Generalized Hilbert fields, II

by

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Abstract. Quadratic form structure is investigated for fields with a unique anisotropic torsion n-fold Pfister form (n-T-local field) and for fields where torsion n-fold Pfister forms are either universal or half-universal (n-T-Hilbert field). It is proved that n-T-locality always implies $(n-1)$-T-Hilbertcity but not conversely. Sufficient conditions are given in terms of the number of orderings of the field ensuring the converse to hold. Numerous examples of T-local fields are given by using techniques of abstract Witt rings. A going-up theorem is proved for n-T-locality, $n = 1$ and 2, in totally positive quadratic extensions.

Introduction. A. Fröhlich [8] and I. Kaplansky [10] investigated quadratic forms over fields that share with P-adic fields and real closed fields the pattern of behaviour of value sets of binary forms. Their axioms imply the characteristic feature of the local theory — the existence of a unique non-split quaternion algebra. The set-up and results of their work have been generalized in [19] to the context of n-fold Pfister forms (theirs corresponding to $n = 1$). The generalization of local theory in [19] retains the relatively trivial behaviour of quadratic forms over formally real Hilbert fields. In case $n = 1$, formally real Hilbert fields are just Euclidean (two square classes) and for $n \geqslant 1$ they are uniquely ordered and there are only two possible value groups for all n-fold Pfister forms. This is in contrast to the situation we encounter in non-real Hilbert fields, where the intersections of value groups of n-fold Pfister forms do not obey any easily formulated rule. Thus it is natural to wonder if there exists a generalized version of the local theory retaining the non-trivial features of non-real generalized Hilbert fields and covering in a unified manner both non-real and formally real fields.

This paper axiomatizes such a generalization of local theory using a fruitful method of transferring properties from non-real to formally real fields, familiar in algebraic theory of quadratic forms. If a property of quadratic forms is important for non-real fields and becomes vacuous or not-so-interesting for formally real fields, it usually turns out to be equally important in the formally real case, when considered only for torsion, instead of all, quadratic forms. In our context this approach reveals a class of formally real fields that share essential properties with straightforward generalization of classical non-real local fields. Let us stress that what we get for formally real fields does not generalize the number-theoretic aspect of Hilberticity or locality (for this, see [19]).

We consider T-Hilbert fields, where torsion n-fold Pfister forms are half-universal or universal, and T-local fields, where there is only one anisotropic torsion $(n+1)$-fold Pfister form. These two notions — known to be equivalent for non-real fields ([19]) — turn out to be no longer equivalent for formally real fields. We show that T-locality is stronger than T-Hilberticity and study the cases where the two are actually equivalent. This is done in Section 1. The second section produces numerous examples of T-local and T-Hilbert fields. Since T-locality and T-Hilberticity are invariant relative to Witt equivalence of fields, one can try to identify T-local and T-Hilbert types among the classes of Witt equivalent fields. It turns out that for any $2^m \leqslant 16$ the number of T-local and T-Hilbert types exceeds one half of the number of all the types of Witt equivalent fields with 2^m square classes. We give constructions suggesting that the occurrence of T-local and T-Hilbert types for any finite number of square classes is equally high. In the final third section we prove a going-up theorem for T-locality in totally positive quadratic extensions for $n = 1$ and $n = 2$. The general case seems to require a new approach and has been left open.

Notation and terminology. We introduce the following unified T-notation.

$TW(F)$ is the torsion ideal of the Witt ring $W(F)$, usually written $W_t F$.

$TI^n F = I^n F \cap TW(F)$.

$TP_n F$ is the set of isometry classes of torsion n-fold Pfister forms over F, often viewed as subset of $W(F)$,

$TR_n F$ is the "nth torsion radical" of F,

$TR_n F = \bigcap \{D_F \Phi\colon \Phi \in TP_n F\}$.

All fields are assumed to have characteristic $\neq 2$.

A field F is said to be *n-T-local* if there is a unique anisotropic torsion n-fold Pfister form over F (up to isometry).

F is said to be *n-T-Hilbert* if every torsion n-fold Pfister form Φ over F is either half-universal (i.e. $|\dot{F}\colon D_F\Phi| = 2$) or universal (i.e. $\dot{F} = D_F\Phi$), and there are some half-universal forms in $TP_n F$.

$F_+ := \sum F^2$ is the set of non-zero sums of squares in F (totally positive elements of F).

X_F is the set of all orderings of F (empty, if F is non-real).

If $K \supset F$ is a field extension and Φ is a form over F, then $\Phi_K := i(\Phi)$, where $i\colon W(F) \to W(K)$ is the functorial ring homomorphism.

Unexplained notation and terminology follows [14].

§ 1. T-Hilbert and T-local fields. In this section we investigate the relationships among the following four properties of a field F. Throughout we assume $n \geqslant 2$ is an integer.

(A) F is an n-T-local field.

(B) F is an $(n-1)$-T-Hilbert field.

(C) F is an n-T-universal field (i.e. $\dot{F} = D_F(\Phi)$ for every $\Phi \in TP_n F$).

(D) F satisfies A_{n+1} (i.e. $|TP_{n+1}F| = 1$, cf. [7])

THEOREM 1.1. (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Leftrightarrow$ (D).

We will need the following two lemmas.

