Some \( m \)-dimensional compacta admitting a dense set of imbeddings into \( R^{2m} \)

by

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Abstract. For each \( m \geq 2 \), an \( m \)-dimensional compact metric space is constructed for which every map into \( 2m \)-dimensional Euclidean space can be approximated by imbeddings.

In [K-L], J. Krasinkiewicz and K. Lorentz explored the notion of a membrane of a map, which generalizes the concept of a compactum essentially imbedded in \( f^p \) introduced in [M-R]. Having discovered an error in the proof of our Lemma 2.7 in [M-R], they gave interesting examples of disjoint essential compacta \( x^p \) disapproving the conclusion of Lemma 3.6. This left Theorem 3.7 of [M-R] in doubt, as its proof relied heavily on Lemma 3.6.

Theorem 3.7 purported to characterize the dimension of compacta in the following way:

\((\ast)\) Let \( X \) be a (metric) compactum; then \( \dim(X) < m \) if and only if the set of imbeddings is dense in the space of continuous maps from \( X \) to \( R^{2m} \).

It turns out that Lemma 3.6 of [M-R] is true in the case \( m = 1 \). This is actually the content of Theorem 3.1 of [M-R], which is reproved as Theorem 3.1 of [K-L].

It follows that the statement \((\ast)\) is true for \( m = 1 \).

On the other hand, \((\ast)\) is false when \( m \geq 2 \); in fact, we shall prove in this paper the following result:

**Theorem.** For each \( m \geq 2 \), there exists a compactum \( X \) with \( \dim(X) = m \) such that every map from \( X \) to \( R^{2m} \) can be arbitrarily closely approximated by imbeddings.

In terms of mapping spaces, this says that the space \( \mathcal{E}(X, R^{2m}) \) of imbeddings from \( X \) to \( R^{2m} \) is dense in the space \( \mathcal{C}(X, R^{2m}) \) of continuous maps to \( R^{2m} \). In this paper, we will actually prove that the space of imbeddings to the \( 2m \)-cube \( I^{2m} \) is dense in the set of maps to \( I^{2m} \); clearly this is sufficient to prove the Theorem.

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Each space $X$ is constructed as the inverse limit of a sequence of finite polyhedra. Crucial to the construction are certain pairs of disjoint essential compacta $M_\epsilon$ and $L_\epsilon$ in the 2m-dimensional ball $D^n \times D^n$. The pairs, constructed and discussed in §2, are a straightforward generalization of the examples given in [K-L].

In §1 we present the facts about inverse limits needed for the construction; in particular, Proposition 1.7 shows that the inverse limit of an inverse system (indexed by the natural numbers) of polyhedra will satisfy the conclusion of the Theorem provided that each map of a polyhedron in the system can be approximated, in an appropriate sense, by an imbedding of a polyhedron "farther out" in the system. In §2 we present the generalization of the examples from [K-L] needed for the construction, in §3, of an inverse system satisfying the hypothesis of Proposition 1.7. Section 2 also contains some other lemmas needed in that construction.

The existence of the spaces we construct here raises many questions. For example, what is the smallest $n$ such that there is an $m$-dimensional compactum whose imbeddings into $R^n$ are dense in the maps to $R^n$? Can the property in the Theorem be characterized topologically? Is this property related to the phenomenon of $m$-dimensional compacta whose products have dimension less than $2m$, whose construction [B] is somewhat similar? We hope that these and related questions can now be approached.

1. Inverse limits. Let $X = (X_t, \epsilon_t)$ be an inverse sequence of compact polyhedra $X_t$ and (continuous) maps $\theta_t^j: X_t \to X_t$. Assume that each $X_t$ has a fixed metric $d = d_t$. Let $X = \lim X_t$; then $X$ is a metrizable compactum and we provide it with a metric $d = d_\theta$. Let $\theta_0: X_0 \to X_0$ denote the projection.

