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On a classification of pointwise compact sets of the first Baire class functions

by

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Abstract. The paper is concerned with compact separable subspaces of the space $B_1(\omega^\omega)$ of the first Baire class functions on irrationals endowed with the pointwise topology, i.e. Rosenthal compacta. We associate to each separable Rosenthal compactum K an ordinal number $\eta(K) \leqslant \omega_1$, which indicates the "Borel complexity" of the compactum. The index $\eta(K)$ is a topological invariant of the function space $C_p(K)$ endowed with the pointwise topology. We construct Rosenthal compacta of arbitrarily large countable index and we use them to give examples of open linear continuous maps raising the Borel class of linear spaces.

§ 1. Introduction. Our terminology follows [En], [Ku] and [Se]. We shall denote by R the real line; ω is the set of natural numbers and ω^ω is the Baire space, i.e. topologically the irrationals.

A map $f\colon X \to Y$, where X and Y are separable metrizable spaces, is of the first Baire class if $f^{-1}(U)$ is an F_σ-set for every open $U \subset Y$ (if Y is a separable Banach space, this means that f is a pointwise limit of a sequence of continuous maps from X into Y), cf. [Ku, § 31]. Given a separable metrizable space X, we denote by $B_1(X)$ the space of real-valued first Baire class functions on X equipped with the topology of pointwise convergence.

This paper is concerned mainly with compact spaces which can be embedded in the space $B_1(\omega^\omega)$ of the first Baire class functions on irrationals, i.e. with Rosenthal compacta, see [Go]. For fundamental facts about Rosenthal compacta we refer the reader to the papers by Bourgain, Fremlin, Talagrand [BFT], Godefroy [Go] and Negrepontis [Ne]. In the sequel we shall often use the deep result by Bourgain, Fremlin and Talagrand [BFT, Th. 3F] stating that Rosenthal compacta K are Fréchet topological spaces, i.e. for every set $A \subset K$ and a point $x \in \bar{A}$ there exists a sequence of points from A which converges to x.

Let us notice that if A is a metrizable space which is a continuous image of irrationals (i.e. A is an analytic space), then compact subspaces of $B_1(A)$ are Rosenthal compacta, as the map $f \to f \circ u$, where $u\colon \omega^\omega \to A$ is a continuous surjection, embeds $B_1(A)$ homeomorphically into $B_1(\omega^\omega)$.

Let K be a separable compact space; given a countable dense subset D of K

we denote by $C_D(K)$ the space of real-valued continuous functions on K endowed with the topology of pointwise convergence on the set D. We shall consider the space $C_D(K)$ as a subspace of the product R^D, identifying a continuous function $f\colon K\to R$ with the restriction f/D.

The results of this paper have been motivated by the following characterization of separable Rosenthal compacta given by G. Godefroy [Go, Th. 4]:

THEOREM (Godefroy). *A separable compact space K is a Rosenthal compactum if and only if for every countable dense subset D of K the space $C_D(K)$ is analytic.*

We shall consider separable Rosenthal compacta K such that for some countable dense subset D of K the function space $C_D(K)$ is a Borel subset of the product R^D. Many "classical" compacta, e.g. the Helly space (cf. [En, Exercise 3.2E]), the two arrows space (cf. [En, Exercise 3.10C]), the space of functions from $[0, 1]$ into $[0, 1]$ of total variation $\leqslant 1$, enjoy this property; however, there exist separable Rosenthal compacta which do not belong to this class, cf. 5.2. If $C_D(K)$ is a Borel set in R^D for some countable dense $D\subset K$, then this is also the case for every such D and, moreover, if D, D' are countable dense subsets of K and α, α' are the Borel classes of $C_D(K)$ and $C_{D'}(K)$, respectively, then $\alpha'\leqslant 1+\alpha$ (see Theorem 2.2; actually, we do not know any example showing that α could differ from α'). This allows us to introduce the index $\eta(K)$ of a separable Rosenthal compactum K — an ordinal number which indicates its "degree of Borel complexity" (we write $\eta(K)=\omega_1$ if $C_D(K)$ is not Borel for countable dense $D\subset K$). We prove that the index $\eta(K)$ is a topological invariant of the function space $C_p(K)$ of real-valued continuous functions on K endowedw ith the pointwise topology (Theorem 3.5). We also show that if the function spaces $C(K)$, $C(L)$ on Rosenthal compacta K and L are homeomorphic in the weak topology, then $\eta(K)\leqslant 1+\eta(L)$ (Theorem 3.1; we do not know examples showing that in this case $\eta(K)$ and $\eta(L)$ could not coincide). There exist separable Rosenthal compacta K with arbitrarily large countable index $\eta(K)$. We construct a family of such compacta in Section 4 and we use these compacta in Sec. 5 to obtain examples of open linear continuous maps $\varphi\colon E\xrightarrow{\text{onto}}F$ between separable linear metric spaces E, F, such that E is an absolute $F_{\sigma\delta}$ and F may have arbitrarily high Borel class (we are not aware of any other similar examples in the literature). We close the paper with some comments and remarks (Sec. 6).

Let us end this section with some explanation of the notation and terminology used throughout this paper. We denote by 2^ω the countable product $\{0, 1\}^\omega$, i.e. topologically the Cantor set.

Given a set X, we denote by R^X the Tikhonov product of the real line, X being the index set. For every $Y\subset X$, $\pi_Y\colon R^X\to R^Y$ is the projection; we shall write π_x instead of $\pi_{\{x\}}$, for $x\in X$.

Given a compact space K, we denote by $C(K)$ the Banach space of real-valued continuous functions on K endowed with the sup norm. The space $C(K)$ equipped with the weak topology will be denoted by $C_w(K)$, while $C_p(K)$ will denote the space $C(K)$ endowed with the pointwise topology.

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§ 2. Borel index of separable Rosenthal compacta. In this section we establish a few simple facts about the Borel structure of function spaces $C_D(K)$ for Rosenthal compacta K and we define the index $\eta(K)$ (for the definition of $C_D(K)$ see Sec. 1).

We begin with an observation related to a theorem from [DGLvM] that the function space $C_p(X)$ is not a $G_{\delta\sigma}$-set in the product R^X, unless the space X is discrete.

