Shape morphisms and spaces of approximative maps

by

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Abstract. A structure of separable metric space is defined on the set $A(X, Y)$ of all approximative maps of a compactum X towards a compactum Y. It is proved that the shape morphisms from X to Y can be represented as path-components of the corresponding topological space. This result is not in general true when the set $A(X, Y)$ is endowed with other natural topologies as it has been already proved.

1. Introduction. Shape morphisms between compacta X and Y can be described as homotopy classes of approximative maps of X towards Y. The notion of an approximative map was introduced by K. Borsuk in his foundational paper of shape theory [2]. This notion is strictly related to the concept of "a map towards a space", used by D. E. Christie [5] in his construction of the the weak homotopy groups. By an *approximative map* of a compactum X *towards* a compactum Y, which is a subset of a space $Q \in \mathrm{AR}$ (that in this paper is assumed to be the Hilbert cube), we understand a sequence of maps f_k: $X \to Q$ satisfying the following condition

For every neighborhood V of Y in Q the homotopy $f_k \simeq f_{k+1}$ in V holds true for almost all k.

We shall denote this approximative map by $\boldsymbol{f} = \{f_k, X \to Y\}$. In [8], the authors introduced two topologies on the set $A(X, Y)$ of all approximative maps of X towards Y, where Y was supposed to lie in the Hilbert cube Q. The first of them was induced by a metric d: $A(X, Y) \times A(X, Y) \to R$ given by the formula

$$d(\boldsymbol{f}, \boldsymbol{g}) = \sup\{\mathrm{dist}(f_k, g_k) |\ k = 1, 2, \ldots\} \quad \text{for } \boldsymbol{f}, \boldsymbol{g} \in A(X, Y).$$

The second one was induced by a pseudometric d^*: $A(X, Y) \times A(X, Y) \to \boldsymbol{R}$ given by

$$d^*(\boldsymbol{f}, \boldsymbol{g}) = \inf\{\sup\{\mathrm{dist}(f_k, g_k) |\ k \geq k'\} | k' = 1, 2, \ldots\}.$$

The paper [8] was devoted to the study of some properties of the obtained spaces $A(X, Y)_d$ and $A(X, Y)_{d^*}$. In particular, it was proved that if two approximative

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maps f and g belong to the same path-component of $A(X, Y)_d$ then f and g are homotopic. However, an example was given proving that there exist compacta X, Y and homotopic approximative maps $f, g \in A(X, Y)$ which do not lie in the same path-component of $A(X, Y)_d$. The same example applies in the case of the space $A(X, Y)_{d^*}$. The authors believe that it could be interesting to have a representation of the shape morphisms between two compacta X, Y as the path-components of a certain space of approximative maps between X and Y. As we have just explained, this is not the case with the spaces $A(X, Y)_d$ and $A(X, Y)_{d^*}$. In this paper we introduce a metric e on the set $A(X, Y)$ which makes $A(X, Y)$ into a topological space with the desired property. The obtained space has some good topological properties. For instance, it is separable and contains the function space Y^X as a closed subset. It also has a property which is in some degree surprising in the present context, namely it is connected.

The reader is supposed to know the most basic facts about shape theory as can be found in [3], [6], [9]. All compacta are assumed to lie in the Hilbert cube, Q. We consider the usual metric on Q.

2. The space $A(X, Y)_e$. Consider compacta X and Y lying in the Hilbert cube, Q. We define a metric on the set $A(X, Y)$ of all approximative maps of X towards Y by the formula

$$e(f, g) = \sup\Big(\Big\{\sum_{k=1}^{\infty} \operatorname{dist}(f_k(x), g_k(x))/2^k \,\Big|\, x \in X\Big\} \cup \{|\operatorname{dist}(f_k(x), Y) - \operatorname{dist}(g_k(x), Y)| \mid x \in X,\ k = 1, 2, \ldots\}\Big)$$

$$\text{for every } f = \{f_k, X \to Y\},\ g = \{g_k, X \to Y\}.$$

This paper is devoted to the study of the corresponding metric space which we shall denote by $A(X, Y)_e$.

Remark 1. If we consider the set $A(X, Y)$ endowed with the metric d studied in [8], it can be easily seen that the identity $1_{A(X,Y)}\colon A(X, Y)_d \to A(X, Y)_e$ is a uniformly continuous map.

The next proposition and its corollaries can be easily proved.

