Existence and nonexistence of universal graphs *

by

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Abstract. By the Four Color Theorem, for every planar graph G there is a map of the vertices of G into K_4 preserving the edge relation. If we require the map to be an embedding, then we get the following negative result: There is no countable planar graph H such that for all countable planar G there is a one-to-one homomorphism of G into H. A similar result holds for the countable graphs of a fixed finite degree. In contrast, we prove existence theorems for universal countable graphs under isometric (hence isomorphic) embeddings for the classes of all countable graphs or those of a fixed diameter or colorability.

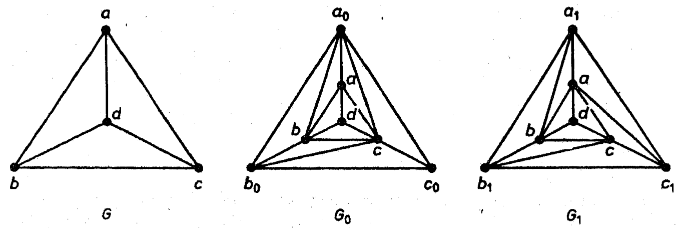
Let $\langle \mathcal{O}, \mathcal{M} \rangle$ be a category, a pair consisting of class of objects and a class of morphisms. An object $U \in \mathcal{O}$ is called *universal* if for every $X \in \mathcal{O}$ there is a morphism $m \in \mathcal{M}$ of X into U. Questions concerning the existence of universal objects arising in graph theory were raised by Rado in [4]. He considered the classes $\mathcal{G}$ of all countable graphs and $\mathcal{G}_{lf}$ of all countable *locally finite* graphs, and the classes $\mathcal{E}_w$ of *weak embeddings* and $\mathcal{E}_s$ of *strong embeddings* of graphs. ("Locally finite" means that each vertex has finite degree. A weak embedding of G into H is a one-to-one map i from the vertex set of G to the vertex set of H such that if x and y are neighbors in G, then $i(x)$ and $i(y)$ are neighbors in H. A strong embedding has the converse property as well.) Obviously, $\langle \mathcal{G}, \mathcal{E}_w \rangle$ has the complete graph on countably many vertices as a universal object, and Rado showed that $\langle \mathcal{G}, \mathcal{E}_s \rangle$ has one also. An argument attributed to de Bruijn showed that $\langle \mathcal{G}_{lf}, \mathcal{E}_w \rangle$ does not have a universal object. A fortiori, $\langle \mathcal{G}_{lf}, \mathcal{E}_s \rangle$ fails to have one also.

The purpose of this paper is to investigate the situation with various classes of countable graphs, including planar graphs, graphs of a fixed finite degree, graphs of a fixed finite diameter, and k-colorable graphs. We will consider the classes of mappings mentioned above as well as the graph homomorphisms and the isometric embeddings.

1. There is no universal planar graph. We consider first the class $\mathcal{G}_p$ of countable *planar* graphs. Let $\mathcal{H}$ be the class of graph homomorphisms. These are the maps which take edges to edges; they are not necessarily one-to-one. By the Four Color

* This research was supported in part by NSF grant DMS-85-01752.

Theorem [1], $\langle \mathcal{G}_p, \mathcal{H} \rangle$ has K_4 as a universal object. That is, a four coloring of the vertices gives a map into K_4, and the fact that neighboring vertices get different colors insures that the image does not have loops. (It might be interesting to note that this is the only way I know to see that $\langle \mathcal{G}_p, \mathcal{H} \rangle$ has a universal object.) We will show in contrast that $\langle \mathcal{G}_p, \mathcal{E}_w \rangle$ does not have a universal object.



Consider the three graphs G, G_0, and G_1 shown in the figure. We regard G as a subgraph of both G_0 and G_1 in the obvious way. Note that these are triangulations, so no new edges can be added to G_0 or G_1 without destroying planarity.

LEMMA 1. *Suppose i is a weak embedding of G into a planar graph H. Then there is at most one j extending i such that j is a weak embedding of either G_0 or G_1 into H.*

Proof. Suppose toward a contradiction that j and k are different maps with both of these properties. There are essentially three cases, depending on the domains of j and k. The first case is when the domains are both G_0. There are two subcases here, depending on whether

$$\{j(a_0), j(b_0), j(c_0)\} = \{k(a_0), k(b_0), k(c_0)\}$$

or not. If the two sets were equal, then since j are k different and injective, we will show by considering $j(a_0)$ that the image $k[G_0]$ would have all the vertices and more edges than G_0. If $j(a_0) = k(a_0)$, then $j(b_0) = k(c_0)$. Since $j(b)$ is a neighbor of $j(b_0)$, $k(b) = j(b)$ would be a neighbor of $k(c_0)$. If $j(a_0) = k(b_0)$, then $k(a)$ would be a neighbor of $k(b_0)$. Finally, If $j(a_0) = k(c_0)$, then $k(a)$ would be a neighbor of $k(c_0)$. Any of these possibilities contradicts the planarity of H. The second subcase is when the sets above are different. Suppose without loss of generality that $j(a_0)$ does not belong to the second set, in particular that it differs from $k(a_0)$. Then the subgraph of H induced by $j[G_0] \cup k[G_0]$ has a contraction to a graph containing $K_{3,3}$ (and possibly more edges) in the following way:

$$j(a_0), \quad k(a_0), \quad i(d); \quad i(a) \quad \{j(b_0), k(b_0), i(b)\} \quad \{j(c_0), k(c_0), i(c)\}\,.$$

