on S^{m-1}, the map $\varrho_k|S^{m-1}$ is also the identity. We conclude that $d(g|S^{m-1}, h|S^{m-1}) < \varepsilon$, so that $g|S^{m-1} \simeq \mathrm{id}$ on S^{m-1}. The homotopy extension property then implies that $\mathrm{id}\colon S^{m-1} \to S^{m-1}$ extends to a map of X_k to S^{m-1}, contradicting the hypothesis.

From Theorem 5 of Chapter I, § 5.2 of [M–S], we obtain the following.

PROPOSITION 1.5. *For each open cover $\mathscr{U}$ of X, there exists a k_0 and an open cover $\mathscr{V}$ of X_{k_0} such that $\varrho_{k_0}^{-1}(\mathscr{V})$ refines $\mathscr{U}$.*

It follows that if $k \geqslant k_0$ in the preceding, then the open cover $\mathscr{V}_0 = \varrho_{k_0,k}^{-1}(\mathscr{V})$ of X_k is such that $\varrho_k^{-1}(\mathscr{V}_0)$ refines $\mathscr{U}$. We therefore have the following.

PROPOSITION 1.6. *For all $\varepsilon > 0$ there exists k_0 such that if $k \geqslant k_0$, then $\varrho_k\colon X \to X_k$ is an ε-map.*

PROPOSITION 1.7. *Let N be a fixed positive integer. Suppose that for each k and for each map $h\colon X_k \to I^N$ and for each $\delta > 0$, there exists $l \geqslant k$ and an imbedding $\gamma\colon X_l \to I^N$ such that $h \circ \varrho_{k,l}\colon X_k \to I^N$ is δ-close to γ. Then $\mathscr{E}(X, I^N)$ is a dense G_δ in $\mathscr{C}(X, I^N)$.*

Proof. We shall show that if $\varepsilon, \delta > 0$ and $f\colon X \to I^N$ is a map, then there exists an ε-map $g\colon X \to I^N$ with $d(f, g) < \delta$. The result will then follow from Proposition 1.1.

Using Proposition 1.6, choose k_0 so that if $k \geqslant k_0$, then $\varrho_k\colon X \to X_k$ is an ε-map. Next, apply Proposition 1.3 to find $k \geqslant k_0$ and a map $h\colon X_k \to I^N$ such that $d(h \circ \varrho_k, f) < \delta/2$.

Choose $l \geqslant k$ and γ as in the hypothesis so that $\gamma\colon X_l \to I^N$ is an imbedding, and $h \circ \varrho_{k,l}$ is $(\delta/2)$-close to γ.

Since $l \geqslant k \geqslant k_0$, then $\varrho_l\colon X \to X_l$ is an ε-map, and since γ is an imbedding, then $\gamma \circ \varrho_l$ is also an ε-map. Using commutativity in the inverse system, $h \circ \varrho_{k,l} \circ \varrho_l = h \circ \varrho_k$. Since $d(h \circ \varrho_k, f) < \delta/2$, it follows that $d(h \circ \varrho_{k,l} \circ \varrho_l, f) < \delta/2$.

On the other hand, $d(h \circ \varrho_{k,l}, \gamma) < \delta/2$ implies that $d(h \circ \varrho_{k,l} \circ \varrho_l, \gamma \circ \varrho_l) < \delta/2$. Therefore $d(f, \gamma \circ \varrho_l) < \delta$. This completes the proof.

2. Auxiliary results. This section contains lemmas needed for the construction of the inverse system in § 3. We begin with a straightforward generalization of the fundamental examples from [K–L] to higher dimensions.

