Index pairs and the fixed point index for semidynamical systems with discrete time

by

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Abstract. We show that index pairs of an isolated invariant set of a dynamical system with discrete time exist but, contrarily to the continuous case, they do not provide a homotopic invariant of the isolated invariant set. Nevertheless the relationship between the fixed point index and index pairs extends from the continuous case to the discrete case.

Introduction. Since the publication of the paper on isolated invariant sets by C. Conley and R. Easton [6] a series of articles appeared in which various kinds of closely related topological indices of isolated invariant sets were constructed (the cohomological index, the homotopy index, the Morse index). The results of these articles, usually designated as the Conley index theory because of the significant role played by C. Conley in its development, has become an important tool in the qualitative study of differential equations (see [1], [7], [8], [9] for instance). The prime root of this theory is the famous Ważewski Retract Theorem (see [33] or [17, Ch. X, §§ 2–3]). The foundations of the theory, in a locally compact setting, were established in the papers of R. Churchill [4] and J. Montgomery [21] (the cohomological index), in the book [5] of C. Conley (the homotopy index and the Morse index) and in the paper of H. Kurland [19] (the Morse index). Later, K. Rybakowski [27], [28] generalized the theory to the case of a non-locally compact space, which admitted direct applications in partial differential equations and functional differential equations. The more recent papers concerning the development of the Conley theory are [30] and [15]. The paper [24] contains some work towards generalizing the Conley theory to the case of non-compact invariant sets.

As pointed out by K. Rybakowski in [27, § 4, Remarks] there is a series of similarities between the homotopy index and the fixed point index (for the definition of the fixed point index see [10], [11, Ch. VII, § 5] or [16]). The fixed point index is defined for an isolated set of fixed points. It is used to establish the existence of fixed points in a similar way as the Conley index is used to establish the existence of invariant sets. Like the homotopy index, the fixed point index satisfies the additivity property and the homotopy property.

The aim of this paper is to show that the above similarities are not only formal. First we show that the notion of index pair, which plays the fundamental role in the construction of the homotopy index, can be extended to the case of a discrete time semidynamical system given by a continuous map and the theorem on existence of index pairs can be proved. Recently J. Robbin and D. Salamon [26] obtained independently a similar result for diffeomorphisms. We give examples that the homotopy type $[M/N, \{N\}]$ of an index pair (M, N) depends not only on the invariant set inside M but also on the pair (M, N). This shows that there is no direct way of carrying over the Conley theory from the continuous to the discrete case, though indirect generalization are possible (see Robbin and Salamon [25] and the author's forthcoming paper [26]). The discrete version of the Conley theory is interesting in applications to difference equations (see [5, § IV, 8.2]).

We show that the formula expressing the fixed point index of the translation operator of the flow in terms of the Euler characteristic of the index pair – known in the case of special index pairs called isolating blocks (see [22], [23], [32]) – does have a counterpart in the discrete case. Namely, we will show that the fixed point index of a given mapping in a neighborhood of an isolated invariant set K equals the Lefschetz number of a certain map associated with every index pair of K. This enables us to generalize the results concerning the fixed point index in [21], [22] and [28] to the case of semiflows with compact attractor on arbitrary ANR's.

The results of this paper are closely related to some similar results of Fenske and Peitgen in [12] and of Fenske in [13] and in some sense generalize those results. The formulae on fixed point index in [12] and [13] concern ejective fixed points, whereas analogous formulae in this paper apply to isolated invariant sets, which allow simultaneous attraction and repulsion and include ejective fixed points as a very special case.

Similarly to [12] we consider in this paper a broad class of maps of compact attraction (see the definition below) in order to ensure applications in non-locally compact spaces.

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1. Preliminaries. The set of real numbers will be denoted by $\boldsymbol{R}$ and the set of natural numbers by N. $\boldsymbol{Z}^-$ and $\boldsymbol{Z}^+$ will stand for non-positive and non-negative integers respectively. (X, ϱ) will denote a fixed metric space. For a subset A of X we will use the notation $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ for the interior, the closure and the boundary of A respectively. If f maps an open subset U of X into X, then $\operatorname{Fix} f$ will stand for the set of fixed points of f. The homotopy type of a pointed space (X, x_0) will be denoted by $[X, x_0]$. The singular homology functor with rational coefficients on the category of topological pairs will be denoted by H_*. $H_*(f)$ for a map $f\colon (X, A) \to (Y, B)$ will be briefly denoted by f_*. Assume Y is another metric space. The map $f\colon X \to Y$ is said to be *compact* iff there is a compact set $K \subseteq Y$ such that $f(X) \subseteq K$. It is said to be *locally compact* iff for every point $x \in X$ there exists a neighborhood U of x such that $f|_U$ is compact. The map $f\colon X \to X$ is *of compact attraction* iff it is locally compact and there exists a compact $A \subseteq X$ such that for every $x \in X$

$$\operatorname{cl}\{f^n(x) \mid n \in \boldsymbol{Z}^+\} \cap A \neq \emptyset .$$

We have the following easy to prove

Remark 1. Assume $f\colon X \to X$ is of compact attraction. Then $\operatorname{Fix} f$ is compact. If $Y \subseteq X$ is closed and $f(Y) \subseteq Y$, then $f|_Y\colon Y \to Y$ is also of compact attraction.