LEMMA 1.2 ([6], Corollary 1). *For arbitrary field F, if $\Phi \in P_n F$ and $2\Phi = 0$ in $W(F)$, then $\Phi = \langle\langle -y, y_2, \ldots, y_n\rangle\rangle$ for some $y \in D_F\langle 1, 1\rangle$ and $y_i \in \dot{F}$.*

LEMMA 1.3. *If $n \geqslant 2$, F is $(n-1)$-T-Hilbert and $\Phi \in TP_n F$, then $2\Phi = 0$ in $W(F)$.*

Proof. Suppose $2^{k+1}\Phi = 0$ and $2^k\Phi \neq 0$ for some $k \geqslant 0$. By Lemma 1.2, $2^k\Phi = \langle\langle -y, y_2, \ldots, y_{n+k}\rangle\rangle$, where $y \in D_F\langle 1, 1\rangle$ and $y_i \in \dot{F}$. Consider

$$\Psi := \langle\langle -y, y_2, \ldots, y_{n-1}\rangle\rangle \in TP_{n-1}F.$$

If Ψ is universal, then it follows that $2^k\Phi = 0$, a contradiction. If Ψ is half-universal, then $y_n \notin D_F(\Psi)$, since otherwise $\Psi \otimes \langle\langle y_n\rangle\rangle = \Psi \otimes \langle\langle 1\rangle\rangle = 0$ and $2^k\Phi = 0$. Hence $D_F(\Psi \otimes \langle\langle y_n\rangle\rangle) = D_F(\Psi) \cup y_n \cdot D_F(\Psi) = \dot{F}$ and it follows that $2^k\Phi = 0$ if $k > 0$. Hence $k = 0$ and $2\Phi = 0$, as desired.

Proof of Theorem 1.1. (*A*) $\Rightarrow$ (B). Suppose $0 \neq \Phi \in TP_n F$. Thus F does not satisfy A_n and so there exists an n-fold Pfister form Ψ killed by 2 in $W(F)$ ([7], Lemma 4.2). By (A), we must have $\Phi = \Psi$. Hence $2\Phi = 0$ and by Lemma 1.2, there are torsion non-universal $(n-1)$-fold Pfister forms over F (for instance, $\langle\langle -y, y_2, \ldots, y_{n-1}\rangle\rangle$). If σ is such a form and $a, b \in \dot{F}$, $a \notin D_F(\sigma)$, $b \notin D_F(\sigma)$, then $\langle\langle -a\rangle\rangle \otimes \sigma$ and $\langle\langle -b\rangle\rangle \otimes \sigma$ are anisotropic torsion n-fold Pfister forms, hence isometric, by (A). It follows that $ab \in D_F(\sigma)$ and this proves that σ is half-universal.

(B) $\Rightarrow$ (C). Let $\Phi \in TP_n F$. By Lemma 1.3, we have $2\Phi = 0$ and then, by Lemma 1.2, Φ has a factor $\Psi \in TP_{n-1}F$. If Ψ is universal, it follows that $\Phi = 0$. If Ψ is half-universal and $\Phi \neq 0$, then a diagonal entry of Φ is not represented by Ψ and so Φ is universal. In any case Φ is universal, as needed.

(C) $\Rightarrow$ (D). Let $\Phi \in TP_{n+1}F$ and suppose $\Phi \neq 0$. Choose $k \geqslant 0$ so that $2^{k+1}\Phi = 0$ and $2^k\Phi \neq 0$. By Lemma 1.2, $2^k\Phi = \langle\langle -y, y_2, \ldots, y_{n+k+1}\rangle\rangle$, with $y \in D_F\langle 1, 1\rangle$ and $y_i \in \dot{F}$. Then $\Psi := \langle\langle -y, y_2, \ldots, y_n\rangle\rangle \in TP_n F$, hence by (C), Ψ is universal. It follows that $\Psi \otimes \langle\langle y_{n+1}\rangle\rangle = 2\Psi = 0$, whence $2^k\Phi = 0$, a contradiction. This proves $TP_{n+1}F = 0$.

(D) $\Rightarrow$ (C). Suppose $\Phi \in TP_n F$. Then by A_{n+1}, $\Phi \otimes \langle\langle -a\rangle\rangle = 0$ for every $a \in \dot{F}$. It follows that $a \in D_F(\Phi)$ for every $a \in \dot{F}$, that is, Φ is universal.

If F is an n-T-local field, we will write ${}_F\Phi$ for the unique anisotropic torsion n-fold Pfister form over F.

COROLLARY 1.4. *If F is an n-T-local field, then the form ${}_F\Phi$ is universal and $2 \cdot {}_F\Phi = 0$ in $W(F)$.*

Properties (A) and (B) are known to be equivalent if the field F is non-real ([19], Theorem B). However, for formally real fields this is no longer true. The "minimal" counterexample has 16 square classes (compare Example 2.4 in Section 2).

EXAMPLE 1.5. Let K be a formally real, uniquely ordered, 1-T-universal field with at least 8 square classes (the existence of such fields for any given finite number of square classes $\geqslant 4$ is proved in [18], The case 2.5.I, p. 217). Let $F = K((t))$ be the formal power series field over K. Then F is 1-T-Hilbert but is not 2-T-local. Indeed, torsion 1-fold Pfister forms over F are $\langle 1, -a\rangle$, where $a \in K_+ = R_1 K$, and these are half-universal over F. Moreover, $K_+ = R_1 K$ and K uniquely ordered implies $|\dot{K}: R_1 K| = 2$, hence K_+ consists of at least four square classes. Now, if $a, b \in K_+ \setminus \dot{K}^2$ lie in distinct square classes of K, then $\langle\langle -a, t\rangle\rangle$ and $\langle\langle -b, t\rangle\rangle$ are anisotropic torsion forms and are not isometric since $ab \notin D_F\langle 1, t\rangle$. Thus F is not 2-T-local.