2.1. THEOREM. *Let K be an infinite separable Rosenthal compactum. For every countable dense subset D of K the space $C_D(K)$ is not a $G_{\delta\sigma}$-set in R^D.*

Proof. Let S be the subset of the countable product R^ω of the real line consisting of sequences converging to zero; the set S is not a $G_{\delta\sigma}$-set in R^ω, cf. [BM], [DGLvM, Th. 6]. It is enough to show that S embeds in $C_D(K)$ as a G_δ subset.

Since the compactum K is a Fréchet space (see Introduction), there exists a sequence of distinct points $(d_n)_{n\in\omega}$ converging to some $x\in K\setminus\{d_n\colon n\in\omega\}$. Let us fix a sequence of continuous functions $f_n\colon K\to[0, 1]$ such that $f_n(d_n)=1$ and $f_n^{-1}((0, 1])\cap f_m^{-1}((0, 1])=\emptyset$ for all $m\neq n$. Now, we can define a map $\varphi\colon S\to C_D(K)$ by the formula

$$\varphi((a_n))=\sum_{n\in\omega}a_nf_n\quad\text{for } (a_n)\in S$$

(notice that for every sequence $(a_n)\in S$ the series $\sum a_nf_n$ is uniformly convergent). It can be easily verified that φ is a homeomorphic embedding. Moreover, we have

$$\begin{aligned}\varphi(S)=\{f\in C_D(K)\colon f(x)=0,\ \forall n\in\omega\ &\forall d\in f_n^{-1}((0, 1])\ f(d)=f(d_n)\cdot f_n(d) \text{ and}\\ &\forall d\in K\setminus\bigcup\{f_n^{-1}((0, 1])\colon n\in\omega\}\ f(d)=0\}\\ =\bigcap_{k\in\omega}\bigcup_{n>k}\{f\colon |f(d_n)|<k^{-1}\}\cap\bigcap\{\{f\colon f(d)&=f(d_n)\cdot f_n(d)\}\colon d\in f_n^{-1}((0, 1]),\ n\in\omega\}\\ &\cap\bigcap\{\{f\colon f(d)=0\}\colon d\in K\setminus\bigcup\{f_n^{-1}((0, 1])\colon n\in\omega\}\}.\end{aligned}$$

Hence $\varphi(S)$ is a G_δ-set in $C_D(K)$.

The next theorem shows that for a Rosenthal compactum K, and for any D, D' countable dense subsets of K, the Borel classes of $C_D(K)$ and $C_{D'}(K)$ almost coincide.

2.2. THEOREM. *Let K be a separable Rosenthal compactum and let D, E be countable dense subsets of K. If $C_D(K)$ is a Borel set of the class α then $C_E(K)$ is a Borel set of class $\leqslant 1+\alpha$. If, in addition, $D\subset E$ then $C_E(K)$ is of Borel class $\leqslant\alpha$.*

Proof. It is enough to consider the case when the compactum K is infinite. Firstly, let us assume that $D\subset E$. Let $\varphi\colon C_D(K)\to C_E(K)$ be the identity map. The map φ is a bijection of the first Baire class. Indeed, for every $e\in E$ the composition of the projection $\pi_e\colon R^E\to R$ (see Sec. 1) and the map φ is a pointwise limit of the sequence of continuous functions $\pi_{d_n} \circ \varphi$, where (d_n) is a sequence of points of D convering to e (by Fréchet property of K, cf. Sec. 1). The inverse map φ^{-1} is continuous. From [Ku, § 35. VII] it follows that we can find $F_{\sigma\delta}$-sets $A \subset R^D$ and $B \subset R^E$ such that $C_D(K) \subset A$, $C_E(K) \subset B$ and a bijection $\psi\colon A \to B$ of the first Baire class such that ψ^{-1} is continuous and ψ extends φ. Using Theorem 2.1 we infer that the space $C_E(K) = \psi(C_D(K)) \cap B$ is a Borel set of class $\leqslant \alpha$. Similarly, if $C_E(K)$ is of the Borel class β then $\alpha \leqslant 1+\beta$.

From these inequalities we obtain the required conclusion for the case of arbitrary countable dense subsets D, E; it is enough to consider the set $D \cup E$.

Now, for each separable Rosenthal compactum K we define an index $\eta(K)$ (an ordinal number $\leqslant \omega_1$) in the following way:

$$\eta(K) = \begin{cases} \min\{\alpha\colon \text{there exists a countable dense subset } D \subset K \text{ such that} \\ \quad \text{the space } C_D(K) \text{ is a Borel set of class } \alpha \text{ in } R^D\} \text{ if such} \\ \quad \text{sets } D \text{ exist}, \\ \omega_1 \quad \text{in the opposite case}. \end{cases}$$

There exist Rosenthal compacta K with $\eta(K) = \omega_1$ (cf. 5.2).

We shall check that if K is an infinite separable compact subset of a space $B_1(M)$ with compact metrizable M, and K has a dense subset, which consists of functions continuous on M, then $\eta(K) = 2$. We prove this fact in a more general form, which will be used in Section 5.2.

2.3. THEOREM. *Let M be a metrizable compact space and let K be a compact subset of the product R^M such that there exists a countable dense subset $D \subset K$ consisting of functions continuous on M. Then the space $C_D(K)$ is an $F_{\sigma\delta}$-set in R^D.*

Proof. Let us observe that for each function $f\colon D \to R$, $f \in C_D(K)$ if and only if

$$(\forall m \in \omega)\ (\exists n \in \omega)\ (\exists t_1, \ldots, t_k \in M)\ (\forall d, e \in D)$$
$$\big(\max_{i \leqslant k} |d(t_i) - e(t_i)| < n^{-1} \Rightarrow |f(d) - f(e)| \leqslant m^{-1}\big).$$

Let $F(m, n, k) = \{(f, t_1, \ldots, t_k) \in R^D \times M^k\colon (\forall d, e \in D)\big(\max_{i \leqslant k} |d(t_i) - e(t_i)| \geqslant n^{-1}$ or $|f(d) - f(e)| \leqslant m^{-1}\big)\}$. For $m, n \in \omega$, $k = 1, 2, \ldots$, $F(m, n, k)$ is a closed subset of the product $R^D \times M^k$. Hence the projection $\pi_{R^D}(F(m, n, k))$ parallel to the compact axis M^k is also closed. Now, one can describe the space $C_D(K)$ by the formula

$$C_D(K) = \bigcap_{m \in \omega} \bigcup_{n \in \omega} \bigcup_{k \in \omega} \pi_{R^D}(F(m, n, k)),$$

which shows that $C_D(K)$ is an $F_{\sigma\delta}$-subset of R^D.