(It is possible that $j(b_0) \neq k(b_0)$, but both are neighbors of $i(b)$. The same is true for c_0.) This again contradicts the planarity of H.

The second case is when the domains of j and k are both G_1, and the final case is when both domains occur. These are argued as the first, with a few different details. ■

Notice that this result did not use much about d, only that the interior of G is a connected planar graph containing neighbors of a, b, and c. For example, it would apply if the interior of G looked like G_0 or G_1. We will therefore iterate the original construction, defining a tree of graphs. First we fix some notation. We use s and t as variables ranging over sequences of 0's and 1's — that is, over maps from sets of the form $\{0, \ldots, n\}$ into $\{0, 1\}$. Given a sequence s, we use $s^\frown 0$ and $s^\frown 1$ to denote the natural one point extensions.

Define graphs G_s by recursion on s as follows: If s is the empty sequence λ, then G_s is a copy of K_4 with vertex set $\{a_\lambda, b_\lambda, c_\lambda, d\}$. Suppose that G_s is given and has vertices of the form a_t, b_t, c_t, d for t an initial segment of s. Then $G_{s^\frown 0}$ is an extension of G_s having additional vertices $a_{s^\frown 0}$, $b_{s^\frown 0}$, and $c_{s^\frown 0}$. It has additional edges

$$\begin{array}{lll} \{a_{s^\frown 0}, b_{s^\frown 0}\}\,, & \{b_{s^\frown 0}, c_{s^\frown 0}\}\,, & \{a_{s^\frown 0}, c_{s^\frown 0}\}\,, \\ \{a_s, a_{s^\frown 0}\}\,, & \{b_s, b_{s^\frown 0}\}\,, & \{c_s, c_{s^\frown 0}\}\,, \\ \{b_s, a_{s^\frown 0}\}\,, & \{c_s, a_{s^\frown 0}\}\,, & \{c_s, b_{s^\frown 0}\}\,. \end{array}$$

The graph $G_{s^\frown 1}$ is also an extension of G_s via three new vertices. This time the vertices are $a_{s^\frown 1}$, $b_{s^\frown 1}$, and $c_{s^\frown 1}$. The additional edges are

$$\begin{array}{lll} \{a_{s^\frown 1}, b_{s^\frown 1}\}\,, & \{b_{s^\frown 1}, c_{s^\frown 1}\}\,, & \{a_{s^\frown 1}, c_{s^\frown 1}\}\,, \\ \{a_s, a_{s^\frown 1}\}\,, & \{b_s, b_{s^\frown 1}\}\,, & \{c_s, c_{s^\frown 1}\}\,, \\ \{a_s, c_{s^\frown 1}\}\,, & \{b_s, a_{s^\frown 1}\}\,, & \{c_s, b_{s^\frown 1}\}\,. \end{array}$$

An obvious induction on sequences shows that each G_s is maximal planar and contractible to K_4. So Lemma 1 applies to G_s, $G_{s^\frown 0}$ and $G_{s^\frown 1}$. If t is a subsequence of s then we identify G_t with a subgraph of G_s in the obvious way.

LEMMA 2. *Suppose i is a weak embedding of G_λ into a planar graph H, and let s be any sequence. Then there is at most one map j extending i such that j is a weak embedding of either $G_{s^\frown 0}$ or $G_{s^\frown 1}$ into H.*

Proof. By induction on s, using the generalization of Lemma 1 noted above. That is, this lemma clearly holds when $s = \lambda$. Suppose it held for s, and let j and k be two extensions of i which weakly embed (say) $G_{s^\frown 0}$ into H. By induction hypothesis, the restrictions of j and k to G_s agree. Now $j = k$ by the generalization of Lemma 1. ■

Given a function $f\colon \omega \to \{0, 1\}$, we define G_f to be the union $U_n G_{f|n}$, where $f|n$ is the restriction of f to the set $\{0, \ldots, n-1\}$. Each G_f is a well defined countable planar graph.

THEOREM 3. *There is no universal countable planar graph under weak embeddings.*

Proof. Suppose H is any countable planar graph. To see that H is not universal we show that there is some f such that G_f does not embed weakly into H. In fact we claim that only countably many of the G_f do embed weakly into H, so that there are uncountably many which do not.