Proposition 1.1. If for all $\epsilon_0 > 0$, the set of $\epsilon$-maps from $X$ to $X^\epsilon$ is dense in $\mathfrak{Q}(X, X^\epsilon)$, then $\delta(X, X^\epsilon)$ is a dense $G_\delta$ in $\mathfrak{Q}(X, X^\epsilon)$.

This proposition follows from the contents of Chapter V, §3 of [H-W]. The next is 27.9 of [N].

Proposition 1.2. If $\dim(X_0) \leq m$ for all $\epsilon_0$, then $\dim(X) \leq m$.

Proposition 1.3. For every $\epsilon > 0$ and for each compact ANR $P$ and map $h: X \to P$, there exists $k$ and a map $f: X_k \to P$ such that $d(f \circ \theta_0, h) < \epsilon$.

For Proposition 1.3, see (CRI) of Theorem 8 in Chapter I, §5.2 of [M-S].

Proposition 1.4. Suppose that for each $k$, $S^{n-1} \times X_k$. Suppose further that for $j > k$, $\theta_j|S^{n-1} = \epsilon_0|S^{n-1} =: \epsilon: S^{n-1} \to S^{n-1}$ and that $X_0$ does not retract to $S^{n-1}$. Then $\dim(X) \geq m$.

Proof. Clearly, the sequence $S = \{S^{n-1}, \theta_k|S^{n-1}: S^{n-1} \to S^{n-1}\}$ is an inverse sequence whose limit $S^{n-1}$ is a closed subspace of $X$. We need only show that $S^{n-1} \to S^{n-1}$ does not extend to a map $k: X \to S^{n-1}$. Suppose, to the contrary, that such a map $k$ does exist. Choose $\epsilon > 0$ such that if $S^{n-1} \to S^{n-1}$ is any map and $d(f, h) < \epsilon$, then $f = h$. Apply Proposition 1.3 to get an index $k$ and a map $f: X_k \to S^{n-1}$ with $d(g \circ \theta_0, h) < \epsilon$. Since all $\theta_0|S^{n-1}$ are the identity on $S^{n-1}$, the map $\theta_0|S^{n-1}$ is also the identity. We conclude that $d(g|S^{n-1}, h|S^{n-1}) < \epsilon$.

so that $g|S^{n-1} \simeq id$ on $S^{n-1}$. The homotopy extension property then implies that $id: S^{n-1} \to S^{n-1}$ extends to a map of $X_0$ to $S^{n-1}$, contradicting the hypothesis.

From Theorem 5 of Chapter I, §5.2 of [M-S], we obtain the following.

Proposition 1.5. For each open cover $\mathcal{U}$ of $X$, there exists a $k_0$ and an open cover $\mathcal{V}$ of $X_{k_0}$ such that $\mathcal{U} = \mathcal{W}_k(\mathcal{V})$ refines $\mathcal{U}$.

It follows that if $k \geq k_0$, then the open cover $\mathcal{U}(\mathcal{V})$ refines $\mathcal{V}$. Thus it is true if $k 

Proposition 1.6. For all $\epsilon > 0$ there exists a $k_0$ such that if $k \geq k_0$, then $q_0: X \to X$ is an $\epsilon$-map.

Proposition 1.7. Let $N$ be a fixed positive integer. Suppose that for each $k$ and for each map $h: X_k \to I^n$ and for each $\delta > 0$, there exists $l \geq k$ and an imbedding $\gamma: X_l \to I^n$ such that $h \circ \theta_l: X_l \to I^n$ is $\delta$-close to $\gamma$. Then $d(X, I^n)$ is a dense $G_\delta$ in $\mathfrak{Q}(X, I^n)$.

Proof. We shall show that if $\epsilon > 0$ and $f: X \to I^n$ is a map, then there exists an $\epsilon$-map $g: X \to I^n$ with $d(f, g) < \epsilon$. The result will then follow from Proposition 1.1.

Using Proposition 1.6, choose $k_0$ so that if $k \geq k_0$, then $q_0: X \to X_l$ is an $\epsilon$-map. Next, apply Proposition 1.3 to find $k \geq k_0$ and a map $h: X \to I^n$ such that $d(h \circ \theta_k, f) < \epsilon/2$.