EXAMPLE 1.6. Here we show that (C) does not imply (B). Let F be an algebraic number field with exactly one ordering. Then F is 2-T-universal yet is not 1-T-Hilbert. In fact, if F is any algebraic number field, then the index $|\dot{F}: D_F\langle 1, -a\rangle|$ is infinite for every $a \in \dot{F} \setminus \dot{F}^2$. This can be proved with the aid of [9], Satz 169. Let us mention in passing that $D_F\langle 1, -a\rangle$ consists always of infinitely many square classes and for any $a \in \dot{F}$ we have

$$\dot{F} = \bigcup \{D_F\langle 1, -b\rangle : b \in D_F\langle 1, -a\rangle \setminus \dot{F}^2\}$$

(cf. [20], Theorem 3.1).

EXAMPLE 1.7. F is 0-T-Hilbert iff F is non-real and has only two square classes. F is 1-T-local iff F_+ consists of two square classes iff $TW(F) = Z/2Z$. Thus for $n = 1$, (B) implies (A) but not conversely.

PROPOSITION 1.8. *The following are equivalent*:

(i) *F is 2-T-local.*

(ii) *$TI^2F = Z/2Z$ and $u(F) = 4$.*

(iii) *$TI^2F = Z/2Z$.*

Proof. (i) $\Rightarrow$ (ii). 2-T-locality implies A_3 by Theorem 1.1, and A_3 implies $TI^3F = 0$ by [6], Theorem 3. On the other hand, TI^2F is generated by 2-fold torsion Pfister forms, hence by 2-T-locality, $TI^2F = Z/2Z$. Hence every form in TI^2F is a 2-fold Pfister form and I^3F is torsion free. By [5], Proposition 1.8(3), we get $u(F) = 4$.

(ii) $\Rightarrow$ (iii) is trivial and to prove (iii) $\Rightarrow$ (i) assume $q \in TI^2F$, $q \neq 0$. Again q is a sum of 2-fold torsion Pfister forms, hence $0 \neq TP_2F \subset TI^2F = \{0, q\}$. It follows $q \in TP_2F$ and F is 2-T-local.

Remark 1.9. If F is 1-T-Hilbert, then $u(F) = 4$. We proceed as in the proof of (i) $\Rightarrow$ (ii) above to get $TI^3F = 0$. Then, using Lemma 1.12 below, we see that any two torsion 2-fold Pfister forms are linked, and universal (the latter by Theorem 1.1). Hence any sum of torsion 2-fold Pfister forms equals a 2-fold Pfister form in $W(F)$. Again by [5], Prop. 1.8(3), we conclude $u(F) = 4$.

For the balance of this section, we focus our attention on the cases where (B) $\Rightarrow$ (A). We will assume $n \geqslant 2$ and F formally real (cf. the comment following Corollary 1.4).

THEOREM 1.10. *Let F be formally real field and $n \geqslant 2$. If F is $(n-1)$-T-Hilbert and $|X_F| < 2^{n-1}$, then F is n-T-local.*

COROLLARY 1.11. (i) *If F has just one ordering, then* (B) $\Rightarrow$ (A) *for $n \geqslant 2$.*

(ii) *If F has 2 or 3 orderings, then* (B) $\Rightarrow$ (A) *for $n \geqslant 3$.*

(iii) *If $|\dot{F}: F_+| = 2^m$, then* (B) $\Rightarrow$ (A) *for $n \geqslant m+1$.*

Here (iii) follows from the fact that $m \leqslant |X_F| \leqslant 2^{m-1}$.

LEMMA 1.12. *Let F be $(n-1)$-T-Hilbert field, $n \geqslant 2$, and Φ_1 and Ψ_1 be anisotropic n-fold Pfister forms over F. Suppose $\Phi_1 = \Phi \otimes \langle 1, a\rangle$, $\Psi_1 = \Psi \otimes \langle 1, b\rangle$, where $a, b \in \dot{F}$ and $\Phi, \Psi \in TP_{n-1}F$. Then there exists $e \in \dot{F}$ such that*

$$\Phi_1 = \Phi \otimes \langle 1, e\rangle \quad \text{and} \quad \Psi_1 = \Psi \otimes \langle 1, e\rangle .$$

Proof of the Lemma proceeds exactly as in [19], proof of Proposition 2.3, and will be omitted.

Proof of Theorem 1.10. Suppose Φ_1 and Ψ_1 are anisotropic forms in TP_nF. By Lemmas 1.3 and 1.2, we can write

$$\Phi_1 = \langle\langle -y, a_2, \ldots, a_n\rangle\rangle, \quad \Psi_1 = \langle\langle -z, b_2, \ldots, b_n\rangle\rangle,$$
$$\text{where } y, z \in D_F\langle 1, 1\rangle .$$