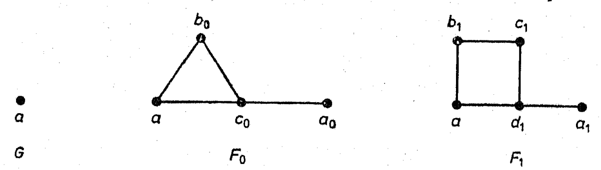
Suppose toward a contradiction that there is an uncountable set $\mathscr{F}$ of functions and an $\mathscr{F}$-indexed set $\{i_f\colon f\in\mathscr{F}\}$ such that for all $f\in\mathscr{F}$, i_f is a weak embedding of G_f into H. The set $\{i_f|G_\lambda\colon f\in\mathscr{F}\}$ of restrictions is a subset of the set of maps from G_λ into H. This set is countable, so there are $f\neq f'$ from $\mathscr{F}$ such that $i_f|G_\lambda = i_{f'}|G_\lambda$. Let s be the largest sequence such that s is a subfunction of both f and f'. We may assume without loss of generality that $s^\frown 0$ is a subfunction of f and and $s^\frown 1$ is a subfunction of f'. But then $i_f|G_{s^\frown 0}$ and $i_{f'}|G_{s^\frown 1}$ are weak embeddings of $G_{s^\frown 0}$ and $G_{s^\frown 1}$, and they extend $i_f|G_\lambda$. This contradicts Lemma 2. ■

Let $\mathscr{H}_s$ be the class of *strong homomorphisms* between graphs. These are the maps which take edges to edges and conversely. They need not be one-to-one. As a corollary to Theorem 3, we see that there is no universal countable planar graphs under strong homomorphisms. This follows from the proof last theorem and the observation that for all f, every strong homomorphism of G_f into *any* graph is one-to-one. Indeed, if G is any graph such that every two distinct vertices x and y of G have distinct sets of neighbors, then every strong homomorphism i of G is one-to-one. For if z is a neighbor of x but not of y, then $i(x)\neq i(y)$ lest $i(z)$ be a neighbor of $i(y)$. This remark will apply as well to the negative result in Theorem 4 below.

2. There is no universal graph of fixed finite degree. For Δ finite, let $\mathscr{G}_\Delta$ be the class of countable graphs of maximum degree Δ. It is well known that each finite graph of degree Δ is $(\Delta+1)$-colorable, and for infinite graphs this follows by the König Infinity Lemma. Therefore $\langle\mathscr{G}_\Delta, \mathscr{H}\rangle$ has $\mathscr{K}_{\Delta+1}$ as a universal structure. For the weak embeddings we can use the method of the last section to prove a non-existence result.

THEOREM 4. *For finite Δ, there is no universal countable graph of degree Δ under weak embeddings.*

Proof. For $\Delta = 2$ this is immediate, so we will assume $\Delta\geqslant 2$. Let G be a one point graph whose vertex is a, and let F_0 and F_1 be the graphs shown in the figure. We regard each of these as an extension of G in the obvious way.



For any graph F and any set S of its vertices, let $\mathscr{A}(F, S, \Delta)$ be the graph obtained by adding new vertices and edges to insure that all vertices in S have degree Δ. That is, if $x\in S$ has degree $\Delta'<\Delta$, then we add $\Delta-\Delta'$ points, and join each of these to x. None of the new points are joined to each other or to any of the other points of F.

Define graphs G_0^n by recursion on $n\in\omega$:

$$G_0^0 = F_0$$
$$G_0^{n+1} = \mathscr{A}(G_0^n, G_0^n - \{a_0\}, \Delta).$$

We regard G_0^n as a subgraph of G_0^{n+1}, and we set $G_0 = \bigcup_n G_0^n$. G_1 is defined from F_1 in the same way. Note that every vertex of G_0 except a_0 has degree Δ in G_0, and G_0 has no cycles except the triangle. Similar remarks apply to G_1.

LEMMA 5. *Suppose i is a weak embedding of G into a graph H of degree Δ. Then there is at most one map j extending i such that*

(a) *j is a weak embedding of either F_0 or F_1 into H;*

(b) *If the domain of j is F_0, then j extends to a weak embedding of G_0, and similarly for F_1 and G_1;*

(c) *If the domain of j is F_0, then $j(a_0)$ lies on a cycle, and similarly for F_1 and $j(a_1)$.*

