Generalized Hilbert fields, II

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Abstract. Quadratic form structure is investigated for fields with a unique anisotropic torsion $n$-fold Pfister form ($n$-T-local field) and for fields where torsion $n$-fold Pfister forms are either universal or half-universal ($n$-T-Hilbert field). It is proved that $n$-T-locality always implies $(n-1)$-T-Hilberticity but not conversely. Sufficient conditions are given in terms of the number of orderings of the field ensuring the converse to hold. Numerous examples of $T$-local fields are given by using techniques of abstract Witt rings. A going-up theorem is proved for $n$-T-locality, $n = 1$ and 2, in totally positive quadratic extensions.

Introduction. A. Fröhlich [8] and I. Kaplansky [10] investigated quadratic forms over fields that share with $P$-adic fields and real closed fields the pattern of behaviour of value sets of binary forms. Their axioms imply the characteristic feature of the local theory — the existence of a unique non-split quaternion algebra. The set-up and results of their work have been generalized in [19] to the context of $n$-fold Pfister forms (those corresponding to $n = 1$). The generalization of local theory in [19] retains the relatively trivial behaviour of quadratic forms over formally real Hilbert fields. In case $n = 1$, formally real Hilbert fields are just Euclidean (two square classes) and for $n \geq 1$ they are uniquely ordered and there are only two possible value groups for all $n$-fold Pfister forms. This is in contrast to the situation we encounter in non-real Hilbert fields, where the intersections of value groups of $n$-fold Pfister forms do not obey any easily formulated rule. Thus it is natural to wonder if there exists a generalized version of the local theory retaining the non-trivial features of non-real generalized Hilbert fields and covering in a unified manner both non-real and formally real fields.

This paper axiomatizes such a generalization of local theory using a fruitful method of transferring properties from non-real to formally real fields, familiar in algebraic theory of quadratic forms. If a property of quadratic forms is important for non-real fields and becomes vacuous or not-so-interesting for formally real fields, it usually turns out to be equally important in the formally real case, when considered only for torsion, instead of all, quadratic forms. In our context this approach reveals a class of formally real fields that share essential properties with straightforward generalization of classical non-real local fields. Let us stress that what we get for
formally real fields does not generalize the number-theoretic aspect of Hilberticity or locality (for this, see [19]).

We consider $T$-Hilbert fields, where torsion $n$-fold Pfister forms are half-universal or universal, and $T$-local fields, where there is one, one anisotropic torsion $(n+1)$-fold Pfister form. These two notions — known to be equivalent for non-real fields ([19]) — turn out to be no longer equivalent for formally real fields. We show that $T$-locality is stronger than $T$-Hilberticity and study the cases where the two are actually equivalent. This is done in Section 1. The second section produces numerous examples of $T$-local and $T$-Hilbert fields. Since $T$-locality and $T$-Hilberticity are invaria nt relative to Witt equivalence of fields, one can try to identify $T$-local and $T$-Hilbert types among the classes of Witt equivalent fields. It turns out that for any $2^n \leq 16$ the number of $T$-local and $T$-Hilbert types exceeds one half of the number of all the types of Witt equivalent fields with $2^n$ square classes. We give constructions suggesting that the occurrence of $T$-local and $T$-Hilbert types for any finite number of square classes is equally high. In the final third section we prove a going-up theorem for $T$-locality in totally positive quadratic extensions for $n = 1$ and $n = 2$. The general case seems to require a new approach and has been left open.

### Notation and terminology

We introduce the following unified $T$-notation.

- $TW(F)$ is the torsion ideal of the Witt ring $W(F)$, usually written $W_F$.
- $TP_n F$ is the set of isometry classes of torsion $n$-fold Pfister forms over $F$, often viewed as subset of $W(F)$.
- $TR_n F$ is the “$n$th torsion radical” of $F$.
- $TR_n F = \cap \{D_F \mapsto \Phi \in TP_n F\}$.

All fields are assumed to have characteristic $\neq 2$.

A field $F$ is said to be $n$-T-local if there is a unique anisotropic torsion $n$-fold Pfister form over $F$ (up to isometry).

A field $F$ is called $n$-T-Hilbert if every torsion $n$-fold Pfister form $\Phi$ over $F$ is either half-universal (i.e. $F^2 \Phi = 2$) or universal (i.e. $F^2 \Phi = D_F \Phi$), and there are some half-universal forms in $TP_n F$.

- $T^* F = \sum F^2$ is the set of non-zero sums of squares in $F$ (totally positive elements of $F$).
- $x_F$ is the set of all orderings of $F$ (empty if $F$ is non-real).
- If $K/F$ is a field extension and $\Phi$ is a form over $F$; then $i: W(F) \hookrightarrow W(K)$ is the functorial ring homomorphism.

Unexplained notation and terminology follow [14].

### § 1. $T$-Hilbert and $T$-local fields

In this section we investigate the relationships among the following four properties of a field $F$: Throughout we assume $n \geq 2$ is an integer.

- $(A)$ $F$ is an $n$-T-local field.
- $(B)$ $F$ is an $(n-1)$-T-Hilbert field.

(C) $F$ is an $n$-T-universal field (i.e. $F = D_F \Phi$ for every $\Phi \in TP_n F$).

(D) $F$ satisfies $A_{n+1}$ (i.e. $|TP_{n+1} F| = 1$), cf. [7].