We apply Lemma 1.12 with $\Phi = \langle\langle -y, a_2, \ldots, a_{n-1}\rangle\rangle$ and $\Psi = \langle\langle -z, b_2, \ldots, b_{n-1}\rangle\rangle$ and get

$$\Phi_1 = \langle\langle -y, a_2, \ldots, a_{n-1}, e_n\rangle\rangle, \quad \Psi_1 = \langle\langle -z, b_2, \ldots, b_{n-1}, e_n\rangle\rangle$$

and then apply Lemma 1.12 again with $\Phi = \langle\langle -y, e_n, a_2, \ldots, a_{n-2}\rangle\rangle$, $\Psi = \langle\langle -z, e_n, b_2, \ldots, b_{n-2}\rangle\rangle$ to get

$$\Phi_1 = \langle\langle -y, e_{n-1}, e_n, a_2, \ldots, a_{n-2}\rangle\rangle, \quad \Psi_1 = \langle\langle -z, e_{n-1}, e_n, b_2, \ldots, b_{n-2}\rangle\rangle .$$

Continuing this procedure we arrive at

$$\Phi_1 = \langle\langle -y, e_2, \ldots, e_n\rangle\rangle, \quad \Psi_1 = \langle\langle -z, e_2, \ldots, e_n\rangle\rangle$$

for some $e_2, \ldots, e_n \in \dot{F}$. Thus any pair of torsion n-fold Pfister forms over F are $(n-1)$-linked. Observe that $-1 \in D_F\langle 1, -y\rangle$ and $-1 \in D_F\langle 1, -z\rangle$, hence

(1.10.1) $$\Phi_1 = \langle\langle -y, c_2 e_2, \ldots, c_n e_n\rangle\rangle, \quad \Psi_1 = \langle\langle -z, c_2 e_2, \ldots, c_n e_n\rangle\rangle$$
$$\text{for any choice of } c_i \in \{1, -1\},\ i = 2, \ldots, n .$$

We now prove that at least one of the forms $\langle\langle c_2 e_2, \ldots, c_n e_n\rangle\rangle$ is torsion, if $|X_F| < 2^{n-1}$. We have the following identity:

$$\sum \langle\langle c_2 e_2, \ldots, c_n e_n\rangle\rangle = 2^{n-1}\langle 1\rangle ,$$

where $(c_2, \ldots, c_n)$ runs through $\{1, -1\}^{n-1}$ ([16], p. 73), hence for every $P \in X_F$ we have

$$\sum \operatorname{sgn}_P \langle\langle c_2 e_2, \ldots, c_n e_n \rangle\rangle = 2^{n-1},$$

where $\operatorname{sgn}_P\colon W(F) \to Z$ is the signature determined by the ordering P. It follows that for every $P \in X_F$ exactly one of the 2^{n-1} summands $\langle\langle c_2 e_2, \ldots, c_n e_n \rangle\rangle$ has signature at P equal to 2^{n-1} while all the others have signature 0 at P. Since $|X_F| < 2^{n-1}$, there is at least one form $\Theta = \langle\langle c_2 e_2, \ldots, c_n e_n \rangle\rangle$ which has signature zero at every $P \in X_F$, hence Θ is torsion. With this choice of Θ in (1.10.1) we apply Lemma 1.12 once again with $\Phi = \Psi = \Theta$ and get finally $\Phi_1 = \Psi_1$. Thus F is n-T-local.

Remark 1.13. The result in Theorem 1.10 is the best possible. In the next section, for every $n \geqslant 2$ we produce an example of an $(n-1)$-T-Hilbert field with $|X_F| = 2^{n-1}$ which is not n-T-local (cf. Theorem 2.7).

§ 2. Construction of T-local and T-Hilbert fields. We will supply evidence here that T-Hilbert and T-local fields occur very often among fields with finite number of square classes. For instance, there are 78 distinct Witt rings for fields with at most 16 square classes. Of these 70 have non-zero torsion and 47 of them are T-local or T-Hilbert of rank $n \leqslant 4$.

More generally, we prove that for any finite number of square classes there are n-T-local (hence $(n-1)$-T-Hilbert) fields of all admissible ranks n (cf. Proposition 2.1 and Theorem 2.5). We also show that counter-examples to the implication (B) $\Rightarrow$ (A) exist for any finite number of square classes $\geqslant 16$ and all admissible ranks but one. All these examples of fields have Witt rings of elementary type in the sense of [15], p. 122.

We begin with an upper bound for the Hilbert, or local rank in terms of the number of square classes.

Proposition 2.1. *If F is n-T-Hilbert or $(n+1)$-T-local field, $n \geqslant 1$ and $|\dot F/\dot F^2| = 2^m$, then $n+1 \leqslant m$. Moreover, if F is formally real, then $n+1 < m$.*

Proof. For non-real fields the result follows from Proposition 3.1 in [19]. So assume F is a formally real field. Then $TP_{n+1}F \neq 0$, hence $2^{n+1} \leqslant u(F)$. By [5], Theorem 2.4, we have $u(F) < 2^m$. Hence $n+1 < m$.

Proposition 2.2. *Let K be a field complete relative to a discrete valuation with residue class field F of characteristic different from 2 and let $n \geqslant 2$.*

(i) *K is n-T-local iff F is $(n-1)$-T-local.*

(ii) *K is n-T-Hilbert iff F is $(n-1)$-T-Hilbert.*

(iii) *K satisfies A_n iff F satisfies A_{n-1}.*

Proof. In view of the identity $\langle\langle a, b\rangle\rangle = \langle\langle a, ab\rangle\rangle$, every n-fold Pfister form Φ over K has either diagonalization $\langle\langle a_1, \ldots, a_n\rangle\rangle$ or $\langle\langle a_1, \ldots, a_{n-1}, a_n\pi\rangle\rangle$, where π is a uniformizer of K and $a_1, \ldots, a_n$ are units. Call the two types of diagonalization the first, and the second type, respectively. The type of diagonalization is invariant under isometry. Moreover, if $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are units in K, then

$$\langle\langle a_1, \ldots, a_{n-1}, a_n\pi\rangle\rangle = \langle\langle b_1, \ldots, b_{n-1}, b_n\pi\rangle\rangle$$

$$\Leftrightarrow \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle = \langle\langle b_1, \ldots, b_{n-1}\rangle\rangle \quad \text{and} \quad a_n b_n \in D_K\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle.$$