Proposition 1. *Let X, X', X'', Y be compacta. If $g\colon X' \to X$ is a map then the function $\gamma_g\colon A(X, Y)_e \to A(X', Y)_e$ defined by*

$$\gamma_g(f) = \{f_k \cdot g, X' \to Y\}, \quad \text{for every } f = \{f_k, X \to Y\} \in A(X, Y)_e$$

is uniformly continuous.

If, besides, $h\colon X'' \to X'$ is another map, then

$$\gamma_h \cdot \gamma_g = \gamma_{g \cdot h}\colon A(X, Y)_e \to A(X'', Y)_e.$$

Finally,

$$\gamma_{1_Y} = 1_{A(X,Y)_e}\colon A(X, Y)_e \to A(X, Y)_e. \ \blacksquare$$

Corollary 1. *Let X and Y be compacta and X_0 a closed subset of X. Then, the restriction function $\varrho\colon A(X, Y)_e \to A(X_0, Y)_e$ defined by*

$$\varrho(f) = f|_{X_0} = \{f_k|_{X_0}, X_0 \to Y\}, \quad \text{for every } f = \{f_k, X \to Y\} \in A(X, Y)_e$$

is uniformly continuous. $\blacksquare$

Corollary 2. *Let X, X', Y be compacta. If X is homeomorphic to X', then $A(X, Y)_e$ is homeomorphic to $A(X', Y)_e$.* $\blacksquare$

The following statement analyzes a situation related to that of Corollary 2 but its proof is not immediate.

Proposition 2. *Let X, Y, Y' be compacta where Y and Y' are Z-sets in Q (see [1] or [4]). If Y is homeomorphic to Y' then $A(X, Y)_e$ is homeomorphic to $A(X, Y')_e$.*

Proof. Let $h_0\colon Y \to Y'$ be a homeomorphism. Then (see Chapman [4], Theorem 11.1) there exists a homeomorphism $h\colon Q \to Q$ such that $h(y) = h_0(y)$, for every $y \in Y$. This homeomorphism allows us to define a function $\varphi_h\colon A(X, Y)_e \to A(X, Y')_e$ by the expression

$$\varphi_h(f) = \{h \cdot f_k, X \to Y'\}, \quad \text{for every } f = \{f_k, X \to Y\} \in A(X, Y)_e.$$

Let us prove that φ_h is continuous. For this purpose, let $f = \{f_k, X \to Y\} \in A(X, Y)_e$ and fix a positive ε. Take $\delta' > 0$ such that

(1) $\operatorname{dist}(h(q_1), h(q_2)) < \varepsilon$, for every $q_1, q_2 \in Q$ such that $\operatorname{dist}(q_1, q_2) < \delta'$.

Consider an index $k_0 \geqslant 1$ such that

(2) $$\sum_{k=k_0+1}^{\infty} \Delta/2^k < \varepsilon/2,$$

where Δ is the diameter of Q, and

(3) $\operatorname{dist}(f_k(x), Y) < \delta'/2$, for every $x \in X$ and $k > k_0$.

From (3) and (1) it follows that

(4) $\operatorname{dist}(h \cdot f_k(x), Y') < \varepsilon$, for every $x \in X$ and $k > k_0$.

Moreover, for every integer k with $1 \leqslant k \leqslant k_0$, there exists $\delta_k' > 0$ such that

(5) $\operatorname{dist}(h(q_1), h(q_2)) < \min\{\varepsilon, \varepsilon \cdot 2^k/2 \cdot k_0\}$ for every $q_1, q_2 \in Q$ with $\operatorname{dist}(q_1, q_2) < \delta_k' \cdot 2^k$.

Consider $\delta = \min\{\delta'/2, \delta_1', \ldots, \delta_{k_0}'\}$. Then, for every approximative map $g = \{g_k, X \to Y\}$ such that $e(f, g) < \delta$ we have

(6) $\operatorname{dist}(f_k(x), g_k(x))/2^k \leqslant e(f, g) < \delta_k'$ for every $x \in X$, $k = 1, \ldots, k_0$

which, by virtue of (5) implies that

(7) $\operatorname{dist}(h \cdot f_k(x), h \cdot g_k(x))/2^k < \varepsilon/2 \cdot k_0$ for every $x \in X$, $k = 1, \ldots, k_0$.