Assume $\varphi = \{\varphi_j\}$ is an endomorphism of a graded vector space E over the field of rational numbers and $N(\varphi) := \bigcup \{\varphi^{-n}(0) \mid n = 1, 2, \ldots\}$. Recall (see [12]) that φ is a *Leray endomorphism* iff $E' := E/N(\varphi)$ is of finite type. For such a φ we define its trace $\operatorname{tr}\varphi$ by $\operatorname{tr}\varphi := \operatorname{tr}\varphi'$, where $\varphi'\colon E' \to E'$ is an induced map. We define also the *Lefschetz number of* φ by

$$\Lambda(\varphi) := \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}\varphi_n .$$

If $f\colon (X, A) \to (X, A)$ is a continuous map of a pair (X, A) of topological spaces into itself such that f_* is a Leray endomorphism, then f is said to be a *Lefschetz map* and in such a case the *Lefschetz number of* f is given by $\Lambda(f) := \Lambda(f_*)$.

Remark 2 (see [3], (1.3)). If φ and ψ are two endomorphisms of graded vector spaces E and F respectively and there exists morphisms $h\colon E \to F$ and $g\colon F \to E$ such that $\varphi = gh$ and $\psi = hg$, then φ is a Leray endomorphism iff ψ is a Leray endomorphism and in that case $\Lambda(\varphi) = \Lambda(\psi)$. Note that the assumptions of this remark are in particular satisfied if there exists an isomorphism $g\colon F \to E$ such that $\varphi g = g\psi$, i.e. if φ and ψ are conjugate.

For a map $f\colon (X, A) \to (X, A)$, $f_X\colon X \to X$ and $f_A\colon A \to A$ will denote the restrictions of f to $(X, \emptyset)$ and $(A, \emptyset)$ respectively. The following remark follows directly from Remark 1, Lemma (4.1) in [12] and Theorem 2.1 in [14].

Remark 3. Assume X and A are metric ANR's and A is a closed subset of X. If $f\colon (X, A) \to (X, A)$ is a map of compact attraction then f, f_X, f_A are Lefschetz maps and

$$\Lambda(f) = \Lambda(f_X) - \Lambda(f_A) .$$

The notion of the Lefschetz number is strongly related to the notion of the fixed point index. The fixed point index is an integer-valued function Ind defined on a certain subclass of all admissible maps, i.e. continuous maps $f\colon U \to X$, where U is open in X and $\operatorname{Fix} f$ is compact (see [10] for the detailed axiomatic definition of the fixed point index). We will use the fixed point index defined on the class of all locally compact admissible maps $f\colon U \to X$, where X is a metric ANR. The existence of such an index is proved in [16], Theorem (12.1).

Assume U is open in X and $f\colon U\to X$ is locally compact. We will say that an open set $V\subseteq X$ *isolates the fixed points of f* iff $\operatorname{Fix} f\cap V$ is compact. In such a situation $f|_V$ is a locally compact admissible map, thus $\operatorname{Ind}(f|_V)$ is defined. It will be convenient to introduce the notation

$$\operatorname{ind}(f, V) := \operatorname{Ind}(f|_V).$$

We give below three properties of the fixed point index which will be useful for us. For the list of all properties of the fixed point index of locally compact admissible maps see [16]. (X and X' denote metric ANR's.)

(1) (additivity). Assume $f\colon U\to X$ is a locally compact map, V_1, V_2 and V isolate the fixed points of f, $V\cap \operatorname{Fix} f\subseteq V_1\cup V_2\subseteq V$, $V_1\cap V_2\cap \operatorname{Fix} f=\emptyset$. Then

$$\operatorname{ind}(f, V) = \operatorname{ind}(f, V_1)+\operatorname{ind}(f, V_2).$$

(2) (commutativity). Assume $U\subseteq X$ and $U'\subseteq X'$ are open and $f\colon U\to X'$ and $g\colon U'\to X$ are continuous. If f is locally compact and gf is admissible then both fg and gf are admissible, locally compact and

$$\operatorname{ind}(fg, g^{-1}(U)) = \operatorname{Ind}(fg) = \operatorname{Ind}(gf) = \operatorname{ind}(gf, f^{-1}(U')).$$

(3) (normalization). The normalization property in the form quoted here is proved in [12], (3.5). If $f\colon X\to X$ is a map of compact attraction, then f_* is a Leray endomorphism and

$$\operatorname{ind}(f, X) = \Lambda(f).$$

We note that the following remark is a straightforward consequence of (1).