Proof. Again suppose that j and k were different maps with these three properties. Suppose first that the domains were both F_0. Let j^* and k^* be the extensions to G_0. Let $G' = G_0-\{a_0\}$. By injectivity of j^* and finiteness of Δ we see that for all $x\in G'$, the neighbors of $j^*(x)$ in H are exactly the images under j^* of the neighbors of x. It follows that the cycles in $j^*[G']$ are exactly the images of cycle in G'. Also, the only neighbor of $j(a_0)$ in $j^*[G']$ is $j(c_0)$. Thus the only triangle in H containing $j(a)$ is $\{j(a), j(b_0), j(c_0)\}$. These remarks hold as well for k, of course, and since $j(a) = i(a) = k(a)$, we see that $\{j(b_0), j(c_0)\} = \{k(b_0), k(c_0)\}$. If $j(b_0) = k(b_0)$ and $j(c_0) = k(c_0)$, then we must have $j(a_0) = k(a_0)$ because $j(a_0)$ is the only neighbor of $j(c_0)$ besides $j(a)$ and $j(b_0)$ which lies on a cycle (and similarly for k). But this would contradict the assumption that $j\neq k$. Therefore $j(b_0) = k(c_0)$, and $j(c_0) = k(b_0)$. By (c), $k(a_0)$ must equal $j(a)$ or $j(c_0)$, since these are the only neighbors of $j(b_0)$ which lie on cycles. But by injectivity, $k(a_0)\neq k(a) = j(a)$ and $k(a_0)\neq k(b_0) = j(c_0)$. Thus the uniqueness is proved when the domains are both G_0.

Virtually the same argument works when the domains of j and k are F_1. If the domain of j is G_0 and the domain of k is G_1, then we get a contradiction since $k[G_1]$ contains no triangles. ■

Notice that this lemma holds when G has vertices other than a provided, say, that a does not lie on a cycle in G. So we can iterate the construction, and define graphs F_s and G_s by simultaneous recursion. To get $F_{s^\frown 0}$ from G_s we add vertices $\{a_{s^\frown 0}, b_{s^\frown 0}, c_{s^\frown 0}\}$ with the obvious 4 edges. Similarly, we construct $F_{s^\frown 1}$ from G_s. We obtain G_s from F_s using $\mathscr{A}$ as above. An easy induction shows that a_s lies on cycle in G_s but it does lie on a cycle of both $G_{s^\frown 0}$ and $G_{s^\frown 1}$. We can also state and prove a lemma which is to Lemma 5 what Lemma 2 is to Lemma 1. And now the proof of Theorem 4 follows the same way Theorem 3 followed from Lemma 2. ■

It is also possible to prove our negative results by diagonalization, the way de Bruijn (in [4]) proves the nonexistence of universal graphs in the class of locally finite graphs. As an example of a proof of this type, we have the following result which is due essentially to de Bruijn.

Theorem 6. *There is no universal locally finite graph under homomorphisms.*

Proof. Let X be a countable graph, and enumerate X as $\{x_n: n \in \omega\}$. Define function $f_n: X \to \omega$ by $f_n(x) = |\{y \in X: d_X(x, y) \leqslant n\}|$. Let f be given by $f(n) = f_n(x_n)+1$. Let G_0 be the one point graph $\{a\}$, and for $n \geqslant 1$, let $G_n = K_{f(n)}$. Form G_X by joining each vertex of each G_n to each vertex of G_{n+1}. Suppose $h: G_X \to X$ is a homomorphism. Since h cannot identify points in a complete subgraph, we have for all n that $f_n(h(a)) \geqslant f(n)$. But if n^* is such that $h(a) = x_{n^*}$, then $f_{n^*}(x_{n^*}) \geqslant f(n^*) = f_{n^*}(x_{n^*})+1$. This contradiction shows that X is not universal. ■

3. Sufficient conditions for universal graphs. In this section, we present a set of sufficient conditions for a class of relational structures to have a countable universal structure under isomorphic embeddings. Our work is a related to classical results of model theory (cf., e.g., [2]), but there is a difference. Our condition (VI) is meant to be an alternative to the following standard condition: If X is a finite subset of the universe $|A|$ of A, and if $A \in \mathscr{C}$, then there is is a finite $B \in \mathscr{C}$ such that $X \subseteq |B|$, and $B \subseteq A$ (B is a substructure of A). This more familiar condition does not hold for any of the classes which we will consider.

Let $\mathscr{C}$ be a class of relational structures of the same similarity type with the following properties:

(I) $\mathscr{C}$ contains at least one but only countably many finite structures (up to isomorphism).

(II) $\mathscr{C}$ is closed under isomorphic images.

(III) If $A, B \in \mathscr{C}$ are finite, then there is some finite $C \in \mathscr{C}$ such that both A and B are isomorphic to substructures of C.

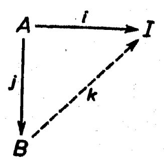
(IV) If $f_1: A \to B_1$ and $f_2: A \to B_2$, are isomorphic embeddings among finite elements of $\mathscr{C}$, then there is some finite $C \in \mathscr{C}$ and isomorphic embeddings $g_1: B_1 \to C$ and $g_2: B_2 \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

(V) $\mathscr{C}$ is closed under unions of chains.

(VI) Let $A, B \in \mathscr{C}$, and suppose that A is finite. Let $S \subseteq |A| \cap |B|$, and suppose that the reducts $A|S$ and $B|S$ are the same. Let $x \in B-S$. Then then there is a finite $C \in \mathscr{C}$ such that $A \cup \{x\} \subseteq |C|$ and $C|(S \cup \{x\}) = B|(S \cup \{x\})$.