For a form Φ as above, let $\bar\Phi$ be the first residue form of Φ. Then Φ is torsion in $W(K)$ iff $\bar\Phi$ is torsion in $W(F)$, and also, $\Phi \neq 0$ iff $\bar\Phi \neq 0$. All this follows from standard facts about local fields (cf. [14], Chapter VI.1).

(i) If K is n-T-local, then $\Phi = {}_K\Phi$ is necessarily of the second type since otherwise the $(n-1)$-fold torsion factor of Φ multiplied by $\langle\langle\pi\rangle\rangle$ is an anisotropic n-fold Pfister form different from Φ. Since distinct anisotropic forms in torsion n-fold Pfister form different from Φ. Since distinct anisotropic forms in $TP_{n-1}F$ lift to distinct anisotropic forms in $TP_{n-1}K$ and multiplied by $\langle\langle\pi\rangle\rangle$ yield distinct anisotropic forms in TP_nK, it follows that $\bar\Phi$ is the unique anisotropic torsion $(n-1)$-fold Pfister form over F. The same argument proves the converse.

(ii) For $\Phi = \langle\langle a_1, \ldots, a_{n-1}, a_n\pi\rangle\rangle$, where $a_1, \ldots, a_n$ are units in K, we have

$$D_K(\Phi) = D_K\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \cup a_n\pi \cdot D_K\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle.$$

Moreover, $\dot K/\dot K^2 = \dot F/\dot F^2 \times Z/2Z$ and Φ is K-universal iff $\bar\Phi$ is F-universal. Also Φ is K-half-universal iff $\bar\Phi$ is F-half-universal. This is sufficient to prove (ii).

(iii) follows easily from the fact that $\Phi = 0$ iff $\bar\Phi = 0$.

Proposition 2.3. *Let E be a field satisfying A_n, $n \geqslant 1$, and let K and F be fields satisfying*

$$W(K) = W(F) \dot\times W(E) \tag{2.3.1}$$

(direct product in the category of abstract Witt rings).

(i) *K is n-T-local iff F is n-T-local.*

(ii) *K is n-T-Hilbert iff F is n-T-Hilbert.*

(iii) *K satisfies A_n iff F satisfies A_n.*

Proof. We have $TP_nK = TP_nF \times TP_nE$ and $TP_nE = 0$ by A_n. This proves (i) and (iii). (ii) follows from the fact that for $\Phi = (\Phi_1, \Phi_2) \in P_nF \times P_nE$, we have

$$D_K(\Phi) = D_F(\Phi_1) \times D_E(\Phi_2)$$

(here the value groups are regarded as subgroups of groups of square classes rather than subgroups of multiplicative groups of fields). Notice that, given F and E, the field K satisfying (2.3.1) always exists according to [12] (see also earlier papers cited there).

Example 2.4. Using the tables of quadratic form schemes in [17] and Propositions 2.2 and 2.3 above, we have compiled the following data confirming the repeated occurrence of T-local and T-Hilbert types among fields with a finite number of square classes. While in the column below the number of square classes the total number of objects is given, we split the total to show the number of objects coming from non-real fields (the first summand) and from formally real fields (the second summand). Thus, for instance, $29 = 15+14$ in the last column, last row but one, means there are 29 n-T-Hilbert types altogether and 15 come from non-real fields, while 14 come from formally real fields.

Number of:					
square classes	1	2	4	8	16
Witt rings	1	$3 = 2+1$	$6 = 4+2$	$17 = 10+7$	$51 = 27+24$
Witt rings with non-zero torsion	1	$2 = 2+0$	$5 = 4+1$	$15 = 10+5$	$47 = 27+20$
T-local or T-Hilbert types	1	$2 = 2+0$	$3 = 2+1$	$10 = 6+4$	$31 = 15+16$
n-T-local types $n \geqslant 2$	0	0	$2 = 2+0$	$9 = 6+3$	$28 = 15+13$
n-T-Hilbert types $n \geqslant 1$	0	0	$2 = 2+0$	$9 = 6+3$	$29 = 15+14$
1-T-local types	0	$2 = 2+0$	$1 = 0+1$	$1 = 0+1$	$2 = 0+2$

THEOREM 2.5. *Let m and n be two positive integers and $2 \leqslant n+1 \leqslant m$.*

(i) *There exists a non-real $(n+1)$-T-local field with 2^m square classes.*

(ii) *There exists a formally real n-T-local field with 2^m square classes.*

Proof. (i) is proved in [19], Theorem 3.2.(i). (ii) Here the proof is similar to that of Theorem 3.2 in [19]. Starting with a formally real algebraic extension of $\boldsymbol{Q}$ with 4 square classes represented by ± 1, ± 2, which is 1-T-local, we double the number of square classes and keep the same local rank using Proposition 2.3 with $E = \boldsymbol{R}$. Using Proposition 2.2 we double the number of square classes and enlarge by 1 the local rank of a given T-local field F. Combining these two methods furnishes a complete proof of the Theorem.