Fix an integer $m \geqslant 2$. For each integer n, fix a simplicial map $\gamma_n\colon S^{m-1} \to S^{m-1}$ of degree n. Define M_n to be the mapping cylinder of γ_n and L_n to be the mapping cone of γ_n. We regard M_n as formed from the disjoint union $S^{m-1} \times I \cup S^{m-1}$ by identifying $(x, 0)$ to $\gamma_n(x)$ for every $x \in S^{m-1}$, and L_n as formed from the disjoint union $D^m \cup S^{m-1}$ by identifying x to $\gamma_n(x)$ for every $x \in S^{m-1} = \partial D^m$. Denote by $\partial_0 M_n$ the image of $S^{m-1} \times \{1\}$ in M_n, and by $\partial_0 L_n$ the image of S^{m-1} in L_n. Note that $\partial_0 M_n$ and $\partial_0 L_n$ are topologically $(m-1)$-spheres.

LEMMA 2.1. *A map $f\colon \partial_0 M_n \to S^{m-1}$ extends to M_n if and only if $\deg(f)$ is divisible by n.*

Proof. Let $q\colon S^{m-1} \times I \to M_n$ denote the quotient map. Suppose that f extends to a map $g\colon M_n \to S^{m-1}$. Let $j_t\colon S^{m-1} \to S^{m-1} \times \{t\}$, $(t \in [0, 1])$ be defined by

$j_t(x) = (x, t)$. Let q_t denote the restriction of q to $S^{m-1} \times \{t\}$ and let g_t denote the restriction of g to the image of q_t. We have $f \circ q_1 \circ j_1 = g_1 \circ q_1 \circ j_1 \simeq g_0 \circ q_0 \circ j_0$. This, together with the fact that q_1, j_1, and j_0 are homeomorphisms, implies that

$$|\deg(f)| = |\deg(f \circ q_1 \circ j_1)| = |\deg(g_0 \circ q_0 \circ j_0)|$$
$$= |\deg(g_0 \circ q_0)| = |\deg(q_0)|\,|\deg(g_0)| = n|\deg(g_0)|\,.$$

Therefore $\deg(f)$ is divisible by n.

Conversely, suppose that $\deg(f) = kn$. Then $f \simeq \gamma_k \circ \gamma_n$, so f extends to M_n if and only if $\gamma_k \circ \gamma_n$ does. But the composite $\gamma_k \circ \gamma_n \circ \mathrm{proj}_{S^{m-1}}\colon S^{m-1} \times I \to S^{m-1}$ factors through M_n, providing the extension. This completes the proof.

LEMMA 2.2. *For each n, there are imbeddings $\varphi\colon M_n \to D^m \times D^m$ and $\psi\colon L_n \to D^m \times D^m$ with the following properties*:

(i) *$\varphi(M_n)$ and $\psi(L_n)$ are disjoint, and*

(ii) *φ carries $\partial_0 M_n$ homeomorphically onto $S^{m-1} \times \{0\}$, and ψ carries $\partial_0 L_n$ homeomorphically onto $\{0\} \times S^{m-1}$.*

Proof. Using the idea of [K–L], we define $\varphi'\colon M_n \to D^m \times D^m$ by $\varphi'([z, t]) = (tz, (1/2+t/2)\gamma_n(z))$. Next, extend γ_n to $\overline{\gamma_n}\colon D^m \to D^m$ by letting $\overline{\gamma_n}(tz) = t\gamma_n(z)$ for $z \in S^{m-1}$, then define $\psi'\colon D^m \to D^m \times D^m$ by $\psi'(y) = (y, -\overline{\gamma_n}(y))$. It is easy to see that these are imbeddings. Moreover,

(1) $$\varphi'(M_n) \cap \psi'(D^m) \text{ is empty}\,.$$

For suppose that $\varphi'([t, z]) = \psi'(y)$. Then we have

$$\frac{1}{2} + \frac{t}{2} = \left|\left(\frac{1}{2} + \frac{t}{2}\right)\gamma_n(z)\right| = |-\overline{\gamma}_n(y)| = |y| = |tz| = t$$

and hence $t = 1$ and $y = z$. But then, $\gamma_n(z) = -\overline{\gamma_n}(z) = -\gamma_n(z)$, a contradiction.