**Theorem 1.1.** $(A) \implies (B) \implies (C) \implies (D)$

We will need the following two lemmas.

**Lemma 1.2.** $(\text{Prop. 1.5.})$

For arbitrary field $F$, if $\Phi \in TP_n F$ and $2\Phi = 0$ in $W(F)$, then $\Phi = \langle (-y_1, y_2, \ldots, y_n) \rangle$ for some $y \in D_F(1, 1)$ and $y_i \in F$.

**Lemma 1.3.** If $n \geq 2$, $F$ is $(n-1)$-T-Hilbert and $\Phi \in TP_n F$, then $2\Phi = 0$ in $W(F)$.

**Proof.** Suppose $2^{n+1} \Phi = 0$ and $2^n \Phi \neq 0$ for some $k \geq 0$. By Lemma 1.2, $2^n \Phi = \langle (-y_1, y_2, \ldots, y_n) \rangle$, where $y \in D_F(1, 1)$ and $y_i \in F$.

If $\Psi$ is universal, then it follows that $2^{n+1} \Phi = 0$, a contradiction. If $\Psi$ is half-universal, then $y_1 \neq D_F(\Psi)$, since otherwise $\Psi \otimes \langle (-y) \rangle = \Psi \otimes \langle (-1) \rangle = 0$ and $2^n \Phi = 0$. Hence $D_F(\Psi \otimes \langle (-y) \rangle) = D_F(\Psi) \cup y_i D_F(\Psi) = F$ and it follows that $2^n \Phi = 0$ for $k \geq 0$. Hence $k = 0$ and $2^n \Phi = 0$, as desired.

**Proof of Theorem 1.1.** $(A) \implies (B)$.

Suppose $0 \neq \Phi \in TP_n F$. Thus $F$ does not satisfy $A_n$ and so there exists an $n$-fold Pfister form $\Psi$ killed by 2 in $W(F)$ ([7], Lemma 4.2). By (A), we must have $\Psi = \Phi$. Hence $2\Phi = 0$ and by Lemma 1.2, there are torsion non-universal (n-1)-fold Pfister forms over $F$ (for instance, $\langle (-y_1, y_2, \ldots, y_{n-1}) \rangle$). If $\sigma$ is such a form and $a, b \in F$, $\sigma(a \Phi \otimes b \Phi)$, then $\langle (-a) \rangle \otimes \sigma$ and $\langle (-b) \otimes \sigma$ are anisotropic torsion $n$-fold Pfister forms, hence isomorphic, by (A). It follows that $\langle a \rangle = D_F(\sigma)$ and this proves that $\sigma$ is half-universal.

$(B) \implies (C)$. Let $\Phi \in TP_n F$. By Lemma 1.3, we have $2^n \Phi = 0$ and then, by Lemma 1.2, $\Phi$ has a factor $\Psi \in TP_{n-1} F$. If $\Psi$ is universal, it follows that $\Phi = 0$. If $\Phi$ is half-universal and $\Phi \neq 0$, then a diagonal entry of $\Phi$ is not represented by $\Psi$ and so $\Phi$ is universal. In any case $\Phi$ is universal, as needed.

$(C) \implies (D)$. Let $\Phi \in TP_{n+1} F$ and suppose $\Phi \neq 0$. Choose $k \neq 0$ so that $2^{n+1} \Phi = 0$ and $2^n \Phi \neq 0$. By Lemma 1.2, $2^n \Phi = \langle (-y_1, y_2, \ldots, y_{n+1}) \rangle$, with $y \in D_F(1, 1)$ and $y_i \in F$. Then $\Psi := \langle (-y_1, y_2, \ldots, y_n) \rangle \in TP_n F$, hence by (C), $\Psi$ is universal. It follows that $\Psi \otimes \langle \zeta \rangle = 2^n \Psi = 0$, whence $2^n \Phi = 0$, a contradiction. This proves $TP_n F = 0$.

$(D) \implies (C)$. Suppose $\Phi \in TP_n F$. Then by $A_{n+1}$, $\Phi \otimes \langle (-\sigma) \rangle = 0$ for every $\sigma \in F$. It follows that $a \Phi = D_F(\Phi)$ for every $a \in F$, that is, $\Phi$ is universal.

If $F$ is an $n$-T-local field, we will write $\Psi F$ for the unique anisotropic torsion $n$-fold Pfister form over $F$.

**Corollary 1.4.** If $F$ is an $n$-T-local field, then the form $\Psi F$ is universal and $2^n \Phi = 0$ in $W(F)$.

Properties (A) and (B) are known to be equivalent if the field $F$ is non-real ([19], Theorem B). However, for formally real fields this is no longer true. The "minimal" counterexample has 16 square classes (compare Example 2.4 in Section 2).
Example 1.5. Let $K$ be a formally real, uniquely ordered, 1-T-universal field with at least 8 square classes (the existence of such fields for any given finite number of square classes $\geq 4$ is proved in [18], The case 2.5.1, p. 217). Let $F = K((t))$ be the formal power series field over $K$. Then $F$ is 1-T-Hilbert but is not 2-T-local. Indeed, torsion 1-fold Pfister forms over $F$ are $(1, -a)$, where $a \in K$, $R = R_1 R_2 K$, and these are half-universal over $F$. Moreover, $K = R_1 R_2 K$ and $K$ uniquely ordered implies $[K : R_1 R_2 K] = 2$, hence $K$ consists of at least four square classes. Now, if $a, b \in K$, $K^{2,1}$ lie in distinct square classes of $K$, then $\langle \langle -a, 1 \rangle \rangle$ and $\langle \langle -b, 1 \rangle \rangle$ are anisotropic torsion forms and are not isometric since $ab \notin D_F(1, 1)$. Thus $F$ is not 2-T-local.