Remark 4. Assume $f\colon U\to X$ is locally compact and $V\cap \operatorname{Fix} f=\emptyset$ for some open V. Then $\operatorname{ind}(f, V)=0$. If V_1 and V_2 isolate the fixed points of f and $V_1\cap\operatorname{Fix} f = V_2\cap \operatorname{Fix} f$ then

$$\operatorname{ind}(f, V_1) = \operatorname{ind}(f, V).$$

2. Index pairs for discrete dynamical system. Let (X, ϱ) denote a fixed metric space. We will consider a fixed continuous map $f\colon X\to X$.

The sequence $\sigma\colon \mathbf{Z}^-\to X$ will be called the *left solution to f through x* iff $f(\sigma(i-1))=\sigma(i)$ for every $i\in\mathbf{Z}^-$ and $\sigma(0)=x$.

For a given set $N\subseteq X$ the sets

$$\operatorname{Inv}^+ N := \{x\in X \mid \forall i\in \mathbf{Z}^+\ f^i(x)\in N\},$$
$$\operatorname{Inv}^- N := \{x\in X \mid \exists\sigma\colon \mathbf{Z}^-\to N \text{ a left solution to } f \text{ through } x\},$$
$$\operatorname{Inv} N := \operatorname{Inv}^- N\cap \operatorname{Inv}^+ N$$

will be called the *positively invariant*, the *negatively invariant* and the *invariant part of N (relative to f)* respectively. We will say that N is *positively invariant, negatively invariant* or *invariant* iff $N=\operatorname{Inv}^+N$, $N=\operatorname{Inv}^-N$, $N=\operatorname{Inv}N$ respectively.

The following three remarks are easy to prove.

Remark 5. $\operatorname{Inv}N$ is invariant, i.e. $\operatorname{Inv}(\operatorname{Inv}N)=\operatorname{Inv}N$. It is the largest invariant subset of N.

Remark 6. If N is closed then Inv^+N is closed.

We will say that a closed set $N\subseteq X$ satisfies the *Rybakowski condition* iff for every pair of sequences $\{x_n\}_{n=1}^\infty\subseteq N$ and $\{m_n\}_{n=1}^\infty\subseteq \mathbf{Z}^+$ such that

$$\{f^i(x_n) \mid i=0, 1, \dots, m_n\}\subseteq N$$

and $m_n\to\infty$, the sequence $\{f^{m_n}(x_n)\}_{n=1}^\infty$ is precompact, i.e. relatively compact.

Remark 7. If N satisfies the Rybakowski condition then Inv^-N and $\operatorname{Inv}N$ are compact. If $\{x_n\}$ and $\{m_n\}$ satisfy the assumptions of the Rybakowski condition then every cluster point of $\{f^{m_n}(x_n)\}$ belongs to Inv^-N.

Assume K is a closed invariant set. If K is the largest invariant set in some its neighborhood N, then K is called an *isolated invariant set* and N is said to *isolate K*. If N is closed, it is called an *isolating neighborhood for K*. If additionally N satisfies the Rybakowski condition then N is said to be an *isolating neighborhood of Rybakowski type*. If an isolated invariant set admits an isolating neighborhood of Rybakowski type, then it is called the *isolated invariant set of Rybakowski type*. Notice that if K is an isolated invariant set then $\operatorname{Fix} F\cap K$ is an isolated set of fixed points.

We have the following

THEOREM 1. *If $f\colon X\to X$ is a locally compact map then every compact isolated invariant set of f is of Rybakowski type.*

Proof. Let K be a compact isolated invariant set of f and M be an isolating neighborhood for K. For every $p\in K$ choose a closed neighborhood $W_p\subseteq M$ of p such that f restricted to W_p is compact. Using compactness of K find a finite collection P of points in K such that $K\subseteq W:=\bigcup\{W_p \mid p\in P\}\subseteq M$. Then W is obviously an isolating neighborhood for K and $f(W)\subseteq K_1$ for some compact $K_1\subseteq X$. Assume $\{x_n\}_{n=1}^\infty\subseteq W$ and $\{m_n\}_{n=1}^\infty$ satisfy the assumptions of the Rybakowski condition. Then, for almost all n we have $f^{m_n}(x_n)=f(f^{m_n-1}(x_n))\in K_1$. This shows that the sequence $\{f^{m_r}(x_n)\}$ is precompact, which finishes the proof. ∎