Note also that

(2) $$\varphi'(M_n) \cap \partial(D^m \times D^m) \subseteq S^{m-1} \times S^{m-1}\,,$$

(3) $$\psi'(D^m) \cap \partial(D^m \times D^m) \subseteq S^{m-1} \times S^{m-1}\,.$$

Now, define

$$\varphi([z, t]) = \begin{cases} \varphi'([z, 2t]) & \text{if } 0 \leqslant t \leqslant 1/2\,, \\ (z, (2-2t)\gamma_n(z)) & \text{if } 1/2 \leqslant t \leqslant 1\,, \end{cases}$$

and

$$\psi([y]) = \begin{cases} \psi'(2y) & \text{if } 0 \leqslant |y| \leqslant 1/2\,, \\ \left((2-2|y|)\dfrac{y}{|y|}, -\gamma_n\left(\dfrac{y}{|y|}\right)\right) & \text{if } 1/2 \leqslant |y| \leqslant 1\,. \end{cases}$$

It is easily checked that φ and ψ satisfy (ii). Also, we observe that

(4) $$\varphi(M_n) = \varphi'(M_n) \cup (\varphi(M_n) \cap (S^{m-1} \times D^m))\,,$$
$$\varphi(M_n) \cap S^{m-1} \times S^{m-1} = \varphi'(\partial_0 M_n)\,,$$

(5) $$\psi(L_n) = \psi'(D^m) \cup (\psi(L_n) \cap (D^m \times S^{m-1}))\,,$$
$$\psi(L_n) \cap S^{m-1} \times S^{m-1} = \psi'(\partial D^m)\,.$$

Property (i) follows from (4), (5), and (1). This completes the proof of Lemma 2.2.

Suppose that Z is a finite polyhedron. Define $\mathscr{F}_Z$ to be the space of maps from Z to $D^m \times D^m$.

LEMMA 2.3. *$\mathscr{F}_Z$ contains a countable dense set.*

Proof. Choose triangulations K of Z and L of $D^m \times D^m$. The set of simplicial maps from some barycentric subdivision of K to some barycentric subdivision of L is countable. By the Simplicial Approximation Theorem [S, Theorem 3.4.8], this set of simplicial maps is dense in the set of all maps.

LEMMA 2.4. *Let Z be an m-dimensional finite simplicial complex containing a subcomplex of Z^{m-1} which has a fixed identification with S^{m-1}. Suppose that Z does not retract to S^{m-1}. Let B be a closed m-ball in $Z - Z^{(m-1)}$. Then there exists an integer $n \geqslant 2$ such that the identification space $(Z - \mathrm{int}(B)) \cup M_n$, obtained by identifying ∂B with $\partial_0 M_n$, does not retract to S^{m-1}.*

Proof. Orient ∂B arbitrarily. We will first show that either the set

$$D = \{d \mid \text{there is a retraction from } Z - \mathrm{int}(B) \text{ to } S^{m-1} \text{ whose restriction to } \partial B \text{ has degree } d\}$$

is empty, or it consists of a single nonzero integer, or there are integers $0 < a < b$ so that $D = \{a + nb \mid n \in Z\}$.

Write S for ∂B and Z_0 for $Z - \mathrm{int}(B)$. If $f\colon Z_0 \to S^{m-1}$ is a retraction, define $d(f)$ to be the degree of the restriction of f to S. The degree of this restriction must be nonzero, otherwise f would extend to Z contrary to hypothesis.