Then, as a consequence of (7) and (2), we get

(8) $$\sum_{k=1}^{\infty} \operatorname{dist}(h\cdot f_k(x), h\cdot g_k(x))/2^k < k_0\cdot\varepsilon/2\cdot k_0+\varepsilon/2 = \varepsilon \quad \text{for every } x\in X.$$

On the other hand, the relation $e(f, g) < \delta$ also implies that

(9) $$|\operatorname{dist}(f_k(x), Y)-\operatorname{dist}(g_k(x), Y)| < \delta \leqslant \delta'/2 \quad \text{for every } x\in X \text{ and } k = 1, 2, \ldots$$

From (9) and (3) we deduce that

(10) $$\operatorname{dist}(g_k(x), Y) < \delta', \quad \text{for every } x\in X \text{ and } k > k_0$$

and by virtue of (10) and (1) the relation

(11) $$\operatorname{dist}(h\cdot g_k(x), Y') < \varepsilon, \quad \text{for every } x\in X \text{ and } k > k_0$$

holds. Now, from (4) and (11) we get

(12) $$|\operatorname{dist}(h\cdot f_k(x), Y')-\operatorname{dist}(h\cdot g_k(x), Y')| < \varepsilon \text{ for every } x\in X \text{ and } k > k_0.$$

Moreover, (6) and (5) imply that

(13) $$|\operatorname{dist}(h\cdot f_k(x), Y')-\operatorname{dist}(h\cdot g_k(x), Y')| \leqslant \operatorname{dist}(h\cdot f_k(x), h\cdot g(x)) < \varepsilon, \quad \text{for every } x\in X \text{ and } k = 1, \ldots, k_0.$$

Then, from (8), (13) and (12) we conclude that

$$e(\varphi_h(f), \varphi_h(g)) \leqslant \varepsilon$$

which proves the continuity of φ_h.

In an analogous way, it is possible to consider the continuous mapping $\varphi_{h^{-1}}\colon A(X, Y')_e\to A(X, Y)_e$ and it can be easily seen that

$$\varphi_{h^{-1}}\cdot\varphi_h = 1_{A(X,Y)_e}, \quad \varphi_h\cdot\varphi_{h^{-1}} = 1_{A(X,Y')_e}.$$

Hence, φ_h is a homeomorphism and the proof is finished. ■

The following result is an immediate consequence of Corollary 2 and Proposition 2.

COROLLARY 3. *Let X, X', Y, Y' be compacta where Y and Y' are Z-sets in Q. If X is homeomorphic to X' and Y is homeomorphic to Y', then $A(X, Y)_e$ is homeomorphic to $A(X', Y')_e$.* ■

Each map $f\colon X\to Y$ can be identified in a natural way with an approximative map $f\in A(X, Y)_e$. We just consider $f = \{f_k = f, X\to Y\}$. In this way we define an injection $i\colon Y^X\to A(X, Y)$.

PROPOSITION 3. *The function i embeds Y^X as a closed subset of $A(X, Y)_e$. Under this correspondence, the set Φ of all homeomorphisms $f\colon X\to Y$ is mapped onto a G_δ-subset of $A(X, Y)_e$.*

Proof. The first part of the proof is left to the reader. The second half is a consequence of the fact that the homeomorphisms are a G_δ-set in Y^X (see Kuratowski [7], Vol. II, p. 91). ■

The next proposition asserts the separability of the space $A(X, Y)_e$.

PROPOSITION 4. *The space $A(X, Y)_e$ is separable for any compacta X and Y.*

Proof. Choose an approximative map $g = \{g_k, X\to Y\}\in A(X, Y)_e$. It is well known that the space Q^X endowed with the metric of uniform convergence is separable. Let D_0 be a countable dense subset of Q^X.

For every index $k_0\geqslant 2$ let D_{k_0} be the set of all approximative maps $h = \{h_k, X\to Y\}\in A(X, Y)_e$ such that

$$h_k = g_k, \quad \text{for every } k\geqslant k_0,$$
$$h_k\in D_0, \quad \text{for every } k < k_0.$$

D_{k_0} is countable since there exists a bijection from $D_0\times \overset{(k_0-1)}{\ldots\ldots\ldots}\times D_0$ onto D_{k_0}.