Choose $l \geq k$ and $\gamma$ as in the hypothesis so that $\gamma: X_l \to I^n$ is an imbedding, and $h \circ \theta_l$ is $(\epsilon/2)$-close to $\gamma$. Since $l \geq k \geq k_0$, then $q_0: X \to X_l$ is an $\epsilon$-map, and since $\gamma$ is an imbedding, then $\gamma = q_0$ is also an $\epsilon$-map. Using compactness in the inverse system, $h \circ \theta_k \circ \theta_l \simeq h \circ \theta_k \circ \theta_l \simeq h \circ \theta_l$, and by Proposition 1.6 we have $d(h \circ \theta_k \circ \theta_l, \gamma) < \epsilon/2$ implies that $d(h \circ \theta_k \circ \theta_l, \gamma) < \epsilon/2$. Therefore $d(f, g) < \epsilon$. This completes the proof.

2. Auxiliary results. This section contains lemmas needed for the construction of the inverse system in §3. We begin with a straightforward generalization of the fundamental examples from [K-L] to higher dimensions.

Fix an integer $m \geq 2$. For each integer $n$, fix a simplicial map $\gamma: S^{n-1} \to S^{n-1}$ of degree $n$. Define $M_0$ to be the mapping cone of $\gamma$, and $L_0$ to be the mapping cone of $\gamma$. We regard $\gamma$ as formed from the disjoint union $S^{n-1} \times I \cup S^{n-1}$ by identifying $(x, 0) \simeq \gamma(x)$ for every $x \in S^{n-1}$, and $L_0$ as formed from the disjoint union $D^n \cup S^{n-1}$ by identifying $x \simeq \gamma(x)$ for every $x \in S^{n-1} = D^n$. Denote by $\delta_0 M_0$ the image of $S^{n-1} \times \{1\}$ in $M_0$, and by $\delta_0 L_0$ the image of $S^{n-1}$ in $L_0$. Note that $\delta_0 M_0$ and $\delta_0 L_0$ are topologically $(m-1)$-spheres.

Lemma 2.1. A map $f: \delta_0 M_0 \to S^{n-1}$ extends to $M_0$ if and only if $deg(f)$ is divisible by $n$.

Proof. Let $g: S^{n-1} \times I \to M_0$ denote the quotient map. Suppose that $f$ extends to a map $g: M_0 \to S^{n-1}$. Let $j: S^{n-1} \to S^{n-1} \times \{1\}$, $(x \in [0, 1])$ be defined by
Let \( f(x) = (x, t) \). Let \( q_0 \) denote the restriction of \( q \) to \( S^{n-1} \times \{ t \} \) and let \( q_t \) denote the restriction of \( q \) to the image of \( q_t \). We have \( f = q_0 + f_1 = q_1 + f_1 = q_0 + f_0 \). This, together with the fact that \( q_1, f_1, \) and \( f_0 \) are homeomorphic, implies that
\[
|\deg(f)| = |\deg(f_1 + q_1)| = |\deg(q_0 + q_0 + f_0)|
\]
\[
= |\deg(q_0 + q_0)| = |\deg(q_0)| = n|\deg(q_0)|.
\]
Therefore \( \deg(f) \) is divisible by \( n \).

Conversely, suppose that \( \deg(f) = kn \). Then \( f \) extends to \( M_n \) if and only if \( \gamma_k \gamma_k \) does. But the composite \( \gamma_k \gamma_k \rightarrow \prod_{z \in 1} S^{n-1} \times I \rightarrow S^{n-1} \) factors through \( M_n \), providing the extension. This completes the proof.