THEOREM 2.6. *Let m and n be two positive integers, $n \geqslant 2$, $m \geqslant 4$ and $n+1 < m$. There exists a formally real $(n-1)$-T-Hilbert field F with 2^m square classes which is not n-T-local.*

Proof. The result holds for $n = 2$ and $m = 4$ by Example 1.5. Using Proposition 2.3 with $E = \boldsymbol{R}$, we prove the result for $n = 2$ and any $m \geqslant 4$. Starting again with the field in Example 1.5 and using Proposition 2.2 we prove the result for $n = 3$ and $m = 5$ and then applying Proposition 2.3 proves the result for $n = 3$ and any $m \geqslant 5$. Continuing this procedure, the Theorem follows by induction.

The last result in this section shows that the bound for the number of orderings found in Theorem 1.10 to guarantee that (B) $\Rightarrow$ (A), is the best possible, for every $n \geqslant 2$.

THEOREM 2.7. *For every $n \geqslant 2$ there exists a formally real $(n-1)$-T-Hilbert field F with $|X_F| = 2^{n-1}$ which is not n-T-local.*

Proof. The field F in Example 1.5 satisfies the requirements for $n = 2$. By induction, using Proposition 2.2, the field $F((t_3)) \ldots ((t_n))$ satisfies the requirements for any $n \geqslant 3$.

Remark 2.8. Using Theorem 2.7 and Proposition 2.3 one can prove by induction the following more general result:

For every $n \geqslant 2$ and every $i \geqslant 0$ there is a field $F_{n,i}$ with $|X_{F_{n,i}}| = 2^{n-1}+i$ such that $F_{n,i}$ is $(n-1)$-T-Hilbert and not n-T-local.

§ 3. **Going-up theorems for T-local fields.** R. Elman and T. Y. Lam [7], Theorem 4.5, proved a going-up theorem for the property A_n in totally positive quadratic extensions $K \supset F$. Since n-T-local fields satisfy A_{n+1}, it is natural to expect that, if a going-up theorem for n-T-locality exists, the best chance to discover it is to look at totally positive quadratic extensions. On the other hand, K. Kozioł [11], generalizing the results of C. Cordes and J. Ramsey [3], has already proved a going-up theorem for n-T-locality in case of quadratic extensions $K = F(\sqrt{a})$, where F is non-real and $a \notin R_{n-1}F$. He also proves that the going-up result is false if $a \in R_{n-1}F$ (and F is non-real).

To generalize these results to formally real fields it seems natural to replace the radical $R_{n-1}F$ by its "torsion" counterpart $TR_{n-1}F$, the intersection of value groups of all torsion $(n-1)$-fold Pfister forms over F. As [7] suggests, there is no hope for a going-up result for T-locality outside the class of totally positive quadratic extensions. Thus we are led to the following conjecture.

CONJECTURE. *Let F be a field, $a \in F_+ \setminus TR_{n-1}F$ and $n \geqslant 1$. If F is n-T-local, so is $K = F(\sqrt{a})$.*

In this section we prove the conjecture for $n = 1$ and $n = 2$. In crucial points of the proof we make use of results that are not known to hold for arbitrary values of n, hence the approach is not easily generalizable to cover other cases.

We begin with the case $n = 1$.

PROPOSITION 3.1. *Let F be 1-T-local and $a \in F_+$. Then $K = F(\sqrt{a})$ is also 1-T-local.*

Proof. Torsion 1-fold Pfister forms are $\langle 1, -c\rangle$ with $c \in F_+$. Since F is 1-T-local, we have $F_+ = \dot{F}^2 \cup a\dot{F}^2 = D_F\langle 1, 1\rangle$ and ${}_F\Phi = \langle 1, -a\rangle$. Thus there is exactly one totally positive quadratic extension $K = F(\sqrt{a})$. By [7], Corollary 2.18, we have $D_K\langle 1, 1\rangle = K_+$ and it remains to prove that $D_K\langle 1, 1\rangle$ consists of two square classes. By Norm Principle [7], 2.13, the norm $N_{K/F}$ induces a homomorphism

$$N\colon D_K\langle 1, 1\rangle/\dot{K}^2 \to D_F\langle 1, 1\rangle/\dot{F}^2$$

and again by Norm Principle and by (1.1) in [7], its kernel is trivial. Thus N is injective and

$$|D_K\langle 1, 1\rangle/\dot{K}^2| \leqslant |D_F\langle 1, 1\rangle/\dot{F}^2| = 2\,.$$

Now $|D_K\langle 1, 1\rangle/\dot{K}^2| > 1$, since otherwise K is Pythagorean, and then, by a result of Diller and Dress (cf. [7], Corollary 3.9), F is also Pythagorean, a contradiction. Thus K_+ consists of two square classes and K is 1-T-local.

For the balance of this section we assume $n \geqslant 2$. If $K = F(\sqrt{a})$ we write N for the norm $N_{K/F}$. For a form Ψ over K we write $s_*(\Psi)$ for its transfer to F, where s is the F-linear functional on K given by $s(1) = 0$, $s(\sqrt{a}) = 1$.

PROPOSITION 3.2. *Suppose F is an n-T-local field, $a \in F_+ \setminus R_{n-1}F$ and $K = F(\sqrt{a})$. Then*

(i) $\dot{F} \subset D_K(\Phi)$ *for every* $\Phi \in TP_{n-1}F$.