Lemma 7. *Every class $\mathscr{C}$ with properties* (I)–(VI) *above has a countable universal structure.*

Proof. A consequence of (I), (II), (IV), and (V) is the existence of a countable structure I with the following injectivity property: Whenever $i: A \to I$ and $j: A \to B$ are isomorphic embeddings among elements of $\mathscr{C}$, there is an isomorphic embedding $k: B \to I$ such that $k \circ j = i$:



I is constructed by amalgamating the finite structures in $\mathscr{C}$ successively using (IV). We may assume therefore that I has a finite substructure which belongs to $\mathscr{C}$. Hence by (III) and injectivity, we see that I embeds every finite structure in $\mathscr{C}$.

Let I be injective, and let X be a countable element of $\mathscr{C}$. We want to see that X maps isomorphically into I, so we may assume that $|X| \cap |I| = \emptyset$. Fix an enumeration without repetitions $\{x_i: i \in \omega\}$ of the elements of X. We construct a chain $I_0 \subseteq I_1 \subseteq \ldots I_n \subseteq \ldots$ of substructures I which belong to $\mathscr{C}$. We will also find points $\{i_n: n \in \omega\}$ such that $\{i_m: m<n\} \subseteq I_n$, and the map $x_n \mapsto i_n$ will be an isomorphic embedding.

Let I_0 be an arbitrary finite substructure of I. Given I_n and $\{i_m: m<n\}$, we apply (VI) to obtain I_{n+1} and i_n. Take $A = I_n$, $B = X$ with i_m replacing x_m for $m<n$, $S = \{i_m: m<n\}$, and $x = x_n$. Then we get some finite $C \in \mathscr{C}$ extending A and containing x_n such that $C|(S \cup \{x_n\}) \cong X|(\{x_m: m \leqslant n\})$ via the obvious map. Now the inclusion maps of A into I and C are isomorphic embeddings, and by injectivity we get a map $k: C \to I$ which is the identity on A. Let $I_{n+1} = k[C]$, and let $i_n = k(x_n)$. Then $X|\{x_m: m \leqslant n\} \cong I|\{i_m: m \leqslant n\}$ via the map $x_m \mapsto i_m$. ■

Now we will apply this result to deduce the existence of a countable graph which isometrically embeds all countable graphs. Consider the class of *distanced graphs*. The structures are of the form

$$G = \langle V, R, D_0, \ldots, D_n, \ldots\rangle$$

where all of the relation symbols are binary, and subject to the axioms that R is a symmetric irreflexive relation on V, and that $D_n(x, y)$ iff there is a path in V of length n from x to y but there is no path of shorter length. (So D_0 is always interpreted as $=$ and D_1 is always interpreted as R; we have included them to save on notation.) Every graph extends to a distanced graph in a unique way. An *isometric embedding* $f: G \to G'$ of graphs is a map between the respective vertex sets V and V' such that $d_G(x, y) = d_{G'}(f(x), f(y))$ for all $x, y \in V$. Every isomorphic embedding of distanced graphs is an isometric embedding of the underlying graphs, and conversely. But an isomorphic embedding of graphs is usually not an isometric embedding.

Lemma 8. *The class of distanced graphs has properties* (I)–(VI).

Proof. We will only outline the proofs here. The only two properties that need checking are (IV) and (VI). (Property III follows from the fact that both A and B are isometrically embedded in the disjoint union of A and B. It also follows from applying (IV) in the case that A is the one-point graph, $B_1 = A$, and $B_2 = B$.)

For the amalgamation property (IV), take disjoint copies of B_1 and B_2 and identify $f_1(a)$ and $f_2(a)$ for $a \in A$. The maps g_1 and g_2 are the natural ones. For (VI), take C to be A together with x and some new elements. The new elements are chains from x to those $y \in S$ which lie in the same connected component of B as x. The length of each chain is the distance in B from x to y. The chains in C are disjoint, and the only edge relations among the new elements are those on the chains. The verifications that both of these constructions work is by induction using the triangle inequality (cf. [3] or the proofs of Theorems 10 and 11 below). ■

Let U be a countable universal distanced graph. Then U embeds isometrically every countable graph X since X can be expanded to a distanced graph.

We will apply Lemma 7 to construct other universal graphs under isometric embeddings. As in the last result, we will not work with graphs themselves but in some class of structures for an expanded signature.