Let $E = \bigcup_{k_0=2}^{\infty} D_{k_0}$. E is a countable subset of $A(X, Y)_e$. We shall prove that E is dense in $A(X, Y)_e$. To see this, consider $f = \{f_k, X\to Y\}\in A(X, Y)_e$ and $\varepsilon > 0$. There exists an index $k_0\geqslant 2$ such that

(1) $$\sum_{k=k_0}^{\infty} \Delta/2^k < \varepsilon/2 \quad (\Delta = \operatorname{diam} Q)$$

and

(2) $$\max\{\operatorname{dist}(f_k(x), Y), \operatorname{dist}(g_k(x), Y)\} < \varepsilon \quad \text{for every } x\in X, k\geqslant k_0.$$

On the other hand, for every $k = 1, \ldots, k_0-1$, there exists $h'_k\in D_0$ such that

(3) $$\operatorname{dist}(f_k, h'_k) < \min\{\varepsilon, \varepsilon\cdot 2^k/2(k_0-1)\}.$$

Consider $h = \{h_k, X\to Y\}\in D_{k_0}$ defined by

(4) $$h_k = h'_k \quad \text{if } k\leqslant k_0-1,$$

(5) $$h_k = g_k \quad \text{if } k\geqslant k_0.$$

Then, (1), (4) and (3) imply that

(6) $$\sum_{k=1}^{\infty} \operatorname{dist}(f_k(x), h_k(x))/2^k < (k_0-1)\cdot\varepsilon/2\cdot(k_0-1)+\varepsilon/2 = \varepsilon.$$

Moreover by virtue of (2) and (5) we have

(7) $$|\operatorname{dist}(f_k(x), Y)-\operatorname{dist}(h_k(x), Y)| < \varepsilon \quad \text{for every } x\in X, k\geqslant k_0$$

and from (3) and (4) it follows that

(8) $$|\operatorname{dist}(f_k(x), Y)-\operatorname{dist}(h_k(x), Y)| \leqslant \operatorname{dist}(f_k(x), h_k(x)) < \varepsilon, \quad \text{for every } x\in X, k\leqslant k_0-1.$$

From (6), (7) and (8) we deduce that

$$e(f, \boldsymbol{h}) \leqslant \varepsilon .$$

This proves that E is dense in $A(X, Y)_e$.

Hence $A(X, Y)_e$ is separable and the proof is complete. ■

The following example shows that $A(X, Y)_e$ is not, in general, complete.

EXAMPLE 1. Consider two compacta X, Y which admit a sequence of maps $f_k: X \to Y$ such that $\{f_k, X \to Y\}$ is not an approximative map of X towards Y. Let $f: X \to Y$ be a fixed map. We define, for each $n = 1, 2, \ldots$ an approximative map $\boldsymbol{g}^n = \{g_k^n, X \to Y\}$ such that

$$g_k^n = \begin{cases} f_k & \text{if } k \leqslant n, \\ f & \text{if } k > n. \end{cases}$$

It is easy to see that the sequence of approximative maps $\boldsymbol{g}^n$, $n = 1, 2, \ldots$ is a Cauchy sequence in $A(X, Y)_e$. However, the sequence $\boldsymbol{g}^n$ has no limit in $A(X, Y)_e$, because if such a limit $\boldsymbol{g} = \{g_k, X \to Y\}$ existed we would have that the sequence of functions g_k^n, $n = 1, 2, \ldots$ would uniformly converge to g_k, for every $k = 1, 2, \ldots$ This would imply that $g_k = f_k$, for each $k = 1, 2, \ldots$ in contradiction with the fact that $\{f_k, X \to Y\}$ is not an approximative map.

We are now going to give a series of results which are related to the exponential law in $A(X, Y)_e$. Consider compacta X, Y, Z and a map $h: Z \to A(X, Y)_e$. We have for each $z \in Z$ an approximative map $h(z) = \{h(z)_k, X \to Y\}$. Hence we can define, for each $k = 1, 2, \ldots$, a function $H_k: X \times Z \to Q$ by $H_k(x, z) = h(z)_k(x)$, for every $(x, z) \in X \times Z$.