Since each metrizable compact space embeds in the space of continuous functions on the unit interval $C_p([0, 1])$ endowed with the pointwise topology, we obtain the following corollary:

2.4. COROLLARY. *If D is a countable dense subset of a compact metrizable space K then the space $C_D(K)$ is an $F_{\sigma\delta}$-set in R^D.*

§ 3. Borel index and homeomorphisms of function spaces. Using a method of factorization of homeomorphisms on function spaces from [Ma] we shall prove the following (cf. also 6.4):

3.1. THEOREM. *Let K, L be compact spaces such that the function spaces $C_w(K)$ and $C_w(L)$ endowed with the weak topology are homeomorphic. If K is a separable Rosenthal compactum, then so is L and $\eta(L) \leqslant 1+\eta(K)$, $\eta(K) \leqslant 1+\eta(L)$; thus, in particular, if $\eta(K) \geqslant \omega$ then $\eta(K) = \eta(L)$.*

From Theorem 3.1 we obtain immediately the following corollary:

3.2. COROLLARY. *Let K, L be compact spaces. If K is a separable Rosenthal compactum and if the Banach spaces $C(K)$ and $C(L)$ are linearly homeomorphic then L is a separable Rosenthal compactum and $\eta(L) \leqslant 1+\eta(K)$, $\eta(K) \leqslant 1+\eta(L)$.*

Before we start the proof of Theorem 3.1, let us state two auxiliary facts:

3.3. LEMMA ([Ma, Lemma 4.1], cf. also [Is, Proof of Theorem 1]). *Let X be a set and let E be a linear subspace of the product R^X. If $f\colon E \to R^\omega$ is a continuous map then f depends on countably many coordinates, i.e. there is a countable set $Y \subset X$ and a continuous map $g\colon \pi_Y(E) \to R^\omega$ such that $f = g \circ \pi_Y|E$.*

3.4. LEMMA. *Let X, Y be sets and let $f\colon E \to F$ be a homeomorphism between linear subspaces $E \subset R^X$ and $F \subset R^Y$. For every countable subsets $S_0 \subset X$, $T_0 \subset Y$ such that the projections $\pi_{S_0}|E\colon E \to R^{S_0}$ and $\pi_{T_0}|F\colon F \to R^{T_0}$ are injective, there exist countable subsets $S_1 \subset X$, $T_1 \subset Y$ such that $S_0 \subset S_1$, $T_0 \subset T_1$ and the function $\pi_{T_1} \circ f \circ (\pi_{S_1}|E)^{-1}$ maps $\pi_{S_1}(E)$ homeomorphically onto $\pi_{T_1}(F)$.*

Lemma 3.4 can be proved in a similar way to Lemma 4.2 in [Ma] and therefore we decided to omit the proof; let us only indicate that the required sets S_1 and T_1 can be obtained by a certain back-and-forth induction based on Lemma 3.3.

Proof of Theorem 3.1. As in the proof of Theorem 2.2, we can assume that the compact spaces K and L are infinite. Let $\varphi\colon C_w(L) \to C_w(K)$ be a homeomorphism and let $M(K) = C(K)^*$, $M(L) = C(L)^*$ be the spaces of Radon measures on the compacta K and L. Let us denote by B_K, B_L the unit balls of $M(K)$, $M(L)$, respectively, endowed with the weak* topology. By a theorem of Godefroy and Talagrand [Go, Proposition 7, Remarque], the ball B_K is a Rosenthal compactum. For arbitrary subsets $S \subset B_K$ and $T \subset B_L$ let $i_S\colon C_w(K) \to R^S$ and $i_T\colon C_w(L) \to R^T$ be defined by the formulas

$$i_S(f)(\mu) = \mu(f) \quad \text{for } \mu \in S,\ f \in C_w(K),$$
$$i_T(f)(\mu) = \mu(f) \quad \text{for } \mu \in T,\ f \in C_w(L).$$

If we identify, using i_{B_K} and i_{B_L}, the spaces $C_w(K)$ and $C_w(L)$ with subspaces of the products R^{B_K} and R^{B_L}, respectively, then we can consider the maps i_S, i_T as projections onto R^S and R^T, respectively.

Let us choose a countable dense subset $D \subset K$ and let $S_0 = \{\delta_d\colon d \in D\}$, where δ_d is the probability measure concentrated at the point d. If the set $S \subset B_K$ contains S_0 then the map $i_S\colon C_w(K) \to R^S$ is injective. We show that there exists a countable subset $T_0 \subset B_L$ such that the map $i_{T_0}\colon C_w(L) \to R^{T_0}$ is also injective. Indeed, applying Lemma 3.3 to the map $i_{S_0} \circ \varphi\colon C_w(L) \to R^{S_0}$ we infer that there exist a countable subset $T_0 \subset B_L$ and a continuous map $g\colon i_{T_0}(C_w(L)) \to R^{S_0}$ such that $i_{S_0} \circ \varphi = g \circ i_{T_0}$. Since the map $i_{S_0} \circ \varphi$ is one-to-one, the set T_0 has the required property. Obviously, for arbitrary subset $T \subset B_L$ containing T_0, the map $i_T\colon C_w(L) \to R^T$ is also injective. Using Lemma 3.4 we can choose countable subsets $S_1 \subset B_K$, $T_1 \subset B_L$ such that $S_0 \subset S_1$, $T_0 \subset T_1$ and the map $\psi_1 = i_{S_1} \circ \varphi \circ i_{T_1}^{-1}$ is a homeomorphism between the spaces $i_{T_1}(C_w(L))$ and $i_{S_1}(C_w(K))$.