DEFINITION 1. We will say that the function $L\colon E\to[0,\infty)$ *grows along trajectories* of f on E iff

$$L(x)>0,\ f(x)\in E\Rightarrow L(f(x))>L(x). \tag{4}$$

Similarly we will say that $L\colon E\to[0,\infty]$ *decreases along trajectories* of f on E iff

$$L(x)>0,\ f(x)\in E\Rightarrow L(f(x))<L(x).$$

DEFINITION 2. Assume K is an isolated invariant set, U is an isolating neighborhood and $\varphi, \gamma\colon U\to[0,\infty]$. We will say that the pair (φ,γ) is a *Lyapunov pair for K* iff

(5) φ decreases and γ grows along trajectories of f on U,

$$K\subseteq \varphi^{-1}(0)\cap\gamma^{-1}(0), \tag{6}$$

(7) for every neighborhood W of K there exists an $\varepsilon > 0$ such that the set

$$H(\varepsilon, \varphi, \gamma) := \{x \in U \mid \varphi(x) < \varepsilon, \gamma(x) < \varepsilon\}$$

satisfies the condition

$$\operatorname{cl} H(\varepsilon, \varphi, \gamma) \subseteq W.$$

We will say that (φ, γ) is a *continuous Lyapunov pair for K* iff (φ, γ) is a Lyapunov pair for K and both φ an γ are continuous.

We have the following

THEOREM 2. *For every non-empty isolated invariant set K of Rybakowski type there exists an open neighborhood V of K, which admits a continuous Lyapunov pair for K.*

The proof of the above theorem is modelled on the proof of a similar theorem on the existence of isolating blocks for flows by K. Rybakowski (see [27], Theorem 2.1). However, some changes are necessary to cope with the discontinuity caused by the discrete character of the semidynamical system f. For this reason we include the proof but postpone it to the last section.

DEFINITION 3. Assume K is an isolated invariant set and (M, N) is a pair of closed subsets of X such that $N \subseteq M$. The pair (M, N) will be called an *index pair for K* or briefly an *index pair* iff the following conditions are satisfied:

$$x \in N,\ f(x) \in M \Rightarrow f(x) \in N, \tag{8}$$

$$x \in M,\ f(x) \notin M \Rightarrow x \in N, \tag{9}$$

$$K := \operatorname{Inv}(M \setminus N) \subseteq \operatorname{int}(M \setminus N). \tag{10}$$

We will say that the pair (M, N) is a *weak index pair for K* iff it satisfies (8), (9) and

$$\operatorname{Fix} f \cap \operatorname{cl}(M \setminus N) \subseteq \operatorname{int}(M \setminus N). \tag{10'}$$

We note the following obvious

Remark 8. Every index pair is also a weak index pair.

We will say that the index pair (weak index pair) (M, N) is *regular* iff the following conditions are satisfied:

(11) there exists a set U open in M such that $N \subseteq U$ and $f(U \setminus N) \subseteq N$,

$$\operatorname{cl}(f(N) \setminus M) \cap \operatorname{cl}(M \setminus N) = \emptyset. \tag{12}$$

Similarly to the case of a semiflow we have the following

THEOREM 3. *Assume that K is a non-empty isolated invariant set which admits a continuous Lyapunov pair on some neighborhood of K. Then for every neighborhood U of K there exists a regular index pair (M, N) for K such that $M \subseteq U$.*

The proof of Theorem 3 presented below depends essentially on Theorem 2. A much shorter proof of Theorem 3 in the smooth case can be found in [26] and in the locally compact case in [25].

Proof. Let $(\varphi, \gamma)\colon V \to [0, \infty] \times [0, \infty]$ be the continuous Lyapunov pair for K. Put $W := U \cap V \cap f^{-1}(V) \cap f^{-2}(V)$. Find $\varepsilon > 0$ such that $\operatorname{cl} H(2\varepsilon, \varphi, \gamma) \subseteq W$ and denote $M := \{x \in V \mid \varphi(x) \leqslant \varepsilon, \gamma(x) \leqslant \varepsilon\}$, $N := \{x \in M \mid \gamma(f(x)) \geqslant \varepsilon\}$, $H := H(2\varepsilon, \varphi, \gamma)$. We will show that (M, N) is a regular index pair for K. It is easy to see that M, N are closed.

Assume $x \in N$ and $f(x) \in M$. Then $\gamma(f(x)) = \varepsilon$ and we get from (4) that $\gamma(f(f(x))) > \gamma(f(x)) \geqslant \varepsilon$, i.e. $f(x) \in N$. Assume in turn that $x \in M$ and $f(x) \notin M$. Then $\varphi(f(x)) < \varphi(x) \leqslant \varepsilon$, thus it must be $\gamma(f(x)) > \varepsilon$ wich means $x \in N$ and (8), (9) are proved.