We claim that if f and g are retractions from Z_0 to S^{m-1}, with $d(f) = r$ and $d(g) = s$, then for any integer n there is a retraction h_n from Z_0 to S^{m-1} such that $d(h_n) = r + n(s-r)$. Triangulate Z_0 and give each simplex an orientation. Choose a vertex $s_0 \in S^{m-1}$. Since S^{m-1} is $(m-2)$-connected, there are maps f' and g', homotopic to f and g respectively, that carry the entire $(m-2)$-skeleton of Z_0 to s_0. For each $(m-1)$-simplex σ of Z_0, define $c_{f'}(\sigma)$ and $c_{g'}(\sigma)$ to be the degrees of the restrictions of f' and g' to σ (regarded as maps from $(\sigma, \partial\sigma)$ to (S^{m-1}, s_0)). Now if Y is a subcomplex of Z_0 such that $|Y|$ is homeomorphic to the $(m-1)$-sphere, and $[Y] = \sum \varepsilon_i \sigma_i$ is a fundamental class determining an orientation for Y, where each σ_i is an $(m-1)$-simplex of Z_0, then the degree of the restriction of f to Y is equal to $\sum \varepsilon_i c_{f'}(\sigma_i)$. Therefore

(1) If τ is any m-simplex of Z_0, then since the restrictions of f' and g' to $\partial\tau$ extend to τ, we have $c_{f'}(\partial\tau) = c_{g'}(\partial\tau) = 0$.

(2) Since the restrictions of f' and g' to S are of degrees r and s respectively, we have $c_{f'}([S]) = r$ and $c_{g'}([S]) = s$.

(3) Since f' and g' are homotopic to retractions to S^{m-1}, we have $c_{f'}([S^{m-1}]) = c_{g'}([S^{m-1}]) = 1$.

Now for any fixed integer n, define $h'_n\colon Z_0^{(m-1)} \to S^{m-1}$ by sending the $(m-2)$-skeleton to s_0 and sending each oriented $(m-1)$-simplex σ by a map of degree $nc_{g'}(\sigma)-(n-1)c_{f'}(\sigma)$. By (1), h'_n extends to all of Z_0. By (2), the degree of h'_n on S is $ns-(n-1)r = r+n(s-r)$. And by (3), the degree of the restriction of h'_n to S^{m-1} is $n-(n-1) = 1$, hence by homotopy extension h'_n is homotopic to a retraction h_n. This completes the proof of the claim.

Assume that there is more than one possibility for the degree of the restriction to S of a retraction from Z_0 to S^{m-1}. Let b be the smallest positive integer for which there are retractions f_1 and f_2 so that $d(f_2)-d(f_1) = b$. By the claim, each of the integers in the set $\{d(f_1)+nb|\ n \in Z\}$ is the degree of the restriction to S of some retraction; since 0 is not a possible degree, this set has the form described in the statement of Lemma 2.4. Since b was chosen to be minimal, no other degrees are possible. This completes the proof that the set D has the stated form.

To complete the proof of Lemma 2.4, choose a positive integer n which does not divide any element of D. Suppose f is any retraction from $Z-\mathrm{int}(B)$ to S^{m-1}. Since the degree of the restriction of f to S is not divisible by n, Lemma 2.1 shows that f cannot extend to M_n.

3. The construction. In this section, we construct for each $m \geqslant 2$ an inverse system $\underline{X} = \{X_k, \varrho_{j,k}\}$ of finite m-dimensional polyhedra whose limit X has dimension m and such that $\underline{X}$ satisfies the hypotheses of Proposition 1.7 with $N = 2m$. This will complete the proof of the Theorem stated in the introduction.

Fix an integer $m \geqslant 2$. Let $X_1 = D^m$. Using Lemma 2.3, choose a sequence $\alpha_{1,k}\colon X_1 \to D^m \times D^m$ such that $\{\alpha_{1,k}\}_{k=1}^{\infty}$ is dense in $\mathscr{F}_{X_1}$.