We now verify that the space $i_{S_1}(C_w(K))$ is analytic, which would ensure that $i_{T_1}(C_w(L))$ is also analytic. Let us consider a map $i_{S_1} \circ i_{S_0}^{-1}\colon i_{S_0}(C_w(K)) \to i_{S_1}(C_w(K))$. For arbitrary $\delta_d \in S_0$, the map $f \to f(\delta_d)$ is a real-valued continuous function on $i_{S_0}(C_w(K))$. As was noticed before, B_K is a Rosenthal compactum and the convex hull W of the set $\{\delta_d, -\delta_d\colon d \in D\}$ is dense in B_K; by the Fréchet property of B_K (see Sec. 1), for every measure $\mu \in S_1$ there exists a sequence of measures $\mu_n \in W$ such that $\mu = \lim_n \mu_n$ and hence $\mu(f) = \lim_n \mu_n(f)$ for each $f \in C_w(K)$. Since each function $\mu_n \circ i_{S_0}^{-1}$ is continuous (each μ_n being a linear combination of elements of S_0), it follows that the function $\mu \circ i_{S_0}^{-1}$ is a pointwise limit of a sequence of continuous functions on $i_{S_0}(C_w(K))$ and therefore the map $i_{S_1} \circ i_{S_0}^{-1}$ is of the first Baire class (cf. [Ku, § 31.VI]). From the theorem of Godefroy [Go, Th. 4] we obtain that the space $i_{S_0}(C_w(K))$ is analytic (recall that $S_0 = \{\delta_d\colon d \in D\}$) and so is the space $i_{S_1}(C_w(K))$, cf. [Ku, § 39.I].

Let $A = i_{T_1}(C_w(L))$; we shall prove that the compactum L can be embedded in the space $B_1(A)$, which will show that L is a Rosenthal compactum; cf. Sec. 1.

Let us define a map $u\colon L \to R^A$ by the formula

$$u(x)(f) = i_{T_1}^{-1}(f)(x) \quad \text{for } x \in L,\ f \in A.$$

The continuous map u separates points of the compact space L and hence it is a homeomorphic embedding. We show that for each $x \in L$ the map $u(x)\colon A \to R$ is of the first Baire class, i.e. $u(L) \subset B_1(A)$.

Let us fix a point $x \in L$. Applying again Lemma 3.4 we can find countable sets $S_2 \subset B_K$ and $T_2 \subset B_L$ such that $S_1 \subset S_2$, $T_1 \cup \{\delta_x\} \subset T_2$ and the function $\psi_2 = i_{S_2} \circ \varphi \circ i_{T_2}^{-1}$ maps $i_{T_2}(C_w(L))$ homeomorphically onto $i_{S_2}(C_w(K))$. Similarly to the preceding case one can verify that the map $h = i_{S_2} \circ i_{S_1}^{-1}$ is of the first Baire class. Let $p\colon i_{T_2}(C_w(L)) \to R$ be the projection onto δ_x-coordinate, i.e. $p(f) = f(\delta_x) = i_{T_2}^{-1}(f)(x)$ for $f \in i_{T_2}(C_w(L))$; since $\delta_x \in T_2$, the function p is continuous. Now, the fact that $u(x)$ is of the first Baire class on A follows from the formula

$$u(x) = p \circ \psi_2^{-1} \circ h \circ \psi_1 .$$

The separability of the Rosenthal compactum L follows from the fact that T_0 is a countable set of measures which separate the points of the space $C(L)$ and each Radon measure μ on L has separable support [Go, Proposition 8], i.e. there is a separable compactum $M \subset L$ such that $\mu(U) = 0$ for arbitrary open subset $U \subset L \setminus M$.

To complete the proof we have to show that $\eta(L) \leqslant 1+\eta(K)$ and $\eta(K) \leqslant 1+\eta(L)$. Since we have already proved that L is a separable Rosenthal compactum, by the symmetry of assumptions it is enough to check that $\eta(L) \leqslant 1+\eta(K)$. Assume that $\eta(K) < \omega_1$. Choose countable dense subsets $E \subset K$ and $F \subset L$ such that the space $C_E(K)$ is a Borel set of class $\eta(K)$. Let $S_3 = \{\delta_e\colon e \in E\} \subset B_K$ and $T_3 = \{\delta_f\colon f \in F\} \subset B_L$. The spaces $i_{S_3}(C_w(K))$ and $i_{T_3}(C_w(L))$ are homeomorphic to the spaces $C_E(K)$ and $C_F(L)$, respectively. From Lemma 3.4 it follows that there exist countable sets $S_4 \subset B_K$ and $T_4 \subset B_L$ such that $S_3 \subset S_4$, $T_3 \subset T_4$ and the map $i_{S_4} \circ \varphi \circ i_{T_4}^{-1}$ is a homeomorphism between the spaces $i_{T_4}(C_w(L))$ and $i_{S_4}(C_w(K))$. The maps $i_{S_4} \circ i_{S_3}^{-1}\colon i_{S_3}(C_w(K)) \to i_{S_4}(C_w(K))$ and $i_{T_4} \circ i_{T_3}^{-1}\colon i_{T_3}(C_w(L)) \to i_{T_4}(C_w(L))$ are injections of the first Baire class, and their inverse maps are continuous. The spaces $i_{S_4}(C_w(K))$ and $i_{T_4}(C_w(L))$ are Borel sets of class $\leqslant \eta(K)$; hence the space $i_{T_3}(C_w(L))$ is a Borel set of class $\leqslant 1+\eta(K)$ (cf. proof of Theorem 2.2), which ends the proof.

We now show that the index $\eta(K)$ of a separable Rosenthal compactum K is a topological invariant of the function space $C_p(K)$ equipped with the pointwise topology (cf. 6.5).

3.5. THEOREM. *Let K and L be compact spaces such that their function spaces $C_p(K)$ and $C_p(L)$ endowed with the pointwise topology are homeomorphic. If K is a separable Rosenthal compactum then so is L and $\eta(K) = \eta(L)$.*

Proof. The fact that L is a separable Rosenthal compactum can be proved in much the same way as in the proof of Theorem 3.1.

We shall show that $\eta(K) \geqslant \eta(L)$; it is enough to consider the case when $\eta(K) < \omega_1$. Let us choose a countable dense subset $D \subset K$ such that the space $C_D(K)$ is a Borel set of class $\eta(K)$ in R^D. Applying Lemma 3.4 one can find countable dense sets $E \subset K$ and $F \subset L$ such that $D \subset E$ and the spaces $C_E(K)$ and $C_F(L)$ are homeomorphic. From Theorem 2.2 and the definition of the index η it follows that $C_E(K)$ and $C_F(L)$ are Borel sets of class $\eta(K)$, hence $\eta(L) \leqslant \eta(K)$. By symmetry, we have $\eta(K) = \eta(L)$.