PROPOSITION 5. *The function H_k is continuous for every $k = 1, 2, \ldots$ Moreover, if the shape of Z is trivial, then $\mathbf{H} = \{H_k, X \times Z \to Y\}$ is an approximative map of $X \times Z$ towards Y.*

Proof. Let k_0 be an index. It is easy to see that the function $p_{k_0}: A(X, Y)_e \to Q^X$ defined by

$$p_{k_0}(\boldsymbol{f}) = f_{k_0}, \quad \text{for every } \boldsymbol{f} = \{f_k, X \to Y\} \in A(X, Y)_e$$

is uniformly continuous if we consider in Q^X the metric of uniform convergence. Hence, for every $k = 1, 2, \ldots$, the composition $p_k h: Z \to Q^X$ is continuous and, as a consequence of the exponential law for function spaces, the function $\varphi_k: X \times Z \to Q$ such that

$$\varphi_k(x, z) = \big((p_k h)(z)\big)(x), \quad \text{for every } (x, z) \in X \times Z$$

is continuous. Now it is plain that $H_k = \varphi_k$ and this proves the first part of the proposition.

To prove the second part consider a neighborhood $V \in \mathrm{ANR}$ of Y in Q. We claim that there exist $V_0 \subset V$, a neighborhood of Y in Q and $\varepsilon > 0$ such that

(1) $$|\mathrm{dist}(q, Y) - \mathrm{dist}(q', Y)| > \varepsilon \quad \text{for every } q \in V_0,\ q' \in Q - V.$$

Indeed, take $\varepsilon > 0$ such that

(2) $$\{q \in Q |\ \mathrm{dist}(q, Y) < 2\varepsilon\} \subset V$$

and $V_0 \subset V$ given by

(3) $$V_0 = \{q \in Q |\ \mathrm{dist}(q, Y) < \varepsilon\} .$$

Then, if $q \in V_0$, $q' \in Q - V$ we have, by virtue of (2) and (3), that

$$\mathrm{dist}(q, Y) < \varepsilon, \quad \mathrm{dist}(q', Y) \geqslant 2\varepsilon .$$

As a consequence

$$|\mathrm{dist}(q, Y) - \mathrm{dist}(q', Y)| = \mathrm{dist}(q', Y) - \mathrm{dist}(q, Y) > 2\varepsilon - \varepsilon = \varepsilon .$$

This proves our claim.

Consider now a $\delta > 0$ such that

(4) $$e\big(h(z), h(z')\big) < \varepsilon \quad \text{for every } z, z' \in Z \text{ with } \mathrm{dist}(z, z') < \delta .$$

Then, there are points $z_0, z_1, \ldots, z_n \in Z$ such that for every $z \in Z$ there exists i, with $0 \leqslant i \leqslant n$, such that $\mathrm{dist}(z, z_i) < \delta$.

On the other hand, there exists an index k_0 such that

(5) $$h(z_i)_k \simeq h(z_i)_{k+1} \text{ in } V_0 \quad \text{for every } k \geqslant k_0,\ i = 0, 1, \ldots, n .$$

Let us see now that $H_k(X \times Z) \subset V$, for every $k \geqslant k_0$. To prove this, take $(x, z) \in X \times Z$ and let $i \leqslant n$ be an index such that $\mathrm{dist}(z, z_i) < \delta$. Then (4) implies that

$$e\big(h(z), h(z_i)\big) < \varepsilon$$

and, in consequence, we have

(6) $$|\mathrm{dist}\big(H_k(x, z), Y\big) - \mathrm{dist}\big(H_k(x, z_i), Y\big)| < \varepsilon \quad \text{for every } k = 1, 2, \ldots$$

Now, by (5) $H_k(x, z_i) \in V_0$ for every $k \geqslant k_0$, and from (6) and (1) we have that

$$H_k(x, z) \in V \quad \text{for every } k \geqslant k_0 .$$

Since $V \in \mathrm{ANR}$, for every $k \geqslant k_0$ there are a neighborhood T_k of Z in Q and a map $\hat{H}_k: X \times T_k \to V$ with

(7) $$\hat{H}_k(x, z) = H_k(x, z) \quad \text{for every } (x, z) \in X \times Z .$$

Moreover, since the shape of Z is trivial, we have for every $k \geqslant k_0$ a map $\alpha_k: Z \times [0, 1] \to T_k$ such that

(8) $$\alpha_k(z, 0) = z, \quad \alpha_k(z, 1) = z_0, \quad \text{for every } z \in Z .$$

Now we define for each $k \geqslant k_0$ a homotopy $F_k: X \times Z \times [0, 1] \to V$ by $F_k(x, z, t) = \hat{H}_k(x, \alpha_k(z, t))$. Obviously, from (7) and (8) it follows that

(9) $$F_k(x, z, 0) = H_k(x, z), \quad F_k(x, z, 1) = H_k(x, z_0) \quad \text{for every } (x, z) \in X \times Z$$

and, from (5) and (9), it can be easily deduced that

$$H_k \simeq H_{k+1} \text{ in } V, \quad \text{for } k \geqslant k_0 .$$

This proves that $\boldsymbol{H} = \{H_k, X\times Z \to Y\}$ is an approximative map and completes the proof of the proposition. ■

It should be noted that the second part of Proposition 5 does not hold without the condition of shape triviality of Z as the following example shows.