Example 1.6. Here we show that (C) does not imply (B). Let $F$ be an algebraic number field with exactly one ordering. Then $F$ is 2-T-universal yet is not 1-T-Hilbert. In fact, if $F$ is an algebraic number field, then the index $[F_d : D_F(1, -a)]$ is finite for every $a \in F^{\times}$. This can be proved with the aid of [9], Satz 169. Let us mention in passing that $D_F(1, -a)$ consists always of infinitely many square classes and for any $a \in F$ we have

$$F = \bigcup \{ D_F(1, -b) : b \in D_F(1, -a) \}$$

(cf. [20], Theorem 3.1).

Example 1.7. $F$ is 0-T-Hilbert iff $F$ is non-real and has only two square classes. $F$ is 1-T-local iff $F$ consists of two square classes iff $TW(F) = Z/2Z$. Thus for $n = 1, 2$, (B) implies (A) but not conversely.

Proposition 1.8. The following are equivalent:

(i) $F$ is 2-T-local.

(ii) $TF = Z/2Z$ and $u(F) = 4$.

(iii) $TF = Z/2Z$.

Proof. (i) $\Rightarrow$ (ii). 2-T-locality implies $A_3$ by Theorem 1.1, and $A_3$ implies $TF = 0$ by [6], Theorem 3. On the other hand, $TF$ is generated by 2-fold torsion Pfister forms, hence by 2-T-locality, $TF = Z/2Z$. Hence every form in $TF$ is a 2-fold Pfister form and $F$ is torsion free. By [5], Proposition 1.3(i), we get $u(F) = 4$.

(ii) $\Rightarrow$ (iii) is trivial and to prove (iii) $\Rightarrow$ (i) assume $q \in TF$, $q \neq 0$. Again $q$ is a sum of 2-fold torsion Pfister forms, hence 0 $\neq TF \subset TF = Z/2Z$. It follows $q \in TF$ and $F$ is 2-T-local.

Remark 1.9. If $F$ is 1-T-Hilbert, then $u(F) = 4$. We proceed as in the proof of (i) $\Rightarrow$ (ii) above to get $TF = 0$. Then, using Lemma 1.12 below, we see that any two torsion 2-fold Pfister forms are linked and universal (the latter by Theorem 1.1). Hence any sum of torsion 2-fold Pfister forms equals a 2-fold Pfister form in $W(F)$. Again by [5], Prop. 1.8(3), we conclude $u(F) = 4$.

For the balance of this section, we focus our attention on the cases where $B = A$. We will assume $n \geq 2$ and $F$ formally real (cf. the comment following Corollary 1.4).

Theorem 1.10. Let $F$ be formally real field and $n \geq 2$. If $F$ is $(n-1)$ T-Hilbert and $|XF| < 2^{n-1}$, then $F$ is $n$-T-local.

Corollary 1.11. (i) If $F$ has just one ordering, then (B) $\Rightarrow$ (A) for $n \geq 2$.

(ii) If $F$ has 2 or 3 orderings, then (B) $\Rightarrow$ (A) for $n \geq 3$.

(iii) If $|F : F_0| = 2^n$, then (B) $\Rightarrow$ (A) for $n \geq m + 1$.

Here (iii) follows from the fact that $m - 1 < |F| < 2^{m-1}$.

Lemma 1.12. Let $F$ be $(n-1)$-T-Hilbert field, $n \geq 2$, and $\Phi_1$ and $\Psi_1$ be anisotropic $n$-fold Pfister forms over $F$. Suppose $\Phi_1 = \Phi \otimes \langle 1, a \rangle$, $\Psi_1 = \Psi \otimes \langle 1, b \rangle$, where $a, b \in F$ and $\Phi, \Psi \in TP_{n-1} F$. Then there exists $e \in F$ such that

$$\Phi_1 = \Phi \otimes \langle 1, e \rangle \quad \text{and} \quad \Psi_1 = \Psi \otimes \langle 1, e \rangle$$

Proof of the Lemma proceeds exactly as in [19], proof of Proposition 2.3, and will be omitted.

Proof of Theorem 1.10. Suppose $\Phi_1$ and $\Psi_1$ are anisotropic forms in $TP_{n} F$. By Lemmas 1.3 and 1.2, we can write

$$\Phi_1 = \langle \langle -y, a_2, \ldots, a_n \rangle \rangle, \quad \Psi_1 = \langle \langle -z, b_2, \ldots, b_n \rangle \rangle,$$

where $y, z \in D_F(1, 1)$.