We have $K = \operatorname{Inv} K \subseteq \operatorname{Inv}(M \setminus N) \subseteq \operatorname{Inv}(V) = K$ and we get from (6) that $K \subseteq \{x \in V \mid \gamma(x) < \varepsilon, \varphi(x) < \varepsilon, \gamma(f(x)) < \varepsilon\} \subseteq \operatorname{int}(M \setminus N)$, which proves (10).

Put $U_0 := M \cap \{x \in W \mid \gamma(f^2(x)) > \varepsilon\}$. Then U_0 is open in M. Take $x \in N$. Then $\gamma(f^2(x)) > \gamma(f(x)) \geqslant \varepsilon$, hence $x \in U_0$, which shows that $N \subseteq U_0$. Assume $x \in U_0 \setminus N$. Let $y = f(x)$. Then $\varphi(y) \leqslant \varphi(x) \leqslant \varepsilon$ and $\gamma(y) = \gamma(f(x)) < \varepsilon$, thus $y \in M$. We have also $\gamma(f(y)) = \gamma(f^2(x)) > \varepsilon$, hence $y \in N$. This proves (11).

In order to prove (12) assume it is not true. Then there exists

$$y \in \operatorname{cl}(M \setminus N) \cap \operatorname{cl}(f(N) \setminus M)$$

and consequently $\gamma(f(y)) \leqslant \varepsilon$ and $\gamma(y) \geqslant \varepsilon$. Thus $\gamma(f(y)) > \gamma(y) \geqslant \varepsilon$, a contradiction. ■

Since $(\emptyset, \emptyset)$ is obviously a regular index pair for $\emptyset$, from the above theorem and Theorems 1 and 2 we get the following corollaries:

COROLLARY 1. *For every isolated invariant set K of Rybakowski type and every neighborhood W of K there exists a regular index pair (M, N) such that $M \subseteq W$ and $K = \operatorname{Inv} M$.*

COROLLARY 2. *Every compact isolated invariant set K of a locally compact map (in particular of a map of compact attraction) admits regular index pairs arbitrarily close to K.*

3. Two examples.

The theorem on the existence of index pairs for isolated invariant sets of flows forms the basis for the construction of the theory of homotopic index for isolated invariant sets (cf. [5], [27]). One of the essential theorems of the theory asserts that the homotopy type of the pointed space $(M/N, \{N\})$ of any index pair (M, N) depends only on the isolated invariant sets inside M. The following examples show that the similar theorem in case of a discrete dynamical system is not true.

EXAMPLE 1. Let $X = \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ and $f\colon X \to X$ be given by

$$f(x) := \begin{cases} x+1 & \text{for } x \in \mathbf{R}, \\ x & \text{otherwise}. \end{cases}$$

Then f is continuous. For every $n \in N$ put $N_n := [n, n+1/2]$ and

$$M_n := \bigcup \{N_i \mid i = 0, 1, 2, \dots, n\}.$$

One can easily verify that for every $n \in N$ (M_n, N_n) is an index pair for f such that $M_n \setminus N_n$ isolates the empty set. On the other hand $[M_n/N_n, \{N_n\}]$ equals the homotopy type of a pointed set of n elements.

EXAMPLE 2 (Smale's Horseshoe). Assume $X = S^2 = R^2 \cup \{\infty\}$, $I := [0, 1]$, $Q := I \times I \subseteq X$. Put

$$C_n := \{\alpha_1/5 + \dots + \alpha_n/5^n + \alpha/5^{n+1} \mid \alpha_i \in \{1, 3\},\ i = 1, 2, \dots, n,\ \alpha \in I\} \subseteq I,$$

$$N_n := I \times \operatorname{cl}(I \setminus C_n).$$

Assume $f\colon X \to X$ is a continuous map such that f maps two rectangles R_0 and R_1 linearly onto rectangles S_0 and S_1, as indicated in Fig. 1. Assume also that f maps $Q \setminus (R_0 \cup R_1)$ into $X \setminus Q$ and $X \setminus Q$ into $X \setminus (S_0 \cup S_1)$. (Compare [31] or [18].) Then

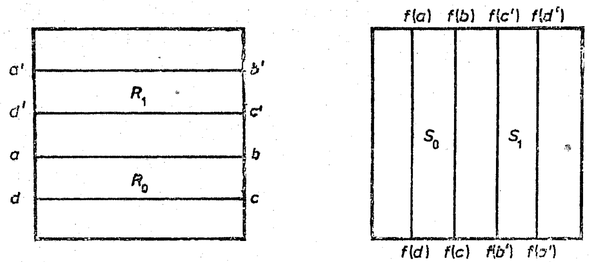

Fig. 1

$K := \operatorname{Inv} Q \subseteq \bigcap \{C_n \times C_n \mid n = 1, 2, \dots\}$. One can easily verify that for all $n \in N$ (Q, N_n) is an index pair for K and that $[Q/N_n, \{N_n\}]$ equals the homotopy type of the wedge sum of 2^n copies of a pointed circle.