§ 4. Rosenthal compacta of arbitrarily large countable Borel index. In this section we construct separable Rosenthal compacta whose function spaces have arbitrarily large Borel class (Sec. 4.1, 4.2). The idea underlying our approach is close to a construction from [LvMP] of function spaces on countable sets, with arbitrarily high Borel classes. We show (Sec. 4.3) that the Rosenthal compacta we obtain can be embedded in the space $B_1(2^\omega)$ of functions of the first Baire class on the Cantor set (not every Rosenthal compactum embeds in $B_1(2^\omega)$, cf. [Po]). Furthermore, we can embed these Rosenthal compacta in compact subspaces of $B_1(2^\omega)$ which have dense subsets consisting of continuous functions (Sec. 4.4). Such special embeddings will be used in Section 5 to obtain some open linear maps between linear metric spaces raising their Borel class.

4.1. THEOREM. *For every countable ordinal number $\alpha \geqslant 2$ there exists a separable Rosenthal compactum K such that $\alpha \leqslant \eta(K) \leqslant 1+\alpha+1$.*

These compacta will be associated in a standard way with almost disjoint families on ω (in particular, the third derived set of these compacta is empty), which, in turn, will be related to Borel subsets A of the Cantor set 2^ω; thus each our compactum $K = K_A$ will correspond to a Borel set $A \subset 2^\omega$.

4.2. Construction of compacta K_A. Let A be a dense Borel subset of additive class $\alpha \geqslant 2$ in the Cantor set 2^ω. We shall associate with A a separable Rosenthal compactum K_A such that $\alpha \leqslant \eta(K_A) \leqslant 1+\alpha+1$.

Let 2^n be the set of functions from $\{0, 1, \ldots, n-1\}$ into $\{0, 1\}$, $n \in \omega$, and let $V_s = \{x \in 2^\omega\colon x|n = s\}$, where $x|n$ is the restriction of a function $x\colon \omega \to \{0, 1\}$ to the set $\{0, 1, \ldots, n-1\}$ and $s \in 2^n$. Let us put $S = \bigcup_{n\in\omega} 2^n$. The open-and-closed sets V_s, for $s \in S$, form a canonical base for the product topology of the Cantor set 2^ω.

For every $s \in 2^n$ and $n \in \omega$, let $f_s\colon A \to R$ be the characteristic function of the set $V_s \cap A$. Let f_x denote the characteristic function of the singleton $\{x\}$, and let 0 be the function identically equal to zero on A. The space $K_A = \{f_s\colon s \in S\} \cup \cup \{f_x\colon x \in A\} \cup \{0\}$ is a compact subset of the Tikhonov cube $\{0, 1\}^A$; K_A is a Rosenthal compactum, since it is contained in the space $B_1(A)$, cf. Sec. 1. Let $T = \{f_s\colon s \in S\}$. For each $x \in 2^\omega$ we define $B_x = \{f_{x|n}\colon n \in \omega\}$, a branch in T. For distinct $x, y \in 2^\omega$ the intersection $B_x \cap B_y$ is finite, hence $\mathscr{B}_A = \{B_x\colon x \in A\}$ is an almost disjoint family of subsets of the countable set T and we can consider K_A as a compactum associated in a standard way with this family, cf. [AU, Ch. V, § 1.3], [En, Exercise 3.6.I].

We shall verify that the space $C_T(K_A)$ is a Borel set of class $\leqslant 1+\alpha+1$ (notice that T is dense in K_A). We shall identify the family 2^T of all subsets of T with the Cantor set. The map which assigns to each point $x \in 2^\omega$ a subset $B_x \subset T$ is a homeomorphism of the Cantor set 2^ω onto the subspace $\mathscr{B} = \{B_x\colon x \in 2^\omega\}$ of the space 2^T and it maps the set A onto $\mathscr{B}_A$.

Let us fix an $n \in \omega$ and a finite subset $F \subset T$. We define a function $\varphi(F, n)\colon \mathscr{B}^n \to 2^T$ in the following way:

$$\varphi(F, n)(B_{x_1}, \ldots, B_{x_n}) = \Big(\bigcup_{i=1}^{n} B_{x_i}\Big) \triangle F$$

for $B_{x_1}, \ldots, B_{x_n} \in \mathscr{B}$, where $\triangle$ is the symmetric difference. The map $\varphi(F, n)$ is continuous. Let

$$H(F, n) = \{C \subset T\colon C = \Big(\bigcup_{i=1}^{n} B_{x_i}\Big) \triangle F \text{ for some } x_1, \ldots, x_n \in A\};$$

we have $\varphi(F, n)^{-1}(H(F, n)) = \mathscr{B}_A^n$. Hence, from a theorem of Saint-Raymond [S-R], [JR, Th. 5.9.12] it follows that $H(F, n)$ is a Borel set of additive class α and so is the set $H = \bigcup \{H(F, n)\colon n \in \omega,\ F \text{ is finite subset of } T\}$. Let $H' = \{T \setminus C\colon C \in H\}$. The function $C \to T \setminus C$ maps homeomorphically H onto H'. One can easily check that $H \cup H' = \{U \cap T\colon U \text{ is an open-and-closed subset of } K_A\}$.

Let Q denote the set of rational numbers. We show that for arbitrary bounded function $v\colon T \to R$

$$(*) \quad v \in C_T(K_A) \text{ if and only if } (\forall p, q \in Q\ \ p < q)\ \big(\exists r \in Q \cap (p, q)\ \ v^{-1}((r, \infty)) \in H \cup H'\big).$$

If the function $u\colon K_A \to R$ is continuous then the set $u(K_A) \subset R$ is compact and scattered, since the compactum K_A is scattered. Hence, for arbitrary $p, q \in Q$ with $p < q$ there exists $r \in \big(Q \cap (p, q)\big) \setminus u(K_A)$ and therefore the set $u^{-1}((r, \infty))$ is open-and-closed in K_A.

Since K_A is a Fréchet space (cf. Sec. 1), the function $v\colon T \to R$ can be extended to a continuous function on K_A if and only if for every sequence (a_n) of points of the set T convergent in K_A, there exists the limit $\lim_n v(a_n)$. Hence, for arbitrary bounded function $v \in R^T \setminus C_T(K_A)$ there exist a point $a \subset K_A$ and sequences (a_n), (b_n) of points of T converging to a such that $v(a_n) < p < q < v(b_n)$ for some $p, q \in Q$. Then for each number $r \in Q \cap (p, q)$ the set $v^{-1}((r, \infty))$ does not belong to $H \cup H'$, since for arbitrary open-and-closed subset $U \subset K_A$, either $a \in U$ and then U contains all but finitely many points of the sequence (a_n) or $a \notin U$ and U contains only finitely many points of the sequence (b_n).