EXAMPLE 2. Suppose $X = \{p\}$ (a point),

$$Y = Z = S^1 = \{z = (z_1, z_2)|\ z_1^2+z_2^2 = 1\},$$

the unit circle in the plane, and let $s\colon S^1 \to S^1$ be the map defined by $s(z_1, z_2) = (-z_1, z_2)$. It is easy to see that the function $h\colon Z \to A(X, Y)_e$ defined by

$$h(z)_k(p) = \begin{cases} z & \text{if } k \text{ is odd},\\ s(z) & \text{if } k \text{ is even}\end{cases}$$

is a continuous mapping, but the associated sequence of maps $H_k\colon X\times Z \to Q$ does not define an approximative map of $X \times Z$ towards Y.

The next result is a converse to the second part of Proposition 5. Consider compacta X, Y, Z, a map $h\colon Z\to A(X, Y)_e$ and, for each $k = 1, 2, \ldots$, the function H_k defined just a line before Proposition 5. If $\boldsymbol{H} = \{H_k, X\times Z\to Y\}$ is an approximative map, then we say that $\boldsymbol{H}$ is the *approximative map associated to h.*

PROPOSITION 6. *Let $\boldsymbol{H} = \{H_k, X\times Z \to Y\}$ be an approximative map, where X, Y, Z are compacta. Then, there exists a map $h\colon Z \to A(X, Y)_e$ whose associated approximative map is $\boldsymbol{H}$.*

Proof. Consider for each $z \in Z$ the partial approximative map $\boldsymbol{\alpha}^z = \{\alpha_k^z, X\to Y\} \in A(X, Y)_e$ such that

(1) $$\alpha_k^z(x) = H_k(x, z), \quad \text{for every } x\in X,\ k = 1, 2, \ldots$$

We define a function $h\colon Z\to A(X, Y)_e$ by

(2) $$h(z) = \boldsymbol{\alpha}^z, \quad \text{for every } z \in Z.$$

We must prove that h is continuous. To see this take $z_0 \in Z$ and $\varepsilon > 0$. Consider an index $k_0 \geqslant 1$ such that

(3) $$\sum_{k=k_0+1}^{\infty} \Delta/2^k < \varepsilon/2, \quad \text{where } \Delta = \operatorname{diam} Q$$

and

(4) $$\operatorname{dist}(H_k(x, z), Y) < \varepsilon, \quad \text{for every } (x, z) \in X\times Z,\ k \geqslant k_0 .$$

On the other hand, for every index $k \leqslant k_0$, there exists a number $\delta_k > 0$ such that

(5) $$\operatorname{dist}(H_k(x, z), H_k(x, z_0)) < \min\{\varepsilon, \varepsilon\cdot 2^k/2\cdot k_0\}$$
$$\text{for every } x\in X \text{ and } z \in B(z_0, \delta_k).$$

Let $\delta = \min\{\delta_1, \ldots, \delta_{k_0}\} > 0$. If $\operatorname{dist}(z, z_0) < \delta$ in Z, then by (3) and (5) we have

(6) $$\sum_{k=1}^{\infty} \operatorname{dist}(H_k(x, z), H_k(x, z_0))/2^k < k_0\cdot \varepsilon/2\cdot k_0 + \varepsilon/2 = \varepsilon, \quad \text{for every } x\in X.$$

By (4) we have

(7) $$|\operatorname{dist}(H_k(x, z), Y) - \operatorname{dist}(H_k(x, z_0), Y)| < \varepsilon \quad \text{for every } x \in X,\ k > k_0$$

and from (5) we get

(8) $$|\operatorname{dist}(H_k(x, z), Y) - \operatorname{dist}(H_k(x, z_0), Y)| \leqslant \operatorname{dist}(H_k(x, z), H_k(x, z_0)) < \varepsilon$$
$$\text{for every } x \in X \text{ and } k \leqslant k_0 .$$

Then, (6), (7), (8), (2) and (1) imply that

$$e(h(z), h(z_0)) \leqslant \varepsilon$$

and, as a consequence, h is continuous. Moreover

$$h(z)_k(x) = \alpha_k^z(x) = H_k(x, z).$$

Hence, $\boldsymbol{H}$ is the approximative map associated to h. This finishes the proof. ■

The next corollary whose proof is left to the reader allows us to identify the spaces $A(X, Y)_e^Z$ and $A(X\times Z, Y)_e$ when the shape of Z is trivial.