We apply Lemma 1.12 with $\Phi = \langle \langle -y, a_2, \ldots, a_n \rangle \rangle$ and $\Psi = \langle \langle -z, b_2, \ldots, b_n \rangle \rangle$ and get

$$\Phi_1 = \langle \langle -y, a_2, \ldots, a_n, c_n \rangle \rangle, \quad \Psi_1 = \langle \langle -z, b_2, \ldots, b_n, c_n \rangle \rangle$$

and then apply Lemma 1.12 again with $\Phi = \langle \langle -y, c_n, a_2, \ldots, a_n \rangle \rangle$,

$$\Psi = \langle \langle -z, e_n, b_2, \ldots, b_n \rangle \rangle$$

to get

$$\Phi_1 = \langle \langle -y, c_n, a_2, \ldots, a_n, d_n \rangle \rangle, \quad \Psi_1 = \langle \langle -z, e_n, b_2, \ldots, b_n, d_n \rangle \rangle$$

and obtain the procedure we arrive at

$$\Phi_1 = \langle \langle -y, c_2, \ldots, c_n \rangle \rangle, \quad \Psi_1 = \langle \langle -z, d_2, \ldots, d_n \rangle \rangle$$

for some $c_2, \ldots, c_n \in F$. Thus any pair of torsion $n$-fold Pfister forms over $F$ are $(n-1)$-linked. Observe that $-1 \in D_F(1, -y)$ and $-1 \in D_F(1, -z)$, hence

$$\langle 1.10.1 \rangle \quad \Phi_1 = \langle \langle -y, c_2, \ldots, c_n \rangle \rangle, \quad \Psi_1 = \langle \langle -z, d_2, \ldots, d_n \rangle \rangle$$

for any choice of $c_i \in \{ 1, -1 \}, i = 2, \ldots, n$.

We now prove that at least one of the forms $\langle \langle c_2, \ldots, c_n \rangle \rangle$ is torsion, if $|XF| < 2^{n-1}$. We have the following identity:

$$\sum \langle \langle c_2, \ldots, c_n \rangle \rangle = 2^{n-1} \cdot 1$$
where \((c_2, ..., c_n)\) runs through \(\{1, -1\}^{n-1}\) ([16], p. 73), hence for every \(P \in \mathcal{X}_P\)
we have
\[
\sum_{\text{sgn}_P} \langle \langle c_2 e_2, ..., c_n e_n \rangle \rangle = 2^{n-1},
\]
where \(\text{sgn}_P : W(F) \to Z\) is the signature determined by the ordering \(P\). It follows that for every \(P \in \mathcal{X}_P\) exactly one of the \(2^{n-1}\) summands \(\langle \langle c_2 e_2, ..., c_n e_n \rangle \rangle\) has signature at \(P\) equal to \(2^{n-1}\) while all the others have signature \(0\) at \(P\). Since \(|\mathcal{X}_P| < 2^{n-1}\), there is at least one form \(\Theta = \langle \langle c_2 e_2, ..., c_n e_n \rangle \rangle\) which has signature zero at every \(P \in \mathcal{X}_P\), hence \(\Theta\) is torsion. With this choice of \(\Theta\) in (1.10) we apply
Lemma 1.12 once again with \(\Phi = \Psi = \Theta\) and get finally \(\Phi \psi = \psi \Phi\). Thus \(F\) is \(n\)-T-local.

Remark 1.13. The result in Theorem 1.10 is the best possible. In the next section, for every \(n \geq 2\) we produce an example of an \((n-1)\)-T-Hilbert field with \(|\mathcal{X}_P| = 2^{n-1}\) which is not \(n\)-T-local (cf. Theorem 2.7).

§ 2. Construction of \(T\)-local and \(T\)-Hilbert fields. We will supply evidence here that \(T\)-Hilbert and \(T\)-local fields occur very often among fields with finite number of square classes. For instance, there are 78 distinct Witt rings for fields with at most 16 square classes. Of these 70 have non-zero torsion and 47 of them are \(T\)-local or \(T\)-Hilbert of rank \(n \leq 4\).

More generally, we prove that for any finite number of square classes there are \(n\)-T-local (hence \((n-1)\)-T-Hilbert) fields of all admissible ranks \(n\) (cf. Proposition 2.1 and Theorem 2.5). We also show that counter-examples to the implication \((B) \Rightarrow (A)\) exist for any finite number of square classes \(\geq 16\) and all admissible ranks but one. All these examples of fields have Witt rings of elementary type in the sense of [15], p. 122.

We begin with an upper bound for the Hilbert, or local rank in terms of the number of square classes.

**Proposition 2.1.** If \(F\) is \((n+1)\)-T-Hilbert field, \(n \geq 1\) and \(|F\eta F| = 2^n\), then \(n+1 \leq m\). Moreover, if \(F\) is formally real, then \(n+1 < m\).

**Proof.** For non-real fields the result follows from Proposition 3.1 in [19]. So assume \(F\) is a formally real field. Then \(TP_{n+1} F \neq 0\), hence \(2^{n+1} \leq w(F)\). By [5], Theorem 2.4, we have \(w(F) < 2^n\). Hence \(n+1 < m\).

**Proposition 2.2.** Let \(K\) be a field complete relative to a discrete valuation with residue class field \(F\) of characteristic different from 2 and let \(n \geq 2\).

(i) \(K\) is \(n\)-T-local iff \(F\) is \((n-1)\)-T-local.

(ii) \(K\) is \(n\)-T-Hilbert iff \(F\) is \((n-1)\)-T-Hilbert.

(iii) \(K\) satisfies \(A_n\) iff \(F\) satisfies \(A_{n-1}\).