For arbitrary number $r \in Q$ we define a map $\psi_r\colon R^T \to 2^T$ in the following way:

$$\psi_r(v) = v^{-1}((r, \infty)) \quad \text{for } v \in R^T.$$

The map ψ_r is of the first Baire class, hence the set $\psi_r^{-1}(H \cup H')$ is a Borel set of additive class $1+\alpha$, cf. [Ku, § 31.III]. Property $(*)$ yields a description of the space $C_T(K_A)$ by the formula:

$$C_T(K_A) = \bigcap \{\bigcup \{\psi_r^{-1}(H \cup H')\colon r \in Q \cap (p, q)\}\colon p, q \in Q,\ p < q\} \cap \bigcup_{m\in\omega} \{v \in R^T\colon v(T) \subset [-m, m]\},$$

which shows that the space $C_T(K_A)$ is a Borel subset of class $\leqslant 1+\alpha+1$ in R^T.

To complete the proof it is enough to show that for arbitrary countable dense subset D of the space K_A, the Borel class of the space $C_D(K_A)$ is at least α.

The set $A' = A \setminus \{x\colon f_x \in D\}$ is also of additive class α (D is countable). For arbitrary $x \in A'$ let $v_x\colon D \to R$ be the characteristic function of the set B_x (notice that the set T is contained in D, since every point $f_s \in T$ is isolated in K_A). It remains to observe that the map $x \to v_x$ is a homeomorphic embedding of the set A' onto a closed subset of the space $C_D(K_A)$. This completes the proof of Theorem 4.1.

4.3. Compact subsets of the space $B_1(2^\omega)$ of arbitrarily large Borel index η. Let A be a dense Borel subset of 2^ω of additive class $\alpha \geqslant 2$, such that every point of A is its point of condensation. (cf. [Ku, § 23.III]), and let K_A be the compact space defined in 4.2. We shall show that the space $B_1(2^\omega)$ contains a subspace L_A homeomorphic to K_A.

Let $Q' = \{x \in 2^\omega: \exists n \in \omega\, \forall k > n\ x(k) = 0\}$; Q' is homeomorphic to the space of rational numbers and $P = 2^\omega \setminus Q'$ is homeomorphic to the space of irrationals. Let $Q' = \{q_n: n \in \omega\}$ and let d be any metric in 2^ω. Let us choose a sequence of distinct points $p_n \in P$, $n \in \omega$, such that $\lim_n d(p_n, q_n) = 0$. Let $u: P \to A$ be a continuous injective map onto A, cf. [Si], [Ku, § 37.I]. We define an extension v of the map u over 2^ω by the formula

$$v(x) = \begin{cases} u(x) & \text{for } x \in P, \\ u(p_n) & \text{for } x = q_n; \end{cases}$$

the map v is of the first Baire class. To see this, we use a classical Baire's criterion; namely, we verify that for every closed subset $F \subset 2^\omega$, the restriction $v|F$ has a point of continuity relative to F, cf. [Ku, § 34.VII]. Given a compactum $F \subset 2^\omega$, either F is contained in Q' and then has an isolated point, which is a point of continuity of $v|F$, or there exists a point $p \in P \cap F$ and then the continuity of u and the definition of the sequence (p_n) implies that $v|F$ is continuous at p.

The mapping which assigns to each function $f \in K_A$ the composition $f \circ v$ is a homeomorphic embedding of the set K_A into the space $B_1(2^\omega)$: the functions f_s are continuous on A and hence each composition $g_s = f_s \circ v$ is of the first Baire class on 2^ω and, for every $x \in A$, the composition $g_x = f_x \circ v$ is a characteristic function of an at most two-element subset of 2^ω. The compactum $L_A = \{g_s: s \in S\} \cup \cup \{g_x: x \in A\} \cup \{0\} \subset B_1(2^\omega)$ is homeomorphic to K_A.

Let us end this section with the following observation, which will be used in the sequel:

(1) $$g_t(x) \leqslant g_{t|n}(x) \quad \text{for } n \in \omega,\ t \in 2^{n+1} \text{ and } x \in 2^\omega;$$

this is so, because $f_t \leqslant f_{t|n}$.

4.4. Some further refinements of the construction. Let L_A be the subspace of $B_1(2^\omega)$ defined in 4.3. For every $s \in S$ we choose a sequence of continuous functions $g_s^n: 2^\omega \to \{0, 1\}$ such that

(2) $$\lim_n g_s^n(x) = g_s(x) \quad \text{for every } x \in 2^\omega,$$

(3) $$g_t^k(x) \leqslant g_{t|n}^{k+1}(x) \quad \text{for every } k, n \in \omega,\ t \in 2^{n+1},\ x \in 2^\omega$$

(observe that (3) can be achieved in view of (1)).

Let us put $D = \{g_s^n: s \in S,\ n \in \omega\} \subset C(2^\omega)$ and let M_A be the closure of D in the space $\{0, 1\}^{2^\omega}$. Evidently, M_A contains the space L_A; we shall show that M_A is contained in $B_1(2^\omega)$. Let us consider

$$N_A = L_A \cup D \cup \{\chi_{\{p_n\}}, \chi_{\{q_n\}}: n \in \omega\} \subset B_1(2^\omega);$$

it is enough to prove that if the characteristic function χ_X of the set $X \subset 2^\omega$ is an accumulation point of the set D then $\chi_X \in N_A$.