(ii) $x \in D_K(\Phi) \Leftrightarrow N(x) \in D_F(\Phi)$ *for every* $x \in \dot{K}$ *and* $\Phi \in TP_{n-1}F$.

(iii) $N(D_K(\Phi)) = D_F(\Phi) \cap D_F\langle 1, -a\rangle$ *for every* $\Phi \in TP_{n-1}F$.

(iv) *If* $z_1 \notin D_K(\Phi)$ *and* $z_2 \notin D_K(\Phi)$, *then* $z_1 z_2 \in D_K(\Phi)$ *for* $z_1, z_2 \in \dot{K}$ *and* $\Phi \in TP_{n-1}F$.

(v) $TP_n K \neq 0$.

(vi) $({}_F\Phi)_K = 0$.

(vii) *If* $0 \neq \Psi \in TP_n K$, *then* $s_*(\Psi) = {}_F\Phi$.

(viii) $s_*(TP_n K) = TP_n F$.

Proof. (i) Let $b \in \dot{F}$ and $b \notin D_F(\Phi)$. Then $\langle\langle -b\rangle\rangle \otimes \Phi \in TP_n F$ and is anisotropic. Since $a \notin R_{n-1}F$, there exists $\Psi \in P_{n-1}F$ such that $\langle\langle -a\rangle\rangle \otimes \Psi$ is anisotropic and it is torsion since $a \in F_+$. By n-T-locality, $\langle\langle -a\rangle\rangle \otimes \Psi = \langle\langle -b\rangle\rangle \otimes \Phi$. Since $\langle\langle -a\rangle\rangle \otimes \Psi$ becomes hyperbolic over K, we get $b \in D_K(\Phi)$.

(ii) $N(x) \in D_F(\Phi) \Leftrightarrow x \in \dot{F} \cdot D_K(\Phi)$ by Norm Principle, and $\dot{F} \cdot D_K(\Phi) = D_K(\Phi)$, by (i).

(iii) follows from (ii).

(iv) If $z_1, z_2 \notin D_K(\Phi)$, then by (ii), $N(z_1), N(z_2) \notin D_F(\Phi)$. Since F is $(n-1)$-T-Hilbert (by Theorem 1.1), $N(z_1 z_2) \in D_F(\Phi)$. Then $z_1 z_2 \in D_K(\Phi)$, by (ii)

(v) If $TP_n K = 0$, then K satisfies A_n and then, by Going–Down Theorem 4.12 in [7], F satisfies A_n, a contradiction.

(vi) As shown in the proof of (i), there is $\Psi \in P_{n-1}F$ such that $\langle\langle -a\rangle\rangle \otimes \Psi$ is torsion and anisotropic. By n-T-locality, $\langle\langle -a\rangle\rangle \otimes \Psi = {}_F\Phi$. Hence $({}_F\Phi)_K = 0$.

(vii) F satisfies A_{n+1} by Theorem 1.1, hence K satisfies A_{n+1} by Going-up Theorem 4.5 in [7], hence K is n-T-universal by Theorem 1.1. It follows that $-1 \in D_K(\Psi)$, hence $2\Psi = 0$. Thus $\Psi = \langle\langle -y\rangle\rangle \otimes \Theta$, where $y \in D_K\langle 1, 1\rangle$ and $\Theta \in P_{n-1}K$. Now $\Theta = \sum \langle a_i\rangle \Phi_i \langle\langle z_i\rangle\rangle$, where $a_i \in \dot{F}$, $\Phi_i \in P_{n-2}F$, $z_i \in \dot{K}$, by Lemma 2 in [6]. Thus

$$s_*(\Psi) = \sum \langle b_i\rangle \Phi_i s_*(\langle\langle -y, z_i\rangle\rangle), \quad \text{for some } b_i \in \dot{F}.$$

Now as in the proof of the Going-up Theorem 4.5 in [7], we conclude that

$$s_*(\langle\langle -y, z_i\rangle\rangle) = \sum \langle\langle d_{ji}, -c_{ji}\rangle\rangle,$$

where $d_{ji} \in \dot{F}$ and $c_{ji} \in F_+$.

Since $\Phi_i \langle\langle d_{ji}, -c_{ji}\rangle\rangle \in TP_n F = \{0, {}_F\Phi\}$ and $2 \cdot {}_F\Phi = 0$, we have

$$s_*(\Psi) = \sum\sum \langle b_i\rangle \Phi_i \langle\langle d_{ji}, -c_{ji}\rangle\rangle = 0 \text{ or } {}_F\Phi.$$

We will show that $s_*(\Psi) \neq 0$. Otherwise, by Theorem 2.3 in [7], there exists $\tau \in P_n F$ such that $\Psi = \tau_K$. Now τ_K torsion implies that τ is torsion, by total positivity of the extension (cf. [7], (1.2)), hence $\Psi \neq 0$ implies $\tau = {}_F\Phi$. By (vi) we get $0 = \tau_K = \Psi$, a contradiction. This proves (vii).

(viii) follows from (vii).

Now we switch to the case $n = 2$. We begin with an analog of a useful result of Cordes ([2], Lemma 1).