Consider first the class of *k-colored distanced graphs*. These are structures of the form $\langle G, C_1, C_2, \ldots, C_k\rangle$ such that G is a distanced graph, and $C_1, \ldots, C_k$ are one place relations on V. The axioms we take are the axioms for distanced graphs together with the axioms

$$(\forall v)[C_1(v) \vee \ldots \vee C_k(v)]$$

$$(\forall v, w)\big[R(v, w) \to \neg[(C_1(v) \wedge C_1(w)) \vee \ldots \vee (C_k(v) \wedge C_k(w))]\big].$$

THEOREM 9. *There is a universal k-colorable graph under isometric embeddings.*

Proof. Again we begin by checking properties (IV) and (VI), for the class of k-colored distanced graphs. The verifications are refinements of the constructions in Lemma 8. (IV) follows from the proof of the amalgamation property for distanced graphs. That is, consider k-colored distanced graphs A, f_1, and f_2, and forget the colors. Let C, g_1, and g_2 arise from the amalgamation of the distanced graph reducts. We need to turn C into a colored graph, and for this, set $C_i(g_1(x))$ iff $C_i(x)$ in B_1 and $C_i(g_2(x))$ iff $C_i(x)$ in B_2. This assignment of colors is consistent on $g_1 \circ f_1[A]$. Since the colors on edges of C are images of colors on edges of either B_1 or B_2, C is a k-colored distanced graph. And now g_1 and g_2 preserve color, and $g_1 \circ f_1 = g_2 \circ f_2$.

For (VI), we again need to refine the argument from the case of distanced graphs by showing how to assign colors to the points on the new chains. For $y \in S$, fix a minimal path in B from x to y. Color the corresponding path in C the same way.

Now that we know that there is a universal k-colored distanced graph I, it follows that the forgetting the colors, I is a universal k-colorable graph for isometric embeddings. That is, let X be k-colorable. Turn X into a k-colored distanced graph and then embed it in I. ■

4. Universal graphs of finite diameter under isometric embeddings. Consider next the class of *N-tethered graphs*. These are structures of the form $\langle G, t\rangle$ such that G is a distanced graph, and $t \in V$ is a distinguished constant. The axioms we take are the axioms for distanced graphs together with the axiom

$$(\forall v)[D_0(v, t) \vee \ldots \vee D_N(v, t)].$$

Thus every N-tethered graph is a distanced graph with a distinguished vertex t such that every vertex v is at most N units from t.

THEOREM 10. *For all N, there is a universal graph of diameter $2N$ under isometric embeddings.*

Proof. We claim first that the class of N-tethered graphs has properties (I)–(VI) above. As with distanced graphs, the only two properties which need checking are (IV) and (VI). We use the notation of (IV) and (VI), and the constructions of Lemma 8. For (IV), note that in C, the amalgamation of the underlying distanced graphs, each vertex is at most N units from $g_1 \circ f_1(t) = g_2 \circ f_2(t)$. So C expands to an N-tethered graph.

For (VI), suppose that as N-tethered graphs, $A|S = B|S$. Then as distanced graphs $A|(S \cup \{t\}) = B|(S \cup \{t\})$. We may assume that $x \neq t$. Let C_0 arise from (VI) applied to the underlying distanced graphs. Thus C_0 contains x, and as distanced graphs $C_0|(S \cup \{t, x\}) = B|(S \cup \{t, x\})$. The problem is that C_0 may not be tethered to t. We therefore add independent chains from t of length N to each point of $C_0 - A$ to get a graph C. We need to see that as distanced graphs $C|(S \cup \{t, x\}) = C_0|(S \cup \{t, x\})$. Clearly $d_C(x, t) = d_{C_0}(x, t)$, and we show in fact that A is isometrically embedded in C. Suppose this were false, and let m be least such that $d_C(u, v) = m < d_A(u, v)$ for some $u, v \in A$. Fix a path p in C from u to v witnessing this. Now p is not a path in C_0, and therefore it must contain t. By minimality of m, we may assume that either u or v is t. Suppose that $v = t$. Also, p must contain a point of $C_0 - A$, so $d_C(u, t) > N$. But since A is N-tethered, $d_A(u, v) \leqslant N$. So $d_A(u, v) < d_C(u, v)$, and this is a contradiction.

We conclude that as distanced graphs $C|(A \cup \{x\}) = C_0|(A \cup \{x\})$; a fortiori $C|(S \cup \{t, x\}) = C_0|(S \cup \{t, x\})$. So as N-tethered graphs, $C|(S \cup \{x\}) = B|(S \cup \{x\})$.

So we conclude on the basis of Lemma 7 that there is a universal N-tethered graph U. We will consider U as merely a graph by forgetting the distance relations, and in order to see that U is universal for graphs of diameter $2N$, we need only show that every graph X of diameter $2N$ is isometrically embedded in a graph Y which can be expanded to an N-tethered graph. Let X have diameter at most $2N$. Form an N-tethered graph Y from X by adding a new point t and independent chains of length N from t to each vertex x of X. Y is tethered to t, and since the only paths between vertices of X using the edges of Y which are not edges of X have length at least $2N$, X is isometrically embedded in Y. ■