If the set $v(X)$ is a singleton then $\chi_X = g_x$ for some $x \in A$, or $X = \{p_n\}$, or else $X = \{q_n\}$, for some $n \in \omega$. Hence $\chi_X \in N_A$. Assume that there exist points $x, y \in X$ such that $v(x) \neq v(y)$. Choose $n \in \omega$ so that for every $s \in 2^n$ either $v(x) \notin V_s$ or $v(y) \notin V_s$. By condition (2), there exists $k \in \omega$ such that for $i \geqslant k$, $g_s^i(x) = 0$ or $g_s^i(y) = 0$ for $s \in 2^n$. Now, it follows from (3) that

(4) $$\forall l \geqslant n+k\ \forall t \in 2^l\ \forall j \in \omega\ \ g_t^j(x) \leqslant g_{t|n}^{j+l-n}(x) = 0 \text{ or } g_t^k(y) \leqslant g_{t|n}^{j+l-n}(y) = 0.$$

Let us consider the neighbourhood $U = \{f \in \{0, 1\}^{2^\omega}: f(x) = f(y) = 1\}$ of the point χ_X. Property (4) yields: $(\forall l \geqslant n+k)(\forall t \in 2^l)(\forall j \in \omega)(g_t^j \notin U)$. Hence χ_X is an accumulation point of the set $\{g_s^i: s \in 2^l,\ l < n+k,\ i \in \omega\}$, and therefore $\chi_X = g_s$ for some $s \in 2^l$, $l < n+k$.

Summing up the construction in this section: we have obtained a compact subspace M_A of $B_1(2^\omega)$ such that continuous functions form a dense subset in M_A and M_A contains a topological copy of the compactum K_A defined in Sec. 4.2.

Remark. Since $\eta(M_A) = 2$ (see Theorem 2.3), the construction applied to any Borel set of class > 2 shows that the index η is not monotone.

§ 5. Open linear maps raising the Borel class of a linear space. Recall that a separable metrizable space is an absolute Borel set of class $\alpha > 1$ (or an absolute $F_{\sigma\delta}$-set) if it can be embedded in a compact metric space as a Borel set of class α (as a $F_{\sigma\delta}$-subset, respectively), cf. [Ku, § 35.IV].

We shall use the spaces M_A described in 4.4 to obtain the following example:

5.1. Example. For every ordinal number $\alpha < \omega_1$ there exist separable locally convex linear metric spaces E, F and an open continuous linear operator $\varphi: E \xrightarrow{\text{onto}} F$ such that E is an absolute Borel space of type $F_{\sigma\delta}$ and F is an absolute Borel set of class $\geqslant \alpha$.

Let us fix a Borel set $A \subset 2^\omega$ of additive class α, satisfying the conditions from Sec. 4.3, and let L_A and M_A be the Rosenthal compacta constructed in Sec. 4.3 and 4.4.

For the dense subset $T' = \{g_s: s \in S\} \subset L_A$, the space $C_{T'}(L_A)$ is homeomorphic to $C_T(K_A)$; hence it is an absolute Borel set of class $\geqslant \alpha$, cf. 4.2. Theorem 2.3 shows that the space $C_D(M_A)$ is an absolute Borel set of type $F_{\sigma\delta}$ and from Theorem 2.2 it follows that this is also the case for the space $C_{D \cup T'}(M_A)$. Let us put $E = C_{D \cup T'}(M_A)$ and $F = C_{T'}(L_A)$. The map $\varphi: E \to F$ defined by the formula $\varphi(f) = f|T'$ for $f \in C_{D \cup T'}(M_A)$ is a linear continuous surjection. We check that φ is open: for every function $f \in C_{D \cup T'}(M_A)$ the basic neighbourhoods of f have the form

$$V = \{g \in C_{D \cup T'}(M_A): |f(x) - g(x)| < \varepsilon \text{ for } x \in F\},$$

where $\varepsilon > 0$ and F is a finite subset of $D \cup T'$. By the Tietze–Urysohn Extension Theorem, $\varphi(V) = \{g \in C_{T'}(L_A): |f(x) - g(x)| < \varepsilon \text{ for } x \in F \cap T'\}$, hence $\varphi(V)$ is open in $C_{T'}(L_A)$.

The above construction can be modified to produce the following example.

5.2. Example. There exists an open continuous linear operator $\varphi: E\to F$ mapping a separable locally convex linear metric space E, which is an absolute $F_{\sigma\delta}$-set, onto a non-Borel locally convex linear metric space F.

Let X be a dense analytic non-Borel subset of 2^ω. Applying the construction from Sec. 4.2 to the set X one obtains an example of a separable Rosenthal compactum $K_X\subset B_1(X)$ for which the space $C_T(K_X)$ is non-Borel, because the set X embeds as a closed subset in $C_T(K_X)$ (cf. 6.3).

Now, we follow the construction from Sec. 4.3. Let u be a continuous map of the space P onto the set X. Let us extend the map u to a map $v: 2^\omega\to X$ of the first Baire class such as in 4.3. The set $L_X=\{f\circ v: f\in K_X\}$ is homeomorphic to K_X and consists of functions of the second Baire class on 2^ω. Let $T'=\{f_s\circ v: s\in S\}$. For every $s\in S$ we choose a sequence of continuous functions $g_s^n: 2^\omega\to\{0,1\}$, $n\in\omega$, which converges pointwise to $f_s\circ v$ (notice that $f_s\circ v$ is of the first Baire class, f_s being continuous), and let $D=\{g_s^n: s\in S,\ n\in\omega\}$. Let M_X be the closure of D in the product space $\{0,1\}^{2^\omega}$. Theorem 2.3 shows that the space $C_D(M_X)$ is an absolute $F_{\sigma\delta}$-set. Following the reasoning from the proof of Theorem 2.2 one can show that the space $C_{D\cup T'}(M_X)$ is also an absolute $F_{\sigma\delta}$-set. Now, we put $E=C_{D\cup T'}(M_X)$, $F=C_T(L_X)$ and define the operator $\varphi: E\to F$ as in the preceding example.

§ 6. Comments and remarks

6.1. Open maps do not raise the Borel index. A continuous image of a separable Rosenthal compactum need not be a Rosenthal compactum; however, open continuous functions map separable Rosenthal compacta onto Rosenthal compacta, cf. Godefroy [Go, Propositions 5 and 6]. We shall show that open continuous maps do not raise the index η of Rosenthal compacta.

Theorem. *Let K and L be separable Rosenthal compacta and let $\varphi: K\to L$ be an open continuous surjection. Then $\eta(L)\leqslant\eta(K)$.*

Proof. It is enough to restrict ourselves to the case when the compactum K is infinite and $\eta(K)<\omega_1$. Let us choose a countable dense subset $D\subset K$ such that $C_D(K)$ is a Borel set of class $\eta(K)$.