4. The fixed point index and the index maps. By the above examples we cannot expect that it is possible to extend directly the Conley index theory to dynamical systems with discrete time. Nevertheless some analogues of Conley index in the discrete case have been constructed recently (see [25] and [26]). The aim of this paper and this section, however, is to show that index pairs carry information about the fixed point index.

We begin with the following

LEMMA 1. *Assume (M, N) is a weak regular index pair for f. Then f maps the pair (M, N) into the pair $(M \cup f(N), N \cup f(N))$ and the inclusion $i_{M,N}\colon (M, N) \to (M \cup f(N), N \cup f(N))$ induces an isomorphism in homology.*

Proof. Assume $y \in f(M)$, i.e. $y = f(x)$ for some $x \in M$. By (9) $f(x) \notin M$ implies $x \in N$, hence $y \in M \cup f(N)$, which proves the first part of the lemma. The second part follows easily from (8), (12) and from the excision property of homology theory (see [11], III. 7.4). ∎

Let $f_{M,N}$ denote the mapping f considered as a mapping of the pair (M, N) into the pair $(M \cup f(N), N \cup f(N))$. The above lemma enables us to define an endomorphism $I_{M,N}\colon H_*(M, N) \to H_*(M, N)$ by $I_{M,N} := (i_{M,N})_*^{-1} \circ (f_{M,N})_*$. We will call this map the *index map of the index pair* (M, N).

The main result of this paper is the following

THEOREM 4. *Assume X is a metric* ANR *and $f\colon X \to X$ is a map of compact attraction. If (M, N) is a regular weak index pair for f and M, N are* ANR's *then $I_{M,N}$ is a Leray endomorphisms and*

$$\operatorname{ind}(f, \operatorname{int}(M \setminus N)) = \Lambda(I_{M,N}).$$

A similar formula can be found in [12], Corollary (4.4). It concerns, however, a very special case of an ejective fixed point.

COROLLARY 3. *Under the assumptions of the above theorem, if additionally $f_{M,N}$ is homotopic to $i_{M,N}$, then the relative Euler characteristic of the pair (M, N) exists and*

$$\operatorname{ind}(f, \operatorname{int}(M \setminus N)) = \chi(M, N).$$

Note that in case of f being a translation operator of a semiflow one can always find index pairs (M, N) for which $f_{M,N}$ is homotopic to $i_{M,N}$. We omit details, because they are based on the technique of isolating blocks not discussed here.

Corollary 3 generalizes Theorem 4.1 in [22], Theorem 2.1 in [23] and Theorem 4.4 in [32], where analogous results are proved, in the locally compact setting, for special index pairs called isolating blocks. Similar formulae, concerning the very special case of repulsing isolated invariant sets can be found in papers of Fenske and Peitgen [12], Proposition [5.3] and Fenske [13], Theorem 1. In the smooth setting the counterpart of Corollary 3 for Hopf index is the main result of Mc Cord [20]. Also the result of Rybakowski for gradient vector fields ([29] Theorem 1), which expresses the Brouwer degree of the field in terms of Betti numbers of Conley index can be viewed, by [32] Theorem 5.1, as a very special case of Corollary 3.

In course of the proof of Theorem 4, we will need the following lemma, which is closely related to Lemma (4.3) in [12].

LEMMA 2. *Assume X, A are metric* ANR's *and A is a closed subset of X. If $f\colon (X, A) \to (X, A)$ is a map of compact attraction, such that there exists some open neighborhood U of A mapped by f into A, then*

$$\operatorname{ind}(f, X \setminus A) = \Lambda(f_X) - \Lambda(f_A) = \Lambda(f).$$

Proof. Put $V := X \setminus A$, $W := X \setminus U$. We have then

$$\operatorname{Fix} f \cap V = \operatorname{Fix} f \cap W, \quad \operatorname{Fix} f \cap U = \operatorname{Fix} f \cap A, \quad \operatorname{Fix} f \cap U \cap V = \emptyset.$$

This shows that U and V isolate the fixed points of f and, by (1) and (3), $\Lambda(f_X) = \operatorname{ind}(f, X) = \operatorname{ind}(f, U)+\operatorname{ind}(f, V)$. Let $i\colon A \to U$ be the inclusion. Since $f(U) \subseteq A$, we get from (2) and (3)

$$\operatorname{ind}(f, U) = \operatorname{ind}(i \circ f|_U, U) = \operatorname{ind}(f|_U \circ i, U \cap A) = \operatorname{ind}(f|_A, A) = \Lambda(f_A).$$