COROLLARY 4. *Let X, Y, Z be compacta and suppose the shape of Z is trivial. Let $\psi\colon A(X, Y)_e^Z \to A(X\times Z, Y)_e$ be the correspondence which assigns to each map $h\colon Z\to A(X, Y)_e$ the associated approximative map $\boldsymbol{H} = \{H_k, X\times Z\to Y\}$. Then ψ is an isometry, if we consider the metric of uniform convergence on $A(X, Y)_e^Z$.* ■

The following result was the main motivation of this paper.

COROLLARY 5. *Two approximative maps $\boldsymbol{f}, \boldsymbol{g}\colon X\to Y$ are homotopic if and only if they lie in the same path-component of $A(X, Y)_e$. As a consequence, the shape morphisms of X to Y are in bijection with the path-components of $A(X, Y)_e$.*

Proof. $\boldsymbol{f}$ and $\boldsymbol{g}$ are homotopic if and only if there exists an approximative map $\boldsymbol{H} = \{H_k, X\times I\to Y\}$ (see Borsuk [3], p. 206) such that.

$$H_k(x, 0) = f_k(x), \quad H_k(x, 1) = g_k(x), \quad \text{for every } x \in X,\ k = 1, 2, \ldots$$

But by Propositions 6 and 5 this is equivalent to the existence of a map $h\colon I\to A(X, Y)_e$ such that $h(0) = \boldsymbol{f}$, $h(1) = \boldsymbol{g}$. ■

As a consequence of Corollary 5 we get a new characterization of shape-triviality. The proof of this result is left to the reader.

COROLLARY 6. *For a compactum Y the following three conditions are equivalent:*

(a) *$A(X, Y)_e$ is path-connected for every compactum X.*
(b) *$A(Y, Y)_e$ is path-connected.*
(c) *The shape of Y is trivial.* ■

In view of the results which we have obtained about the path-components of $A(X, Y)_e$ it can be interesting to know something about the nature of the (connected) components of $A(X, Y)_e$. Here we get a quite surprising result: $A(X, Y)_e$ is connected. To prove it we need before a technical lemma. The symbol $[f]$ means the class of all approximative maps homotopic to f.

LEMMA 1. *Let X, Y be compacta and $f\colon X\to Y$ a map. Consider the approximative map $\boldsymbol{f} = \{f_k, X\to Y\}$ generated by f (i.e. $f_k = f$, for every k) and let $\boldsymbol{g} = \{g_k, X\to Y\}$ be an approximative map such that $\boldsymbol{g} \notin [\boldsymbol{f}]$. Then, for every $\varepsilon > 0$ there exists $\boldsymbol{h} \in [\boldsymbol{g}]$ such that $e(\boldsymbol{f}, \boldsymbol{h}) \leqslant \varepsilon$.*

Proof. Consider an index $k_0 \geqslant 1$ such that

(1) $$\operatorname{dist}(g_k(x), Y) < \varepsilon, \quad \text{for every } x \in X,\ k > k_0,$$

(2) $$\sum_{k=k_0+1}^{\infty} \Delta/2^k < \varepsilon, \quad \text{where } \Delta = \operatorname{diam} Q.$$

We define an approximative map $\boldsymbol{h} = \{h_k, X\to Y\}$ by

(3) $$h_k = \begin{cases} f = f_k & \text{if } k \leqslant k_0, \\ g_k & \text{if } k > k_0 \end{cases}$$

Clearly $\boldsymbol{h} \in [\boldsymbol{g}]$ and from (1), (2) and (3) it follows that

(4) $$\sum_{k=1}^{\infty} \operatorname{dist}(f_k(x), h_k(x))/2^k < \varepsilon \quad \text{for every } x \in X$$

and that

(5) $$|\operatorname{dist}(f_k(x), Y) - \operatorname{dist}(h_k(x), Y)| = \begin{cases} 0 & \text{if } k \leqslant k_0, \\ \operatorname{dist}(g_k(x), Y) & \text{if } k > k_0 \end{cases}$$

for every $x \in X$.