**Proof.** In view of the identity \(\langle \langle a, b \rangle \rangle = \langle a, ab \rangle\), every \(n\)-fold Pfister form \(\phi\) over \(K\) has either diagonalization \(\langle a_1, ..., a_n \rangle\) or \(\langle a_1, ..., a_{n-1}, a_n \rangle\), where \(a_i\) is a uniformizer of \(K\) and \(a_1, ..., a_n\) are units. Call the two types of diagnosticization the first, and the second type, respectively. The type of diagonalization is invariant under isometry. Moreover, if \(a_1, ..., a_n, b_1, ..., b_n\) are units in \(K\), then
\[
\langle \langle a_1, ..., a_{n-1}, a_n \rangle \rangle = \langle \langle b_1, ..., b_{n-1}, b_n \rangle \rangle
\]
\[
\Rightarrow \langle \langle a_1, ..., a_{n-1} \rangle \rangle = \langle \langle b_1, ..., b_{n-1} \rangle \rangle \quad \text{and} \quad a_n b_n \in D_K \langle \langle a_{n-1} \rangle \rangle.
\]

For a form \(\Phi\) as above, let \(\overline{\Phi}\) be the first residue form of \(\Phi\). Then \(\Phi\) is torsion in \(W(K)\) if \(\overline{\Phi}\) is torsion in \(W(F)\), and also, \(\Phi \neq 0\) iff \(\overline{\Phi} \neq 0\). All this follows from standard facts about local fields (cf. [14], Chapter VII).

(i) If \(K\) is \(n\)-T-local, then \(\Phi = \overline{\Phi}\) is necessarily of the second type since otherwise the \((n-1)\)-fold torsion factor of \(\Phi\) multiplied by \(\langle \rangle\) is an anisotropic torsion \(n\)-fold Pfister form different from \(\Phi\). Since distinct anisotropic forms in \(TP_{n-1} F\) lift to distinct anisotropic forms in \(TP_{n-1} K\) and multiplied by \(\langle \rangle\) yield distinct anisotropic forms in \(TP_{n} K\), it follows that \(\Phi\) is the unique anisotropic torsion \((n-1)\)-fold Pfister form over \(F\). The same argument proves the converse.

(ii) For \(\Phi = \langle \langle a_1, ..., a_{n-1}, a_n \rangle \rangle\), where \(a_1, ..., a_n\) are units in \(K\), we have
\[
D_K(\Phi) = D_K(\langle \langle a_1, ..., a_{n-1} \rangle \rangle) \cup a_n D_K(\langle \langle a_{n-1} \rangle \rangle).
\]

Moreover, \(K^0/\mathbb{K}^2 = F/\mathbb{K}^2 \times \mathbb{Z}/2\mathbb{Z}\) and \(\Phi\) is \(K\)-universal iff \(\overline{\Phi}\) is \(F\)-universal. Also \(\Phi\) is \(K\)-half-universal iff \(\overline{\Phi}\) is \(F\)-half-universal. This is sufficient to prove (i).

(iii) follows easily from the fact that \(\Phi = 0\) iff \(\overline{\Phi} = 0\).

**Proposition 2.3.** Let \(E\) be a field satisfying \(A_n, n \geq 1\), and let \(K\) and \(F\) be fields satisfying
\[
(2.3.1) \quad W(K) = W(F) \times W(E)
\]

(direct product in the category of abstract Witt rings).

(i) \(K\) is \(n\)-T-local iff \(F\) is \((n-1)\)-T-local.

(ii) \(K\) is \(n\)-T-Hilbert iff \(F\) is \((n-1)\)-T-Hilbert.

(iii) \(K\) satisfies \(A_n\) iff \(F\) satisfies \(A_{n-1}\).

**Proof.** We have \(TP_K F = TP_F F \times TP_E F\) and \(TP_E F = 0\) by \(A_n\). This proves (i) and (ii). (iii) follows from the fact that for \(\Phi = \langle \phi_1, \phi_2 \rangle \in TP_F F \times TP_E F\), we have
\[
D_K(\Phi) = D_K(\phi_1) \times D_K(\phi_2)
\]

(here the value groups are regarded as subgroups of groups of square classes rather than subgroups of multiplicative groups of fields). Notice that, given \(F\) and \(E\), the field \(K\) satisfying (2.3.1) always exists according to [12] (see also earlier papers cited there).

**Example 2.4.** Using the tables of quadratic form schemes in [17] and Propositions 2.2 and 2.3 above, we have compiled the following data confirming the repeated occurrence of \(T\)-local and \(T\)-Hilbert types among fields with a finite number of square classes. While in the column below the number of square classes the total number of objects is given, we split the total to show the number of objects coming
from non-real fields (the first summand) and from formally real fields (the second summand). Thus, for instance, 29 = 15 + 1 + 1 in the last column, last row but one, means there are 29 n-T-Hilbert types altogether and 15 come from non-real fields, while 14 come from formally real fields.