**Lemma 2.2.** For each \( n \), there are embeddings \( \varphi : M_n \rightarrow D^{n} \times D^{n} \) and \( \psi : L_n \rightarrow D^{m} \times D^{m} \) with the following properties:

(i) \( \varphi(M_n) \) and \( \psi(L_n) \) are disjoint, and

(ii) \( \varphi \) carries \( \partial_0 M_n \) homeomorphically onto \( S^{n-1} \times \{ 0 \} \), and \( \psi \) carries \( \partial_0 L_n \) homeomorphically onto \( \{ 0 \} \times S^{m-1} \).

**Proof.** Using the idea of [K-L], we define \( \varphi : M_n \rightarrow D^{n} \times D^{n} \) by \( \varphi([x, t]) = (t, (2-t/2)\gamma_0(x)) \). Next, extend \( \gamma_0 \) to \( \gamma_0 : D^{m} \rightarrow D^{m} \) by letting \( \gamma_0(z) = t_0 \gamma(z) \) for \( z \in S^{m-1} \), then define \( \psi : D^{m} \rightarrow D^{m} \times D^{m} \) by \( \psi(z) = (y, -\gamma_0(z)) \). It is easy to see these are embeddings. Moreover,

\[
\varphi(M_n) \cap \psi(D^{m}) = \emptyset.
\]

For suppose that \( \varphi([t, z]) = \psi(y) \). Then we have
\[
\frac{1}{2} + \frac{t}{2} = \left( \frac{1}{2} + \frac{t}{2} \right)\gamma_0(z) = \frac{1}{2} - \gamma_0(z) = \frac{1}{2} = t
\]
and hence \( t = 1 \) and \( y = z \). But then, \( \gamma_0(z) = -\gamma_0(z) = -\gamma_0(z) \), a contradiction. Note also that

\[
\varphi(M_n) \cap \psi(D^{m}) \subseteq S^{n-1} \times S^{m-1}.
\]

Now, define
\[
\psi((x, t)) = \left\{ \begin{array}{ll}
\varphi([x, 2t]) & \text{if } 0 \leq t \leq 1/2, \\
([x, (2-t)\gamma_0(x)] & \text{if } 1/2 \leq t \leq 1,
\end{array} \right.
\]
and
\[
\psi((y)) = \left\{ \begin{array}{ll}
\psi(y) & \text{if } 0 \leq |y| \leq 1/2,
\gamma_0\left( \left( \frac{|y|}{|y|} \right) y \right) & \text{if } 1/2 \leq |y| \leq 1.
\end{array} \right.
\]

It is easily checked that \( \varphi \) and \( \psi \) satisfy (ii). Also, we observe that

\[
\varphi(M_n) = \psi(M_n) \cup (\varphi(M_n) \cap S^{n-1} \times D^{m}),
\]

\[
\varphi(M_n) \cap S^{n-1} \times S^{m-1} = \psi_0(M_n),
\]

\[
\psi(L_n) = \psi(D^{m}) \cup (\psi(L_n) \cap D^{m} \times S^{m-1}),
\]

\[
\psi(L_n) \cap S^{n-1} \times S^{m-1} = \psi(D^{m}).
\]

Property (i) follows from (4), (5), and (i). This completes the proof of Lemma 2.2.

Suppose that \( Z \) is a finite polyhedron. Define \( F_Z \) to be the space of maps from \( Z \) to \( D^{m} \times D^{m} \).

**Lemma 2.3.** \( F_2 \) contains a countable dense set.

**Proof.** Choose triangulations \( K \) of \( Z \) and \( L \) of \( D^{m} \times D^{m} \). The set of simplicial maps from some barycentric subdivision of \( K \) to some barycentric subdivision of \( L \) is countable. By the Simplicial Approximation Theorem [S, Theorem 3.3.8], this set of simplicial maps is dense in the set of all maps.

**Lemma 2.4.** Let \( Z \) be an \( m \)-dimensional finite simplicial complex containing a subcomplex of \( Z^{n-1} \) which has a fixed identification with \( S^{n-1} \). Suppose that \( Z \) does not retract to \( S^{n-1} \). Let \( B \) be a closed m-ball in \( Z - Z^{n-1} \). Then there exists an integer \( n \geq 2 \) such that the identification space \( Z - \text{int}(B) \times M_n \), obtained by identifying \( \partial B \) with \( \partial_0 M_n \), does not retract to \( S^{n-1} \).