Suppose inductively that $l \geqslant 1$ and there are chosen finite polyhedra $X_1, X_2, \ldots, X_l$ and maps $\varrho_{j-1,j}\colon X_j \to X_{j-1}$ for $j = 2, 3, \ldots, l$. Let $\varrho_{k,k}\colon X_k \to X_k$ equal the identity map, and for $1 \leqslant j < k \leqslant l$, let

$$\varrho_{j,k} = \varrho_{j,j+1} \circ \varrho_{j+1,j+2} \circ \ldots \circ \varrho_{k-1,k}\colon X_k \to X_j .$$

Suppose further that the following hold:

(1_l) For $1 \leqslant j < l$, there is a fixed sequence $\alpha_{j,k}\colon X_j \to D^m \times D^m$ such that $\{\alpha_{j,k}\}_{k=1}^{\infty}$ is dense in $\mathscr{F}_{X_j}$.

(2_l) X_j contains S^{m-1}, X_j does not retract to S^{m-1}, and the restriction of each $\varrho_{j,k}$ to S^{m-1} is the identity map for $1 \leqslant j < k \leqslant l$.

(3_l) $\dim(X_j) = m$ for $1 \leqslant j \leqslant l$.

(4_l) For $j+k \leqslant i \leqslant l$, there is selected an imbedding $\gamma^i_{j,k}\colon X_j \to D^m \times D^m$ such that $d(\gamma^i_{j,k}, \alpha_{j,k} \circ \varrho_{j,i}) < 1/2^i$.

Now choose a dense sequence $\{\alpha_{l,k}\}_{k=1}^{\infty}$ in $\mathscr{F}_{X_l}$.

We wish to construct a finite polyhedron X_{l+1} and a map $\varrho_{l,l+1}\colon X_{l+1} \to X_l$ so that when we adjoin X_{l+1} to the collection $X_1, X_2, \ldots, X_l$, the corresponding properties (1_{l+1}), (2_{l+1}), (3_{l+1}), and (4_{l+1}) will be true.

Write Y_0^l for X_l. Consider the finite collection of maps

$$\{\alpha_{j,k} \circ \varrho_{j,l}|\ j+k \leqslant l+1\} \subseteq \mathscr{F}_{Y_0^l} .$$

Choose self-transverse piecewise-linear approximations $\delta^{l,0}_{j,k}$ of $\alpha_{j,k} \circ \varrho_{j,l}$ (see, for example, [R–S]) so that

(5_l) $d(\delta^{l,0}_{j,k}, \alpha_{j,k} \circ \varrho_{j,l}) < 1/2^{l+2}$.

(6_l) No point in $D^m \times D^m$ is a double point for more than one $\delta^{l,0}_{j,k}$.

The double points $\{z_1, z_2, \ldots, z_t\}$ of the $\delta^{l,0}_{j,k}$ lie in the interior of $D^m \times D^m$. Each z_r arises from a pair of distinct points $\{p_r, q_r\}$ in $Y_0^l - (Y_0^l)^{(m-1)}$. Choose disjoint closed neighborhoods N_r of the z_r in $D^m \times D^m$ and pairwise disjoint neighborhoods P_r of p_r and Q_r of q_r in $Y_0^l - (Y_0^l)^{(m-1)}$ so that

(7) $\mathrm{diam}(N_r) < 1/2^{l+2}$ and N_r is homeomorphic to $B^m \times B^m$ where B^m is the m-ball.

(8) For each r, if z_r is a double point of $\delta^{l,0}_{j,k}$ then $\delta^{l,0}_{j,k}$ carries P_r homeomorphically onto $B^m \times \{0\} \subseteq N_r$ and Q_r homeomorphically onto $\{0\} \times B^m$.

(9) For each r, if z_r is not a double point of $\delta^{l,0}_{j,k}$ then there is a closed neighborhood $V^r_{j,k}$ of $\delta^{l,0}_{j,k}(p_r)$ in the interior of $D^m \times D^m$ such that $\mathrm{diam}(V^r_{j,k}) < 1/2^{l+2}$, the preimage under $\delta^{l,0}_{j,k}$ of $V^r_{j,k}$ is P_r, and $V^r_{j,k}$ is homeomorphic to $B^m \times B^m$ so that $\delta^{l,0}_{j,k}$ carries P_r homeomorphically onto $B^m \times \{0\}$.