Number of:

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<tr>
<td>Witt rings with non-zero torsion</td>
<td>1 2 = 2 + 0 5 = 4 + 1 15 = 10 + 5 47 = 27 + 20</td>
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<tr>
<td>T-local or T-Hilbert types</td>
<td>1 2 = 2 + 0 3 = 2 + 1 10 = 6 + 4 31 = 15 + 16</td>
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<tr>
<td>n-T-local types</td>
<td>n ≥ 2 0 0 2 = 2 + 0 9 = 6 + 3 29 = 15 + 14</td>
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<td>n-T-Hilbert types</td>
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<tr>
<td>1-T-local types</td>
<td>0 2 = 2 + 0 1 = 0 + 1 1 = 0 + 1 2 = 0 + 2</td>
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**Theorem 2.5.** Let m and n be two positive integers and 2 ≤ n + 1 ≤ m.

(i) There exists a non-real (n+1)-local field F with 2n square classes.

(ii) There exists a formally real n-local field F with 2n square classes.

Proof. (i) is proved in [19], Theorem 3.2. (ii) Here the proof is similar to that of Theorem 2.2 in [19]. Starting with a formally real algebraic extension of Q with 4 square classes represented by ±1, ±2, which is 1-local, we double the number of square classes and keep the same local rank using Proposition 2.3 with E = R. Using Proposition 2.2 we double the number of square classes and enlarge by 1 the local rank of a given T-local field F. Combining these two methods furnishes a complete proof of the Theorem.

**Theorem 2.6.** Let m and n be two positive integers, n ≥ 2, m > 4 and n + 1 < m - 1. There exists a formally real (n - 1)-local field F with 2n square classes which is not n-local.

Proof. The result holds for n = 2 and m = 4 by Example 1.5. Using Proposition 2.3 with E = R, we prove the result for n = 2 and any m > 4. Starting again with the field in Example 1.5 and using Proposition 2.2 we prove the result for n = 3 and m = 5 and then applying Proposition 2.3 proves the result for n = 3 and any m > 5. Continuing this procedure, the Theorem follows by induction.

The last result in this section shows that the bound for the number of orderings found in Theorem 1.10 to guarantee that (B) = (A), is the best possible, for every n ≥ 2.

**Theorem 2.7.** For every n ≥ 2 there exists a formally real (n - 1)-T-Hilbert field F with [Xn] = 2n - 1 which is not n-T-local.

Proof. The field F in Example 1.5 satisfies the requirements for n = 2. By induction, using Proposition 2.2, the field F(λ1,...,λn) satisfies the requirements for any n ≥ 3.

Remark 2.8. Using Theorem 2.7 and Proposition 2.3 one can prove by induction the following more general result:

For every n ≥ 2 and every i ≥ 0 there is a field Fni with [Xn] = 2n - 1 + i such that Fni is (n - i)-T-Hilbert and not n-T-local.

§ 3. Going-up theorems for T-local fields. R. E. Elman and T. Y. Lam [7], Theorem 4.5, proved a going-up theorem for the property A1 in totally positive quadratic extensions F/F. Since n-T-local fields satisfy A1, it is natural to expect that, if a going-up theorem for n-T-locality exists, the best chance to discover it is to look at totally positive quadratic extensions. On the other hand, K. Kozioł [11], generalizing the results of C. P. D. and J. Ramsey [3], has already proved a going-up theorem for n-T-locality in case of quadratic extensions K = F(√a), where F is non-real and a ∉ Rn-1, F. He also proves that the going-up result is false if a ∈ Rn-1, F and (F is non-real).

To generalize these results to formally real fields it seems natural to replace the radical Rn-1, F by its "torsion" counterpart TRn-1, F, the intersection of value groups of all torsion (n - 1)-fold Pfister forms over F. As [7] suggests, there is no hope for a going-up result for T-locality outside the class of totally positive quadratic extensions. Thus we are led to the following conjecture.

**Conjecture.** Let F be a field such that a ∈ F, \TRn-1, F and n ≥ 1. If F is n-T-local, then K = F(√a).

In this section we prove the conjecture for n = 1 and n = 2. In crucial points of the proof we make use of results that are not known to hold for arbitrary values of n, hence the approach is not easily generalizable to other cases.

We begin with the case n = 1.

**Proposition 3.1.** Let F be 1-T-local and a ∈ F*. Then K = F(√a) is also 1-T-local.

Proof. Torsion 1-fold Pfister forms are (1, -c) with c ∈ F*. Since F is 1-T-local, we have Fc = F2 ⊕ aF2 = Dc(1, 1) and c = (1, -a). Thus there is exactly one totally positive quadratic extension K = F(√a). By [7], Corollary 2.18, we have Dc(1, 1) = K and it remains to prove that Dc(1, 1) consists of two square classes. By Norm Principle [7, 2.3.13], the norm NxF induces a homomorphism N: Dc(1, 1)/K2 → Dc(1, 1)/F2 and again by Norm Principle and by (1.1) in [7], its kernel is trivial. Thus N is injective and

\[ |Dc(1, 1)/K2| ≤ |Dc(1, 1)/F2| = 2. \]

Now \[ |Dc(1, 1)/K2| > 1, \] since otherwise K is Pythagorean, and then, by a result of Diller and Dress (cf. [7, Corollary 3.9], F is also Pythagorean, a contradiction. Thus K consists of two square classes and K is 1-T-local.