Thus $\operatorname{ind}(f, X \setminus A) = \Lambda(f_X) - \Lambda(f_A)$. The remaining equality follows from Remark 3. ■

5. Proof of Theorem 4. Let $I := [0, 1]$. Put

$$Y := M \times \{0\} \cup N \times I \cup X \times \{1\}.$$

Let U be an open neighborhood of N in M such that $f(U \setminus N) \subseteq N$. Let $W := (M \setminus U) \times \{0\}$ and $Z := N \times I \cup X \times \{1\}$. W and Z are closed and disjoint subsets of Y. Let $\alpha\colon Y \to I$ be continuous and such that $\alpha|_W = 0$, $\alpha|_Z = 1$. We will show that the mapping g given by $g(x, t) := (f(x), \alpha(x, t))$ maps Y into Y.

Indeed, if $(x, t) \in W$ then $x \in M \setminus N$, thus by (9) $f(x) \in M$, i.e. $g(x, t) = (f(x), 0) \in M \times \{0\} \subseteq Y$. If $(x, t) \in Z$ then $g(x, t) = (f(x), 1) \in X \times \{1\} \subseteq Y$. If $(x, t) \in Y \setminus (W \cup Z)$ then $x \in U \setminus N$, thus $f(x) \in N$, i.e. $g(x, t) = (f(x), \alpha(x, t)) \in N \times I \subseteq Y$.

In fact, we proved even more, namely that g is a mapping of the pair (Y, Z) into itself. We will prove that g is a map of compact attraction. Let $(x, t) \in Y$. There exists a neighborhood V of x and a compact $K \subseteq X$, such that $f(V) \subseteq K$. Then $g(V \times I) \subseteq K \times I$, which shows that g is locally compact. Assume A is a compact set in X such that for every $x \in X$ $\operatorname{cl}\{f^n(x)|\ n \in N\} \cap A \neq \emptyset$. Put $A' := A \times I$ and take $(x, t) \in Y$. Then $g^n(x, t) = (f^n(x), s_n)$ for some $s_n \in I$. Let

$$y \in \operatorname{cl}\{f^n(x)|\ n \in N\} \cap A.$$

Choose a sequence $\{n_k\}_{k=1}^{\infty} \subseteq N$ such that $f^{n_k}(x) \to y$ and a subsequence $\{r_k\} \subseteq \{s_{n_k}\}$ such that $r_k \to r$ for some $r \in I$. Then $(y, r) \in \operatorname{cl}\{g^n(x, t)|\ n \in N\} \cap A' \cap Y$. Since $A' \cap Y$ is compact, we proved that g is of compact attraction.

Put $Q := N \times I$, $P := M \times \{0\} \cup Q$, P and Q are closed subsets of Y. Q is an ANR by Theorem (7.2) in [2]. Since $M \times \{0\}$, $N \times \{0\}$ are ANR's and $M \times \{0\} \cap Q = N \times \{0\}$, we infer from Theorem (6.1) in [2] that P is also an ANR. An easy computation shows that (P, Q) is a regular index pair for g. Put $R := \operatorname{int}(M \setminus N)$, $R' := \operatorname{int}(M \setminus N) \times \{0\}$, $\beta\colon R \ni x \to (x, 0) \in R'$, $V := f^{-1}(R)$, $V' := V \times \{0\}$. Then

$$\operatorname{Fix} f \cap R = \operatorname{Fix} f \cap V, \quad \operatorname{Fix} g \cap R' = \operatorname{Fix} g \cap V'$$

and Remark 4 together with (2) applied to $g \circ \beta|_V$ and $\beta|_{V^{-1}}$ show that

(13) $$\operatorname{ind}(f, \operatorname{int}(M \setminus N)) = \operatorname{ind}(g, \operatorname{int}(P \setminus Q)).$$

Observe that, by Theorem (6.1) in [2], Z is an ANR, because $N \times I$, $X \times \{1\}$ and $(N \times I) \cap (X \times \{1\}) = N \times \{1\}$ are ANR's. Similarly Y is an ANR, because

$Y = Z \cup (M \times \{0\})$, $Z \cap (M \times \{0\}) = N \times \{0\}$. Taking $U' := (U \times \{0\}) \cup Z$, an easy computation shows that U' is an open neighborhood of Z, mapped by g into Z. Thus it follows from Lemma 2 that g_* is a Lefschetz map and, since $P \setminus Q = Y \setminus Z$, we have