By Bourgain's result [Bo], in each Rosenthal compactum the set of points with a countable base of neighbourhoods is dense; let $E=\{e_n: n\in\omega\}$ be a dense subset of L consisting of such points and let $\{U_n^k\}_{k\in\omega}$ be a base of neighbourhoods of the point e_n in L.

Let us consider the subspace X of the space $C_D(K)$ consisting of functions which are constant on the fibres of φ. Using the fact that φ is open one can easily show that

$$X=\bigcap_{n\in\omega}\bigcap_{i\in\omega}\bigcup_{k\in\omega}\bigcap\{\{f\in C_D(K): |f(c)-f(d)|\leqslant i^{-1}\}: c,d\in D\cap\varphi^{-1}(U_n^k)\}.$$

Hence the set X is a $F_{\sigma\delta}$-subset of the space $C_D(K)$. From Theorem 2.1 it follows that X is a Borel set of class $\leqslant\eta(K)$ in R^D. It remains to observe that the set X is homeomorphic to the space $C_{\varphi(D)}(L)$.

6.2. Embedding of compacta K_A into the balls $B_{X^{**}}$, where X is a separable Banach space which does not contain l_1. Let us recall the theorem of Odell and Rosenthal [Ne, Th. 1.17] that for a separable Banach space X which does not contain an isomorphic copy of the space l_1, the unit ball $B_{X^{**}}$ of the second dual X^{**} endowed with the weak* topology is a separable Rosenthal compactum (notice that $\eta(B_{X^{**}})=2$, by theorem 2.3).

We shall show that the well-known factorization technique of Davis, Figiel, Johnson and Pełczyński [DFJP] allows one to associate with every compactum M_A constructed in Sec. 4.4 a separable Banach space E_A such that the unit ball $B_{E_A^{**}}$ in its second dual is a Rosenthal compactum containing a topological copy of M_A.

Let A be a Borel subset of the space 2^ω as in Sec. 4.3 and let M_A be the compactum constructed in 4.4 (in the sequel we adopt the notation introduced in Sec. 4). The set D is a dense subset of the space M_A consisting of continuous functions on 2^ω. Let $-D=\{-f: f\in D\}\subset C(2^\omega)$ and let W be the convex hull of the union $D\cup -D$ in the space $C(2^\omega)$. We denote by $B_{C(2^\omega)}$ the unit ball of the Banach space $C(2^\omega)$. Let $\|\cdot\|_n$ be the Minkowski gauge of $2^nW+2^{-n}B_{C(2^\omega)}$ in $C(2^\omega)$, where $n=1,2,\ldots$ For every function $f\in C(2^\omega)$ we define

$$|||f|||=\Big(\sum_{n=1}^{\infty}\|f\|_n^2\Big)^{1/2}.$$

The space $E_A=\{f\in C(2^\omega): |||f|||<\infty\}$ equipped with the norm $|||\cdot|||$ is a separable Banach space, cf. [DFJP, Lemma 1].

Let $i: W\to C(2^\omega)^{**}$ be the embedding defined by the formula:

$$(*)\qquad i(f)(\mu)=\mu(f)=\int f\,d\mu\quad\text{for } f\in W,\ \mu\in C(2^\omega)^*.$$

We prove that the set $i(W)$ is relatively sequentially compact in the dual space $(C(2^\omega)^{**}, w^*)$ equipped with the weak* topology.

Let Z be the closure of the set W in the product R^{2^ω}; [BFT, Theorem 5E] yields that the set Z is a compact subset of the space $B_1(2^\omega)$. Using the formula $(*)$ we can extend the map i to an embedding $i: Z\to(C(2^\omega)^{**}, w^*)$. From the fact that Z is a Fréchet space (cf. Sec. 1) and from the Lebesgue Dominated Convergence Theorem it follows that the map i is a homeomorphic embedding of the set Z into the space $(C(2^\omega)^{**}, w^*)$. Since each sequence of points of the set W has a subsequence convergent in the space Z, every sequence of points of $i(W)$ has a subsequence which converges to some point of $i(Z)\subset C(2^\omega)^{**}$.

Now, [DFJP, Lemma 1 (xii)] yields that the canonical embedding $E_A\to E_A^{**}$ maps the unit ball B_{E_A} onto a relatively sequentially compact subset of (E_A^{**}, w^*). Hence the space l_1 does not embed isomorphically into E_A, cf. [Ne, Theorem 1.19].

Let $j: E_A\to C(2^\omega)$ be an inclusion. Then the map $j^{**}: E_A^{**}\to C(2^\omega)^{**}$ is injective and maps homeomorphically the ball $(B_{E_A^{**}}, w^*)$ onto the weak* closure of the set $i(B_{E_A})$ in the space $C(2^\omega)^{**}$. It can be easily checked that the set W is contained in the ball B_{E_A}; hence, the Rosenthal compactum $(B_{E_A^{**}}, w^*)$ contains a topological copy of the set Z and therefore also copies of the spaces K_A and M_A.

6.3. We do not know if the property $\eta(K) < \omega_1$ characterizes separable Rosenthal compacta which can be embedded topologically in $B_1(2^\omega)$. As we have mentioned, there are separable Rosenthal compacta, which do not embed in $B_1(2^\omega)$, cf. [Po], but all known examples of this kind have the index $\eta = \omega_1$.

6.4. THEOREM 3.1 extends a result of Godefroy [Go, Proposition 10], who proved that if the function spaces $C_w(K)$, $C_w(L)$ of separable compact spaces K, L are homeomorphic and K is a Rosenthal compactum, then so is L.

6.5. The assumption of compactness of the space L in Theorem 3.5 is essential, since, as was shown by Gul'ko and Khmyleva [GKh, Theorem 4], the spaces $C_p([0, 1])$ and $C_p(R)$ are homeomorphic.

6.6. Theorems 3.1 and 3.5 show that if A is a Borel set in 2^ω of additive class $\alpha \geqslant 2$ and B is a Borel subset of 2^ω of the additive class $> 2+\alpha+1$ then the spaces $C_p(K_A)$, $C_p(K_B)$ and also the spaces $C_w(K_A)$, $C_w(K_B)$ are not homeomorphic, where K_A and K_B are compacta described in Sec. 4.2. Let us notice that, by a result of Aharoni and Lindenstrauss [AL], for all uncountable Borel subsets $A \subset 2^\omega$ the Banach spaces $C(K_A)$ are Lipschitz homeomorphic.

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