**Proof.** Orient \( \partial B \) arbitrarily. We will first show that either the set
\[
D = \{ d \mid \text{there is a retraction from } Z - \text{int}(B) \text{ to } S^{n-1}, \text{ whose restriction to } \partial B \text{ has degree } d \}
\]
is empty, or it consists of a single nonzero integer, or there are integers \( 0 < a < b \) so that \( D = \{ a + nh \mid n \in Z \} \).

Write \( S \) for \( \partial B \) and \( Z_0 \) for \( Z - \text{int}(B) \). If \( f: Z_0 \rightarrow S^{n-1} \) is a retraction, define \( d(f) \) to be the degree of the restriction of \( f \) to \( S \). The degree of this restriction must be nonzero, otherwise \( f \) would extend to \( Z \) contrary to hypothesis.

We claim that if \( f \) and \( g \) are retractions from \( Z_0 \) to \( S^{n-1} \), with \( d(f) = r \) and \( d(g) = s \), then for any integer \( n \) there is a retraction \( h_n \) from \( Z_0 \) to \( S^{n-1} \) such that \( d(h_n) = r + n(s - r) \). Triangulate \( Z_0 \) and give each simplex an orientation. Choose a vertex \( s_0 \in S^{n-1} \). Since \( S^{n-1} \) is \((m,n)\)-connected, there are maps \( f' \) and \( g' \), homotopic to \( f \) and \( g \) respectively, that carry the entire \((m,n)\)-skeleton of \( Z_0 \) to \( s_0 \). For each \((m,n)\)-simplex \( x \) of \( Z_0 \), define \( c_0(x) = c_0(x) \) to be the degree of the restrictions of \( f' \) and \( g' \) to \( x \) (regarded as maps from \( x \) to \( s_0 \)). Now if \( Y \) is a subcomplex of \( Z_0 \) such that \( |Y| \) is homeomorphic to the \((m,n)\)-sphere, and \( |Y| = \sum c_0(x) \) is a fundamental class determining an orientation for \( Y \), where each \( c_0(x) \) is an \((m-1)\)-simplex of \( Z_0 \), then the degree of the restriction of \( f \) to \( Y \) is equal to \( \sum c_0(f)(x) \). Therefore

(i) \( f \) is any \( m \)-simplex of \( Z_0 \), then since the restrictions of \( f' \) and \( g' \) to \( \partial \) extend to \( f \), we have \( c_0(\partial f) = c_0(\partial f) = 0 \).
2. Since the restrictions of $f'$ and $g'$ to $S$ are of degrees $r$ and $s$ respectively, we have $c_p((S^r)) = r$ and $c_p(S^s) = s$.

Since $f'$ and $g'$ are homotopic to retractions to $S^{m-1}$, we have $c_p(S^{m-1}) = 1$

Now for any fixed integer $n$, define $h'_n : Z^{m-1} \to S^{m-1}$ by sending the $(m-2)$-skeleton to $s_0$ and sending each oriented $(m-1)$-simplex $\sigma$ by a map of degree $n(c_p(\sigma)) - (n-1)c_p(\sigma)$. By (1), $h'_n$ extends to all of $Z_n$. By (2), the degree of $h'_n$ on $S$ is $ns - (n-1)r = r + n(r-s)$. And by (3), the degree of the restriction of $h'_n$ to $S^{m-1}$ is $n - (n-1) = 1$, hence by homotopy extension $h'_n$ is homotopic to a retraction $h_n$. This completes the proof of the claim.

Assume that there is more than one possibility for the degree of the restriction to $S$ of a retraction from $Z_n$ to $S^{m-1}$. Let $b$ be the smallest positive integer for which there are retractions $f_1$ and $f_2$ so that $d(f_1) - d(f_2) = b$. By the claim, each of the integers in the set $\{d(f_1) + nb : n \in Z\}$ is the degree of the restriction to $S$ of some retraction; since $0$ is not a possible degree, this set has the form described in the statement of Lemma 2.4. Since $b$ was chosen to be minimal, no other degrees are possible. This completes the proof that the set $D$ has the stated form.