LEMMA 3.3. *If F is a 2-T-local field and $\langle 1, a\rangle, \langle 1, b\rangle \in TP_1 F$, then $D_F\langle 1, a\rangle = D_F\langle 1, b\rangle$ iff $ab \in R_1 F$.*

Proof. $\langle\langle a, -c\rangle\rangle, \langle\langle b, -c\rangle\rangle \in TP_2 F$ for every $c \in \dot{F}$ and since $D_F\langle 1, a\rangle = D_F\langle 1, b\rangle$, if one of the forms is isotropic (anisotropic), so is the other. By 2-T-locality, $\langle\langle a, -c\rangle\rangle = \langle\langle b, -c\rangle\rangle$, whence $ab \in D_F\langle 1, -c\rangle$ for every $c \in \dot{F}$. Thus $ab \in R_1 F$. The converse is known to hold for any field F.

LEMMA 3.4. *Suppose F is 2-T-local, $a \in F_+ \setminus R_1 F$ and $K = F(\sqrt{a})$. Then either*

(i) $D_F\langle 1, -a\rangle \neq D_F\langle 1, -b\rangle$ *for some* $b \in F_+ \setminus R_1 F$, *or*

(ii) $F_+ = R_1 F \cup aR_1 F$.

Moreover, (ii) *implies* $a \in TR_1 F$.

Proof. If (i) does not hold, then $D_F\langle 1, -a\rangle = D_F\langle 1, -b\rangle$ for every $b \in F_+ \setminus R_1 F$. By Lemma 3.3, we have $ab \in R_1 F$ for every $b \in F_+ \setminus R_1 F$, and (ii) follows.

Now assume (ii). Since the value set of a quadratic form consists of cosets of $R_1 F$ (cf. [1]), we have

$$D_F\langle 1, a\rangle \subset F_+ = R_1 F \cup aR_1 F \subset D_F\langle 1, a\rangle,$$

whence $F_+ = D_F\langle 1, a\rangle$.

We also have $R_1 F \subset D_F\langle 1, 1\rangle \subset F_+$ and since $R_1 F$ has index 2 in F_+ we have either $R_1 F = D_F\langle 1, 1\rangle$ or $D_F\langle 1, 1\rangle = F_+$. The first possibility is ruled out since it is equivalent with F satisfying A_2 ([7], Theorem 4.3) which is not the case. Thus $D_F\langle 1, 1\rangle = F_+ = D_F\langle 1, a\rangle$. Observe that (ii) implies also $D_F\langle 1, -a\rangle = TR_1 F$. Now we have

$$a \in F_+ = D_F\langle 1, 1\rangle \cap D_F\langle 1, a\rangle \subset D_F\langle 1, -a\rangle = TR_1 F.$$

We are finally ready to prove

GOING-UP THEOREM 3.5. *Let $K = F(\sqrt{a})$, where $a \in F_+ \setminus TR_1 F$. Then, if F is 2-T-local, so is K.*

Proof. By Lemma 3.4, we may assume there is $b \in F_+ \setminus R_1 F$ such that $D_F\langle 1, -a\rangle \neq D_F\langle 1, -b\rangle$. The forms $\langle 1, -a\rangle$ and $\langle 1, -b\rangle$ are not universal $(a, b \notin R_1 F)$ and since they are torsion, they are half-universal, by Theorem 1.1. Thus there exists $d \in D_F\langle 1, -a\rangle \setminus D_F\langle 1, -b\rangle$. It follows that $\langle\langle -b, -d\rangle\rangle = {}_F\Phi$ by 2-T-locality of F, and $d = N(x)$ for some $x \in \dot{K}$. Thus

$${}_F\Phi = \langle\langle -b, -N(x)\rangle\rangle.$$

Now we will prove that every anisotropic torsion 2-fold Pfister form over K is isometric with $\langle\langle -b, -x\rangle\rangle$. So let $0 \neq \Psi \in TP_2 K$. Then $s_*(\Psi) = {}_F\Phi$ by Proposition 3.2(vii) and we observe that also $s_*(\langle\langle -b, -x\rangle\rangle) = {}_F\Phi$. Thus $q = \Psi - \langle\langle -b, -x\rangle\rangle$ has the following properties:

$$q \in TI^2 K \quad \text{and} \quad s_*(q) = 0.$$

From the exactness of the sequence (cf. [7], Corollary 2.10)

$$I^2F \xrightarrow{i} I^2K \xrightarrow{s_*} I^2F$$

where i is the functorial map, we conclude that there is $\Theta \in I^2F$ such that $q = \Theta_K$. Here Θ_K is torsion, hence by total positivity of the extension, also Θ is torsion. Thus $\Theta \in TI^2F = \mathbf{Z}/2\mathbf{Z}$, the latter by Proposition 1.8. Thus $\Theta \in TP_2F = \{0, {}_F\Phi\}$ and in either case, $q = \Theta_K = 0$ (by Proposition 3.2(vi)). This proves $\Psi = \langle\langle -b, -x\rangle\rangle$ and finishes the proof.

Remark 3.6. The result in 3.5 is inapplicable if $F_+ \subset TR_1F$. If this happens, then necessarily $F_+ = D_F\langle 1, 1\rangle$. Indeed, if $b \in F_+ \subset TR_1F$, then $b \in F_+ \subset D_F\langle 1, -b\rangle$, whence $b \in D_F\langle 1, 1\rangle$.

For formally real fields with 8 square classes there are 3 cases where F is 2-T-local (II, IV and V in [13]). Of these, one satisfies $F_+ \subset TR_1F$ (Case IV) and the Going-up Theorem 3.5 does not apply. In the remaining cases we have

$D_F\langle 1, 1\rangle = F_+ \not\subset TR_1F$ (Case II) and

$D_F\langle 1, 1\rangle \neq F_+ \not\subset TR_1F$ (Case V).

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