For the balance of this section we assume n ≥ 2. If K = F(√a) we write N for the norm NxF. Then a formal \( i \) over K we write \( is(\sqrt{a}) \) for its transfer to F, where \( is \) is the F-linear functional on K given by \( is(1) = 0, is(\sqrt{a}) = 1. \)
Proposition 3.2. Suppose $F$ is an $n$-T-local field, $a \in F_\infty \backslash R_{1,-} F$, and $K = F_\sqrt{n}$. Then:

(i) $F \subseteq \Delta(K)$ for every $\Phi \in TP_{\infty}$.
(ii) $x \in \Delta(K) \Rightarrow N(x) \subseteq \Delta(K)$ for every $x \in K$ and $\Phi \in TP_{\infty}$.
(iii) $N(\Delta(K)) = \Delta(K) \cap \Delta(1, -a)$ for every $\Phi \in TP_{\infty}$.
(iv) If $x_1 \notin \Delta(K)$ and $x_2 \notin \Delta(K)$, then $x_1 x_2 \in \Delta(K)$ for $x_1, x_2 \in K$ and $\Phi \in TP_{\infty}$.
(v) $TP_{\infty} K \neq 0$.
(vi) $(\Phi x)_x = x$.
(vii) If $0 \neq \Psi \in TP_{\infty} K$, then $s_\Psi(\Psi) = \Phi$.
(viii) $s_\Psi(\Phi, \Psi) = TP_{\infty} F$.

Proof. (i) Let $b \in F$ and $b \notin \Delta(K)$. Then $\Psi_{-b} \otimes_\Psi TP_{\infty} F$ and is anisotropic. Since $a \notin R_{1,-} F$, there exists $\Psi \in P_{\infty} F$ such that $\Psi_{-a} \otimes_\Psi TP_{\infty}$ is anisotropic and it is torsion since $a \in F_\infty$. By $n$-T-locality, $\Psi_{-a} \otimes_\Psi TP_{\infty} F$ is anisotropic. Since $\Psi_{-a} \otimes_\Psi TP_{\infty} F$ becomes hyperbolic over $K$, we get $b \notin \Delta(K)$.

(ii) $N(\Psi) \subseteq \Phi \otimes_\Psi \Phi \otimes_\Psi \Phi$ by the Norm Principle, and $\Phi \otimes_\Psi \Phi \otimes_\Psi \Phi = \Phi(\Phi)$, by (i).

(iii) follows from (ii).

(iv) If $x_1, x_2 \notin \Delta(K)$, then by (ii), $N(x_1) N(x_2) \notin \Delta(K)$. Since $F$ is $(n-1)$-T-Hilbert (Theorem 1.3), $N(x_1) N(x_2) \notin \Delta(K)$. Then $x_1 x_2 \notin \Delta(K)$, by (ii).

(v) $TP_{\infty} K = 0$, then $K$ satisfies $N_{\infty}$ and then, by Going-Down Theorem 4.12 in [7], $F$ satisfies $N_{\infty}$, a contradiction.

(vi) As shown in the proof of Theorem 4.1 in [7], there exists $\Psi \in P_{\infty} F$ such that $\Psi_{-a} \otimes_\Psi TP_{\infty}$ is torsion and anisotropic. By $n$-T-locality, $\Psi_{-a} \otimes_\Psi TP_{\infty} F = \Phi(\Phi) = 0$. Hence $(\Phi x)_x = x$.

(vii) $F$ satisfies $N_{\infty}$ in Theorem 1.1, hence $K$ satisfies $N_{\infty}$ by Going-up Theorem 4.5 in [7], hence $K$ is $n$-T-universal by Theorem 1.1. It follows that $-1 \notin \Delta(K)$, hence $2 \Psi = 0$. Thus $\Psi = \Psi_{-y} \otimes_\Psi \Theta$, where $y \notin \Delta(1,1)$ and $\Theta \in P_{\infty} K$. Now $\Theta = \sum (a_i) \Phi_i \otimes_\Psi \Theta_i$, where $a_i \in F$, $\Phi_i \in P_{\infty} F$, $x_i \in K$, by Lemma 2 in [6]. Thus $s_\Psi(\Psi) = \sum (b_i) \Phi_i s_{-y}(\Phi_i) s_\Psi(\Theta_i)$, for some $b_i \in F$.

Now as in the proof of the Going-up Theorem 4.5 in [7], we conclude that $s_\Psi(\Phi, \Psi) = \sum s_{-y}(\Phi_i) s_\Psi(\Theta_i)$, where $d_i \in F$ and $c_i \in F_\infty$.

Since $\Phi_i \otimes_\Psi \Theta_i \in TP_{\infty} F = \{0, \Phi\}$ and $2 \Phi = 0$, we have $s_\Psi(\Psi) = \sum (b_i) \Phi_i s_{-y}(\Phi_i) s_\Psi(\Theta_i) = 0$ or $\Phi$.

We will show that $s_\Psi(\Psi) \neq 0$. Otherwise, by Theorem 2.3 in [7], there exists $x \in P_{\infty} F$ such that $\Psi = x$. Now $x$ torsion implies that $x$ is torsion, by total positivity of the extension (cf. [7], (1.2)), hence $\Psi = 0$ implies $x = \Psi$, a contradiction. This proves (vii).

(viii) follows from (vii).

Now we switch to the case $n = 2$. We begin with an analog of a useful result of Corder (2), Lemma 1.

Lemma 3.3. If $F$ is a $2$-T-local field and $\langle 1, a \rangle \subseteq \langle 1, b \rangle$ and $\Phi \in TP_{\infty}$, then $\Delta(1, a) \subseteq \Delta(1, b)$ if $ab \notin R_{1,F}$.