Consider last the class of *N-multitethered graphs*. There are structures of the form $\langle G, T, E_0, E_1, \ldots, E_N\rangle$ such that G is a distanced graph, and $T, E_0, \ldots, E_N$ are one place relations on G. The axioms we take are the axioms for distanced graphs together with the axioms

$$(\forall x, y)[T(x) \wedge T(y) \wedge x \neq y \to R(x, y)]$$

$$(\forall v)(\exists t)[T(t) \wedge [D_0(v, t) \vee \ldots \vee D_N(v, t)]]$$

$$(\forall x)[E_0(x) \leftrightarrow T(x)]$$

$$(\forall x)\{E_i(x) \leftrightarrow [(\exists t)[T(t) \wedge D_i(x, t)] \wedge (\exists t)[T(t) \wedge D_{i-1}(x, t)]]\} \qquad (i \geqslant 1)$$

Every N-multitethered graph is a distanced graph with a set T of vertices such that the induced subgraph determined by T is complete, and every vertex v is at most N

units from some element of T. So the diameter of an N-multitethered graph is at most $2N+1$. In addition, each N-multitethered graph comes with relations $E_0, \ldots, E_N$ with the property that $E_i(x)$ iff the shortest distance from x to any member of T is i. Thus $A \subseteq B$ as N-multitethered graphs iff A is an isometric subgraph of B, $T^A \subseteq T^B$, and for all $x \in A$,

$$\min\{d_A(x, t)\colon\ t \in T^A\} = \min\{d_B(x, t)\colon\ t \in T^B\}.$$

THEOREM 11. *For all N, there is a universal graph of diameter $2N+1$ under isometric embeddings.*

Proof. We follow the proof of the last theorem. Again the only two properties of the N-multitethered graphs which need checking are (IV) and (VI). For (IV), let C_0, g_1, and g_2 arise from amalgamating the underlying distanced graphs. Let $D_i = g_i[B_i]$, so D_0 and D_1 are distanced subgraphs of C_0. As sets of vertices, $C_0 = D_0 \cup D_1$. We will identify A with its image under $g_0 \circ f_0 = g_1 \circ f_1$; so as sets of vertices $D_0 \cap D_1 = A$. Every path in C_0 from an element of $D_0 - D_1$ to an element of D_1 contains an element of A, and vice-versa. Let A, D_0 and D_1 be multitethered to T^A, T^{D_0} and T^{D_1}, respectively. The amalgamation insures that the restrictions of T^{D_0} and T^{D_1} to A are exactly T^A. We will define an N-multitethered graph C containing D_0 and D_1 as N-multitethered subgraphs. The vertices of C are those of C_0. C has the edges of C_0 with additional edges $\{x, y\}$ for $x \in T^{D_0} - T^{D_1}$, $y \in T^{D_1} - T^{D_0}$. Now set $T = T^{D_0} \cup T^{D_1}$. To complete the definition of C, we specify the relations E_i according to the axioms above.

We now check that C has all of the right properties. Clearly the subgraph of C induced by T is complete, and each point of C is at most N units from some element of T. C_0 is not an isometric subgraph of C, but we claim that D_0 and D_1 are still isometric subgraphs of C. Before we show this, however, we will prove that for all $x \in D_0$,

$$\min\{d_{D_0}(x, t)\colon\ t \in T^{D_0}\} = \min\{d_C(x, t)\colon\ t \in T\}.$$

(The same proof establishes the analogous property for D_1, of course.)

In defining C from C_0, the only new edges were between elements of T. So it is sufficient to show that for all $x \in D_0$,

$$\min\{d_{D_0}(x, t)\colon\ t \in T^{D_0}\} = \min\{d_{C_0}(x, t)\colon\ t \in T\}.$$

Suppose towards a contradiction that for some x, the first minimum is greater than the second. No such x can belong to A, since if $x \in A$, the facts that $A \subseteq D_0$ and $A \subseteq D_1$ as N-multitethered graphs would imply the above equality. Fix such a point x which minimizes the second minimum. Let $t^* \in T$ realize the second minimum for x, and let p be a path in C_0 of minimal length between x and t^*. Then $t^* \in D_1 - D_0$, and so p contains some $y \in A$ after x. Also,

$$\min\{d_{D_0}(y, t)\colon\ t \in T^{D_0}\} = \min\{d_{C_0}(y, t)\colon\ t \in T\} = d_{C_0}(y, t^*).$$

Let $t^{**} \in D_0$ realize the first minimum. Thus

$$d_{D_0}(x, t^{**}) \leqslant d_{D_0}(x, y) + d_{D_0}(y, t^{**}) = d_{C_0}(x, y) + d_{C_0}(y, t^*) = d_{C_0}(x, t^*),$$

and this is a contradiction.

Now we will prove that D_0 and D_1 are isometric subgraphs of C. For if not, let m be least such that there are $x, y \in$ (without loss of generality) D_0 such that $d_C(x, y) = m < d_{D_0}(x, y)$. Fix a minimal path p in C between x and y. By minimality, y is the first point on p after x that lies in D_0. Since p is not a path of C_0, it contains two vertices joined by a new edge, say a_0 and a_1 ($a_0 \in T^{D_0} - T^{D_1}$, $a_1 \in T^{D_1} - T^{D_0}$). So p may be written as $x, \ldots, a_1, a_0$. By the last paragraph, there is some $a_2 \in T^{D_0}$ such that $d_{D_0}(x, a_2) \leqslant d_{C_0}(x, a_1)$. Since a_0 and a_2 are neighbors in D_0, $d_{D_0}(x, a_0) \leqslant d_C(x, a_0)$. This once again is a contradiction.