(10) For each r, if z_r is not a double point of $\delta^{l,0}_{j,k}$ then there is a closed neighborhood $W^r_{j,k}$ of $\delta^{l,0}_{j,k}(q_r)$ in the interior of $D^m \times D^m$ such that $\mathrm{diam}(W^r_{j,k}) < 1/2^{l+2}$, the preimage under $\delta^{l,0}_{j,k}$ of $W^r_{j,k}$ is Q_r, and $W^r_{j,k}$ is homeomorphic to $B^m \times B^m$ so that $\delta^{l,0}_{j,k}$ carries Q_r homeomorphically onto $\{0\} \times B^m$.

After slight further adjustment of the maps $\delta^{l,0}_{j,k}$, we may assume that the closed neighborhoods $V^r_{j,k}$ and $W^r_{j,k}$ are pairwise disjoint.

Consider $Y_0^l - \mathrm{int}(P_1)$ and let $S = \partial P_1 \subset Y_0^l$. Now Y_0^l does not retract to S^{m-1}, so by Lemma 2.4 there exists an n so that the space Z_1^l obtained from $Y_0^l - \mathrm{int}(P_1)$ by attaching a copy of M_n using a homeomorphism of $\partial_0 M_n$ with ∂P_1 does not retract to S^{m-1}.

Next, form Y_1^l by attaching a copy of L_n to $Z_1^l - \mathrm{int}(Q_1)$ using a homeomorphism of $\partial_0 L_n$ with ∂Q_1. Again, S^{m-1} may be regarded as a subspace of Y_1^l, and Y_1^l does not retract to S^{m-1} since any retraction of $Z_1^l - \mathrm{int}(Q_1)$ to S^{m-1} cannot have degree zero on ∂Q_1 and so cannot extend to Y_1^l.

Consider the imbeddings $\varphi\colon M_n \to B^m \times B^m$ and $\psi\colon L_n \to B^m \times B^m$ as given in Lemma 2.2. For each pair j, k with $j+k \leqslant l+1$, define $\delta^{l,1}_{j,k}\colon Y_1^l \to D^m \times D^m$ as follows. On the subspace $Y_0^l - (P_1 \cup Q_1)$, $\delta^{l,1}_{j,k} = \delta^{l,0}_{j,k}$. On M_n, if z_1 is a singular point of $\delta^{l,0}_{j,k}$, then $\delta^{l,1}_{j,k} = \varphi$, carrying P_1 into N_1, while if z_1 is not a singular point of $\delta^{l,0}_{j,k}$, then $\delta^{l,1}_{j,k} = \varphi$, carrying P_1 into $V^1_{j,k}$. On Q_1, the definition is similar using ψ. Coordinates are selected on these pieces so that these maps fit together. The singular image of each $\delta^{l,1}_{j,k}$ is contained in $\{z_2, z_3, \ldots, z_{t_l}\}$. There is a projection map $\varrho^l_{0,1}$: $Y^l_1 \to Y^l_0$ defined by extending the identity map on $Y^l_0 - \operatorname{int}(P_1 \cup Q_1)$ using maps from $(M_n, \partial_0 M_n)$ to $(P_1, \partial P_1)$ and from $(L_n, \partial_0 L_n)$ to $(Q_1, \partial Q_1)$. Since the diameters of N_1, $W^1_{j,k}$, and $V^1_{j,k}$ are all less than $1/2^{l+2}$, we have

$$d(\delta^{l,1}_{j,k}, \alpha_{j,k} \circ \varrho_{j,l} \circ \varrho^l_{0,1}) \leqslant d(\delta^{l,1}_{j,k}, \delta^{l,0}_{j,k} \circ \varrho^l_{0,1}) + d(\delta^{l,0}_{j,k} \circ \varrho^l_{0,1}, \alpha_{j,k} \circ \varrho_{j,l} \circ \varrho^l_{0.1})$$
$$< 1/2^{l+2} + 1/2^{l+2} = 1/2^{l+1}.$$