To complete the proof of Lemma 2.4, choose a positive integer $n$ which does not divide any element of $D$. Suppose $f$ is any retraction of $Z - \text{int}(D)$ to $S^{m-1}$. Since the degree of the restriction of $f$ to $S$ is not divisible by $n$, Lemma 2.1 shows that $f$ cannot extend to $M_n$.

3. The construction. In this section, we construct for each $m \geq 2$ an inverse system $X = (X_n, \varphi_{n,m})$ of finite $m$-dimensional polyhedra whose limit $X$ has dimension $m$ and such that $X$ satisfies the hypotheses of Proposition 1.7 with $N = 2m$.

This will complete the proof of the Theorem stated in the introduction.

Fix an integer $m \geq 2$. Let $X_1 = D^m$. Using Lemma 2.3, choose a sequence $\alpha_{1,k} : X_1 \to D^m \times D^m$ such that $\alpha_{1,k} \circ \text{id}_{X_1} = \text{id}_{X_1}$.

Suppose inductively that $l \geq 1$ and there are chosen finite polyhedra $X_1, X_2, \ldots, X_l$ and maps $\varphi_{l-1,j} : X_j \to X_{j-1}$ for $j = 2, 3, \ldots, l$. Let $\alpha_{l,k} : X_k \to X_1$ equal the identity map, and for $1 \leq j < k \leq l$, let

$$\alpha_{l,j} = \alpha_{l,j-1} \circ \varphi_{l-1,j-1} \circ \ldots \circ \varphi_{1,0} \circ \alpha_{1,1} : X_1 \to X_j$$

Suppose further that the following hold:

(i) For $1 \leq j < l$, there is a fixed sequence $\alpha_{l,j} : X_j \to D^m \times D^m$ such that $(\alpha_{l,j})_* = \text{id}_X$.

(ii) $X_j$ contains $S^{m-1}$, $X_j$ does not retract to $S^{m-1}$, and the restriction of each $\varphi_{l,j}$ to $S^{m-1}$ is the identity map for $1 \leq j < k \leq l$.

(iii) $\dim(X_j) = m$ for $1 \leq j \leq l$.

(iv) For $j + k < l \leq l$, there is selected an imbedding $\psi_{l,j} : X_j \to D^m \times D^m$ such that $\psi_{l,j} \circ \alpha_{l,j} \circ \varphi_{l-1,j-1} \circ \ldots \circ \varphi_{1,0} \circ \alpha_{1,1} \circ \varphi_{1,1} \circ \ldots \circ \varphi_{1,0} \circ \alpha_{1,1} < 1/2^j$.

Now choose a dense sequence $(\alpha_{l,k})^m_{k=1}$ in $\mathcal{F}_X$. We wish to construct a finite polyhedron $X_{i+1}$ and a map $\varphi_{i+1,j} : X_{i+1} \to X_j$, so that when we adjoin $X_{i+1}$ to the collection $X_1, X_2, \ldots, X_i$, the corresponding properties $(1_{X_{i+1}}, (X_{i+1}), \beta_{i+1,j}, (1_{X_{i+1}})$, and $(\psi_{i+1,j})$ will be true.

Write $Y_j^2$ for $X_j$. Consider the finite collection of maps

$$\{\alpha_{l,k} \circ \varphi_{l,k} : j + k \leq l + 1\} \subseteq \mathcal{F}_X$$

Choose self-transverse piecewise-linear approximations $\delta_{l,j}^k$ of $\alpha_{l,k} \circ \varphi_{l,k}$ (see, for example, [R-S-I]) so that

$$\delta_{l,j}^k \circ \alpha_{l,j} \circ \varphi_{l-1,j-1} \circ \ldots \circ \varphi_{1,0} \circ \alpha_{1,1} < 1/2^j$$

No point in $D^m \times D^m$ is a double point for more than one $\delta_{l,j}^k$.