As a consequence of (4), (5) and (1),

$$e(\boldsymbol{f}, \boldsymbol{h}) \leqslant \varepsilon$$

and this completes the proof. ■

COROLLARY 7. *Let X, Y be compacta such that there are at least two different approximative classes from X towards Y. Then, there exists a class $[\boldsymbol{f}]$ which is a non-open set in $A(X, Y)_e$ and the rest of the classes are non-closed sets in $A(X, Y)_e$.*

Proof. Let $f\colon X\to Y$ be a map and consider the class $[\boldsymbol{f}]$, where $\boldsymbol{f}$ is the approximative map generated by f. Suppose that $\boldsymbol{g}\colon X\to Y$ is an approximative map such that $[\boldsymbol{g}] \neq [\boldsymbol{f}]$. By Lemma 1 there exists, for every $\varepsilon > 0$, an $\boldsymbol{h} \in [\boldsymbol{g}]$ such that

$$e(\boldsymbol{f}, \boldsymbol{h}) \leqslant \varepsilon.$$

This proves that $[\boldsymbol{f}]$ is non-open and $[\boldsymbol{g}]$ is non-closed. ■

We are now in a position to prove our final result.

PROPOSITION 7. *$A(X, Y)_e$ is connected for all compacta X and Y.*

Proof. If there exists only one approximative class of X towards Y then, by virtue of Corollary 5, $A(X, Y)_e$ is path-connected. Hence, we can assume that there are at least two different approximative classes. If $A(X, Y)_e$ is non-connected there are two non-empty open sets G_1, G_2 of $A(X, Y)_e$ such that

(1) $$A(X, Y)_e = G_1 \cup G_2, \quad G_1 \cap G_2 = \emptyset.$$

Choose a map $f\colon X\to Y$ and consider the approximative map $\boldsymbol{f}\colon X\to Y$ generated by f. Suppose that $\boldsymbol{f} \in G_1$. Since $[\boldsymbol{f}]$ is, by virtue of Corollary 5, a path-connected subspace, we have that

(2) $$[\boldsymbol{f}] \subset G_1.$$

Consider an approximative map $\boldsymbol{g} \in G_2$. From (1) and (2) we get that $\boldsymbol{g} \notin [\boldsymbol{f}]$. Since G_1 is open in $A(X, Y)_e$ there exists $\varepsilon > 0$ such that

(3) $$\boldsymbol{h} \in G_1 \quad \text{for every } \boldsymbol{h} \in A(X, Y)_e \text{ with } e(\boldsymbol{f}, \boldsymbol{h}) \leqslant \varepsilon.$$

By virtue of Lemma 1 there exists an approximative map $\boldsymbol{h} \in A(X, Y)_e$ with

(4) $$\boldsymbol{h} \in [\boldsymbol{g}],$$

(5) $$e(\boldsymbol{f}, \boldsymbol{h}) \leqslant \varepsilon.$$

Now (3) and (5) imply that $\boldsymbol{h} \in G_1$ and the fact that $[\boldsymbol{g}] \subset G_2$ together with (4) imply that $\boldsymbol{h} \in G_2$. This contradicts the hypothesis that $G_1 \cap G_2 = \emptyset$, and we have proved that $A(X, Y)_e$ is connected. ■

References

[1] R. D. Anderson, *On topological infinite deficiency*, Michigan J. Math. 14 (1967), 365–383.
[2] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), 223–254.
[3] — *Theory of Shape*, Monograf. Mat. 59, Polish Scientific Publishers, Warszawa 1975.
[4] T. A. Chapman, *Lectures on Hilbert Cube Manifolds*, Reg. Conf. Ser. in Math. 28, Amer. Math. Soc., 1976.
[5] D. E. Christie, *Net homotopy for compacta*, Trans. Amer. Math. Soc. 56 (1944), 275–308.
[6] J. Dydak and J. Segal, *Shape Theory: An Introduction*, Lecture Notes in Math. 668, Springer-Verlag, Berlin 1978.
[7] K. Kuratowski, *Topology*, Academic Press, New York–London 1968.
[8] V. F. Laguna and J. M. R. Sanjurjo, *Spaces of approximative maps*, Math. Japonica 31 (4) (1986), 623–633.
[9] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam 1982.

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