Repeating this procedure, starting with Y^l_1 and replacing $\delta^{l,0}_{j,k}$ by $\delta^{l,1}_{j,k}$, we obtain a sequence $Y^l_1, Y^l_2, \ldots, Y^l_{t_l}$ of m-dimensional finite complexes, corresponding maps $\delta^{l,i}_{j,k}$: $Y^l_i \to D^m \times D^m$, and projections $\varrho^l_{i-1,i}$: $Y^l_i \to Y^l_{i-1}$ such that the distance between $\delta^{l,t_l}_{j,k}$ and the composite $\alpha_{j,k} \circ \varrho_{j,l} \circ \varrho^l_{0,1} \circ \varrho^l_{1,2} \circ \ldots \circ \varrho^l_{t_l-1,t_l}$ is less than $1/2^{l+1}$. Put $X_{l+1} = Y^l_{t_l}$, $\gamma^{l+1}_{j,k} = \delta^{l,t_l}_{j,k}$, and $\varrho_{l,l+1} = \varrho^l_{0,1} \circ \varrho^l_{1,2} \circ \ldots \circ \varrho^l_{t_l-1,t_l}$. Note that the restriction of $\varrho_{l,l+1}$ to S^{m-1} is the identity, and $\gamma^{l+1}_{j,k}$: $X_{l+1} \to D^m \times D^m$ is an imbedding. We have thus established the desired properties.

Now consider the inverse sequence $\underline{X} = \{X_k, \varrho_{k,j}\}$ just constructed. Let $X = \lim \underline{X}$. Applying Proposition 1.2 to (3_l), we see that $\dim(X) \leqslant m$. On the other hand, Proposition 1.4 and (2_l) yield that $\dim(X) \geqslant m$. Therefore

$$\dim(X) = m. \tag{11}$$

Let us now check that the hypotheses of Proposition 1.7 are satisfied with $N = 2m$. Identify $D^m \times D^m$ with I^{2m}. Let k and a map h: $X_k \to D^m \times D^m$ and $\delta > 0$ be given. Using (1_{k+1}), there is an $\alpha_{k,s}$: $X_k \to D^m \times D^m$ such that

$$d(\alpha_{k,s}, h) < \delta/2. \tag{12}$$

Choose l so large that $k+s \leqslant l$ and $1/2^l < \delta/2$. According to (4_l), there is an imbedding $\gamma^l_{k,s}$: $X_l \to D^m \times D^m$ such that

$$d(\gamma^l_{k,s}, \alpha_{k,s} \circ \varrho_{k,l}) < \delta/2. \tag{13}$$

From (12), we get

$$d(\alpha_{k,s} \circ \varrho_{k,l}, h \circ \varrho_{k,l}) < \delta/2. \tag{14}$$

Then (13) and (14) combine to yield the desired condition that $d(\gamma^l_{k,s}, h \circ \varrho_{k,l}) < \delta$.

References

[B] K. Borsuk, *Concerning the Cartesian product of Cantor-manifolds*, Fund. Math. 38 (1951), 57–72.

[H–W] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, N. J. 1941.

[K–L] J. Krasinkiewicz and K. Lorentz, *Disjoint membranes in cubes*, preprint.

[M–S] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam 1982.

[M–R] D. McCullough and L. Rubin, *Intersections of separators and essential submanifolds of I^N*, Fund. Math. 116 (1983), 131–142.

[N] K. Nagami, *Dimension Theory*, Academic Press, New York 1970.

[R–S] C. Rourke and B. Sanderson, *Introduction to Piecewise-Linear Topology*, Springer-Verlag, New York 1972.

[S] E. Spanier, *Algebraic Topology*, Mc Graw-Hill, New York 1966.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OKLAHOMA
Norman, Oklahoma 73019
U.S.A.

Received 25 February 1988;
in revised form 16 May 1988