The double points $\{x_1, x_2, \ldots, x_n\}$ of the $\delta_{l,j}^k$ lie in the interior of $D^m \times D^m$. Each $x_i$ arises from a pair of distinct points $\{p_i, q_i\}$ in $Y_j^2$ of the form $\delta_{l,j}^k$. Choose disjoint closed neighborhoods $N_i$ of the $x_i$ in $D^m \times D^m$ and pairwise disjoint neighborhoods $P_i$ of $p_i$, and $Q_i$ of $q_i$ in $Y_j^2$ such that

$$\text{diam}(N_i) < 1/2^{j+2}$$

and $N_i$ is homeomorphic to $B^m \times B^m$ where $B^m$ is the $m$-ball.

For each $r$, if $x_i$ is a double point of $\delta_{l,j}^k$ then $\delta_{l,j}^k$ carries $P_i$ homeomorphically onto $B^m \times \{0\} \subseteq N_i$, and $Q_i$ homeomorphically onto $\{0\} \times B^m$.

For each $r$, if $x_i$ is not a double point of $\delta_{l,j}^k$ then there is a closed neighborhood $V_{r,j}^2$ of $\delta_{l,j}^k(x_i)$ in the interior of $D^m \times D^m$ such that $\text{diam}(V_{r,j}^2) < 1/2^{j+2}$, the preimage under $\delta_{l,j}^k$ of $V_{r,j}^2$ is $P_i$, and $V_{r,j}^2$ is homeomorphic to $B^m \times B^m$ so that $\delta_{l,j}^k$ carries $P_i$ homeomorphically onto $B^m \times B^m$.

After slight further adjustment of the maps $\delta_{l,j}^k$, we may assume that the closed neighborhoods $V_{r,j}^2$ and $W_{r,j}^2$ are pairwise disjoint.

Consider $Y_j^2 - \text{int}(P_i)$ and let $S = \partial P_i \subseteq Y_j^2$. Now $Y_j^2$ does not retract to $S^{m-1}$, so by Lemma 2.4 there exists an $n$ so that the space $Z_n^2$ obtained from $Y_j^2 - \text{int}(P_i)$ by attaching a copy of $M_n$ using a homeomorphism of $\partial P_i$, $M_n$, with $\partial P_i$ does not retract to $S^{m-1}$.

Next, form $Y_j^2$ by attaching a copy of $L_n$ to $Z_n^2$ using a homeomorphism of $\partial L_n$ with $\partial Q_i$. Again, $S^{m-2}$ may be regarded as a subspace of $Y_j^2$, and $Y_j^2$ does not retract to $S^{m-1}$ since any retraction of $Z_n^2 - \text{int}(Q_i)$ to $S^{m-1}$ cannot have degree zero on $\partial Q_i$ and so cannot extend to $Y_j^2$.

Consider the imbeddings $\phi : M_n \to B^m \times B^m$ and $\psi : L_n \to B^m \times B^m$ as given in Lemma 2.2. For each pair $j, k$ with $j + k \leq l + 1$, define $\psi_{l,j}^k : Y_j^2 \to D^m \times D^m$ as follows. On the subspace $Y_j^2 - (P_i \cup Q_i)$, $\psi_{l,j}^k = \delta_{l,j}^k$. On $M_n$, if $x_i$ is a singular point of $\delta_{l,j}^k$, then $\psi_{l,j}^k(x_i) = \varphi$, carrying $P_i$ into $N_i$, while if $x_i$ is a singular point of $\delta_{l,j}^k$. This completes the proof of the theorem.
then $\delta^{1}_{1} = \emptyset$, carrying $P_{1}$ into $V^{1}_{n+1}$. On $Q_{1}$, the definition is similar using $\psi$. Coordinates are selected on these pieces so that these maps fit together. The singular image of each $\delta^{1}_{1}$ is contained in $(z_{1}, z_{2}, ..., z_{n})$. There is a projection map $\phi_{0,1}^{1}$: $Y_{1}^{1} \rightarrow Y_{2}^{1}$ defined by extending the identity map on $Y_{2}^{1} \cap (F_{1} \cup Q_{1})$ using maps from $(M_{2}, \partial_{2}M_{2})$ to $(F_{2}, \partial_{2}F_{2})$ and from $(L_{2}, \partial_{2}L_{2})$ to $(Q_{1}, \partial_{0,1})$. Since the diameters of $N_{1}$, $W_{2}$, $2^{1}$, and $V^{1}_{n+1}$ are all less than $1/2^{n+2}$, we have
\[
\phi_{0,1}^{1} = \partial_{0,1}^{1} \circ \phi_{0,1}^{1} + d(\phi_{0,1}^{1} \circ \phi_{0,1}^{1} + \phi_{0,1}^{1} \circ \phi_{0,1}^{1}) < 1/2^{n+1} + 1/2^{n+2} = 1/2^{n+1}.
\]