Proof. $\langle a, -a \rangle, \langle b, -a \rangle \in TP_{\infty} F$ for every $c \in F$ and since $\Delta(1, a) = \Delta(1, b)$ if one of the forms is isotropic (anisotropic), so is the other. By $2$-T-locality, $\langle a, -a \rangle = \langle b, -a \rangle$, whence $ab \in \Delta(1, -a)$ for every $c \in F$. Thus $ab \notin R_{1,F}$. The converse is known to hold for any field $F$.

Lemma 3.4. Suppose $F$ is a $2$-T-local field, $a \in F_\infty \backslash R_{1,F}$ and $K = F_\sqrt{2}$. Then either:

(i) $\Delta(1, a) \subseteq \Delta(1, -b)$ for some $b \in F_\infty \backslash R_{1,F}$.

(ii) $F = F_\sqrt{2}$.

Moreover, (ii) implies $a \in TR_{1,F}$.

Proof. If (i) does not hold, then $\Delta(1, a) \subseteq \Delta(1, -b)$ for every $b \in F_\infty \backslash R_{1,F}$. By Lemma 3.3, we have $ab \in R_{1,F}$ for every $b \in F_\infty \backslash R_{1,F}$, and (ii) follows.

Now assume (ii). Since the value set of a quadratic form consists of cosets of $R_{1,F}$ (cf. [1]), we have $\Delta(1, a) \subseteq F_\infty = R_{1,F} R_{1,F} = \Delta(1, a)$, whence $F_\infty = \Delta(1, a)$.

We also have $R_{1,F} \subseteq \Delta(1, 1)$, $F_\infty$, and since $R_{1,F}$ has index 2 in $F_\infty$ we have $R_{1,F} \subseteq \Delta(1, 1)$ or $R_{1,F} \subseteq F_\infty$. The first possibility is ruled out since it is equivalent with $F$ satisfying $A_2 (\Psi)$, Theorem 4.3 which is not the case. Thus $\Delta(1, -b) = \Delta(1, a)$. Observe that (ii) implies also $\Delta(1, -a) \subseteq TR_{1,F}$.

Now we have $a \in R_{1,F} = \Delta(1, -a) \subseteq TR_{1,F}$.

We are finally ready to prove.

Going-up Theorem 3.5. Let $K = F_\sqrt{2}$, where $a \in F_\infty \backslash R_{1,F}$. Then, if $F$ is $2$-T-local, so is $K$.

Proof. By Theorem 3.4, we may assume there is $b \in F_\infty \backslash R_{1,F}$ such that $\Delta(1, a) \subseteq \Delta(1, -b)$. The forms $(1, a)$ and $(1, -b)$ are not universal $(a, b \notin R_{1,F})$ and since they are torsion, they are half-universal, by Theorem 1.1. Thus there exists $d \in \Delta(1, -a) \Delta(1, -b)$. It follows that $\langle b, -d \rangle = \Phi$ by $2$-T-locality of $F$, and $d = N(x)$ for some $x \in K$. Thus $\Phi = \langle -b, -N(x) \rangle$.

Now we will prove that every anisotropic torsion $2$-fold Pfister form over $K$ is isometric with $\langle b, -d \rangle$. So let $0 \neq \Psi \in TP_{\infty} K$. Then $s_\Psi(\Psi) = \Phi$ by Proposition 3.2(vii) and we observe that also $s_{\Psi}(\langle -b, -x \rangle) = \Phi$. Thus $\Psi = \langle -b, -x \rangle$ has the following properties:

$q \in TP_{\infty} K$ and $s_\Psi(q) = 0$. 
From the exactness of the sequence (cf. [7], Corollary 2.10)

\[ F^2 \to F \to F/K^{2} \to F^2 \]

where \( i \) is the functorial map, we conclude that there is \( \Theta \in \mathcal{I}^2 F \) such that \( \gamma = \Theta_k \).

Here \( \Theta_k \) is torsion, hence by total positivity of the extension, also \( \Theta \) is torsion. Thus \( \Theta \in \mathcal{I}^2 F = \mathbb{Z}/2\mathbb{Z} \), the latter by Proposition 1.8. Thus \( \Theta \in \mathcal{I}^2 F = \{0, \rho \Theta \} \) and in either case, \( \gamma = \Theta_k = 0 \) (by Proposition 3.2(vi)). This proves \( \Psi = \langle \langle -b, -a \rangle \rangle \) and finishes the proof.

Remark 3.6. The result in 3.5 is inapplicable if \( F_\alpha \subseteq T_R F \). If this happens, then necessarily \( F_\alpha = D_\mu(1, 1) \). Indeed, if \( b \in F_\alpha \subseteq D_\mu(1, -b) \), whence \( b \in D_\mu(1, 1) \).

For formally real fields with 8 square classes there are 3 cases where \( F \) is 2-\( T \)-local (II, IV and V in [12]). Of these, one satisfies \( F_\alpha \subseteq T_R F \) (Case IV) and the Going-up Theorem 3.5 does not apply. In the remaining cases we have

- \( D_\mu(1, 1) = F_\alpha \subseteq T_R F \) (Case II); and
- \( D_\mu(1, 1) \neq F_\alpha \subseteq T_R F \) (Case V).

References