Turning to (VI), let C_0 arise from (VI) applied to the underlying distanced graphs. Thus C_0 contains a point x such that as distanced graphs $C_0|(S \cup \{x\}) = B|(S \cup \{x\})$. Let T be the multitether of A, and let $t^* \notin C_0$ be a new point. Form C from C_0 by adding t^* as a new vertex. Also add (1) independent chains of length N from t^* to each point of $C_0 - A$, and (2) edges from t^* to the elements of T. To finish the definition of C, interpret the E_i relations in the natural way. Now C is N-multitethered to $T \cup \{t^*\}$. We need to see that A is isometrically embedded in C. Suppose that m is least so that for some $x, y \in A$, $d_C(x, y) = m < d_A(x, y)$. Fix a path p in C of length m. There are three cases here, depending on p. If p has two subpaths of type (1), then its length is at least $1+N+N+1 = 2N+2$ which is clearly a contradiction. If it has two new edges of type (2), then it has a subpath of the form t_1, t^*, t_2 for some $t_1, t_2 \in T$. But then t_1 and t_2 are neighbors. So we contradict the minimality of the length of p. The most interesting case is when subpaths of both new types occur in p. Suppose that $\{t, t^*\}$ is an edge of type (2) on p and that $t^*, z_1, \ldots, z_N$ is a subpath of p of type (1). By minimality we may suppose that $x = t$. Now $z_N \notin A$, so m, the length of p, is therefore at least $N+2$. But $x = t$ belongs to T, so $d_A(x, y) \leqslant N+1$, and this too is a contradiction. This proves that A is isometrically embedded in C. It remains to check that for all $x \in A$, the distance (in C) from x to t^* is not less than the distance from x to any element of T. This is true because all of the paths of C which are not paths of A either contain some element of T or are at least N units long.

To conclude the proof, we need only see that every graph X of diameter $2N+1$ is isometrically embedded in a graph which can be expanded to an N-multitethered graph. Form an N-multitethered graph Y from X by adding a set $T = \{t_x\colon\ x \in X\}$ of distinct new vertices. Add edges from t_x to t_y for $x \neq y$. For each $x \in X$, add a new chain from x to t_x. The only new paths in Y between elements of X have length at least $2N+1$, and between each pair in Y there is a path of that length. So Y is multitethered to T, and X is isometrically embedded in Y. Interpret the relations E_i in the natural way. ■

We conclude with a short discussion of the homogeneity properties of the universal graphs constructed in this section. Recall that a structure A is homogeneous if every isomorphism of finite substructures extends to an automorphism of A. The universal k-colored distanced graph constructed in Theorem 11 is homogeneous as a k-colored distanced graph. (This and all of our homogeneity results are established by back-and-forth arguments.) If one forgets the colors, then the reduct is not homogeneous as a distanced graph.

The universal graph U_{2N} of diameter $2N$ constructed in Theorem 10 is not homogeneous as a distanced graph. (It is homogeneous as an N-tethered graph.) The vertex t is the only point whose distance from each $v \in U$ is at most N. However, if one considers the subgraph U'_{2N} induced by $\{v \in U\colon d_U(v, t) = N\}$, then the injectivity of U_{2N} implies that U'_{2N} is isometrically embedded in U_{2N}. And the proof of Theorem 10 shows that every graph of diameter $2N$ is isometrically embedded in U'_{2N}. So U'_{2N} is a universal graph of diameter $2N$ (although it is not N-tethered.) Another injectivity argument shows that U'_{2N} is homogeneous as a distanced graph. Similar remarks apply to the universal graph of diameter $2N+1$. (Incidentally, when $N = 1$, U'_{2N} is exactly Rado's graph. Also, U'_{2N} may be described in the following way: Let U be the universal homogeneous distanced graph, and let x be an arbitrary point of U. Then U'_{2N} is the subgraph of U induced by the points whose distance from x is exactly N.)

All of our universal graphs have the property that any finite set of vertices is contained in a finite isometric subgraph. Using this fact and a back-and-forth argument, one can show that the homogeneous universal graphs in the various categories are unique.

The table summarizes the results concerning universal graphs which we cited or proved.

Class of countable graphs	Class of morphisms		
	homomorphisms	homomorphic embeddings	isometric embeddings
all graphs	√	√	√
diameter D	√	√	√
k-colorable	√	√	√
planar	√	×	×
degree Δ	√	×	×
locally finite	×	×	×

Existence (√) and nonexistence (×) of universal graphs.

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Received 22 June 1987;
in revised form 7 September 1987