Repeating this procedure, starting with $Y_{1}^{1}$ and replacing $\delta^{0}_{1}$ by $\delta^{1}_{1}$, we obtain a sequence $Y_{1}^{1}, Y_{2}^{1}, ..., Y_{n+1}^{1}$ of $m$-dimensional finite complexes, corresponding maps $\delta^{1}_{j+1}$: $Y_{j}^{1} \rightarrow Y_{j+1}^{1}$, and projections $Y_{j}^{1} \rightarrow Y_{j-1}^{1}$ such that the distance between $\delta^{1}_{j} \cup 1$ and the composite $\phi_{n+1}^{1} \circ \phi_{n+1}^{1} \circ \phi_{n+1}^{1} \circ ... \circ \phi_{n+1}^{1}$ is less than $1/2^{n+1}$. Put $X_{n+1} = Y_{n+1}^{1}$, $\delta^{1} = \delta^{1}_{n+1}$, and $\phi_{0,1}^{1} = \phi_{0,1}^{1} \circ \phi_{0,1}^{1} \circ \phi_{0,1}^{1} \circ ... \circ \phi_{0,1}^{1}$. Note that the restriction of $\phi_{0,1}^{1}$ to $S^{n+1}$ is the identity, and $\delta^{1}_{n+1}$: $X_{n+1} \rightarrow \delta^{1}$ is an imbedding. We have thus established the desired properties.

Now consider the inverse sequence $X = \{X_{i}, \phi_{n+1}^{1}\}$ just constructed. Let $X = \lim X$. Applying Proposition 1.2 to (3), we see that $\dim(X) \leq m$. On the other hand, Proposition 1.4 and (2) yield that $\dim(X) \geq m$. Therefore
\[
\dim(X) = m.
\]

Let us now check that the hypotheses of Proposition 1.7 are satisfied with $N = 2m$. Identify $D^{n} \times D^{m}$ with $I^{2m}$. Let $\sigma$ and a map $h$: $X_{n+1} \rightarrow D^{n} \times D^{m}$ and $\delta > 0$ be given. Using (1), there is an $a_{n+1}^{1}$: $X_{n+1} \rightarrow D^{n} \times D^{m}$ such that
\[
d(a_{n+1}^{1}, h) < \delta/2.
\]

Choose $l$ so large that $k + s < l$ and $1/2^{l} < \delta/2$. According to (4), there is an imbedding $\phi_{l}^{1}$: $X_{l} \rightarrow D^{n} \times D^{m}$ such that
\[
d(\phi_{l}^{1}, a_{l}^{1} \circ \phi_{l}^{1}) < \delta/2.
\]

From (12), we get
\[
d(\phi_{l}^{1} \circ a_{l}^{1} \circ h, \phi_{l}^{1}) < \delta/2.
\]

Then (13) and (14) combine to yield the desired condition that $d(\phi_{l}^{1} \circ a_{l}^{1} \circ h, \phi_{l}^{1}) < \delta$.

References