(14) $$\operatorname{ind}(g, \operatorname{int}(P \setminus Q)) = \operatorname{ind}(g, \operatorname{int}(Y \setminus Z)) = \Lambda(g_*).$$

Consider now the following commutative diagram

$$\begin{tikzcd}
H_*(Y, Z) \arrow[r, "g_*"] & H_*(Y, Z) & H_*(Y, Z) \arrow[l, "\mathrm{id}"'] \\
H_*(P, Q) \arrow[u, "j_{1*}"] \arrow[r, "g_{(P,Q)*}"] \arrow[d, "p_{1*}"] & H_*(P \cup g(Q), Q \cup g(Q)) \arrow[u, "j_{2*}"] \arrow[d, "p_{2*}"] & H_*(P, Q) \arrow[l, "(i_{P,Q})_*"'] \arrow[u, "j_{1*}"] \arrow[d, "p_{1*}"] \\
H_*(M, N) \arrow[r, "(f_{M,N})_*"] & H_*(M \cup f(N), N \cup f(N)) & H_*(M, N) \arrow[l, "(i_{M,N})_*"']
\end{tikzcd}$$

in which $j_1\colon (P, Q) \to (Y, Z)$ and $j_2\colon (P \cup g(Q), Q \cup g(Q)) \to (Y, Z)$ are inclusions and

$p_1\colon (P, Q) \ni (x, t) \to x \in (M, N)$,

$p_2\colon (P \cup g(Q), Q \cup g(Q)) \ni (x, t) \to x \in (M \cup f(N), N \cup f(N))$

are projections. Since $\operatorname{cl}(X \times \{1\} \setminus P) \subseteq X \times \{1\} \subseteq \operatorname{int} Z$, we see that j_1 is excision, thus j_{1*} is an isomorphism. By Lemma 1 applied to (P, Q), $(i_{P,Q})_*$ is an isomorphism hence j_{2*} is an isomorphism. This shows that g_* and $I_{P,Q}$ are conjugate and since g_* is a Lefschetz map, we see by Remark 2 that $I_{P,Q}$ is also a Lefschetz map and

(15) $$\Lambda(g_*) = \Lambda(I_{P,Q}).$$

One can easily verify that the mapping $(M, N) \ni x \to (x, 0) \in (P, Q)$ is the homotopy inverse of p_1, thus p_{1*} is an isomorphism. Since $(i_{P,Q})_*$ and $(i_{M,N})_*$ are isomorphisms, we see that also p_{2*} is an isomorphism. This shows that $I_{M,N}$ and $I_{P,Q}$ are conjugate and since $I_{P,Q}$ is a Lefschetz map, so is $I_{M,N}$ and

(16) $$\Lambda(I_{P,Q}) = \Lambda(I_{M,N}).$$

The assertion follows now from (13), (14), (15) and (16). ■

6. Proof of Theorem 2. For a subset $A \subseteq X$ define the mapping

$$\omega_A\colon A \ni x \to \begin{cases} \infty & \text{if } \forall i \in Z^+\ f^i(x) \in A, \\ \max\{n \in Z^+|\ f^i(x) \in A \text{ for } i = 0, 1, \ldots, n\} & \text{otherwise}. \end{cases}$$

We shall need the following two easy to prove remarks.

Remark 9. If A is open then ω_A is lower semi-continuous (l.s.c.) and at every point x of discontinuity of ω_A, $f^{\omega_A(x)+1}(x) \in \operatorname{bd} A$. If A is closed then ω_A is upper semi-continuous (u.s.c.) and at every point x of discontinuity of ω_A, $f^l(x) \in \operatorname{bd} A$ for some $l \in \{0, 1, \ldots, \omega_A(x)\}$. If U is open and $N = \operatorname{cl} U$ then $\omega_U \leqslant \omega_N$ and if for some $x \in U$ $\omega_U(x) = \omega_N(x)$ then both ω_U and ω_N are continuous at x.

Remark 10. Assume $F\colon X\times Z^+\to R$ is continuous, $A\subseteq X$, $\omega\colon A\to Z^+\cup\{\infty\}$ is l.s.c. at $x_0\in X$ and

$$F_1\colon A\ni x\to\sup\{F(x,n)\mid n\in Z^+, n\leqslant\omega(x)\}\in[0,\infty],$$

$$F_2\colon A\ni x\to\inf\{F(x,n)\mid n\in Z^+, n\leqslant\omega(x)\}\in[0,\infty].$$

Then F_1 is l.s.c. and F_2 is u.s.c at x_0.

For every open set $U\subseteq X$ such that $\operatorname{Inv}U$ is compact define the function

$$G_U\colon X\ni x\to\varrho(x,\operatorname{Inv}U)/(\varrho(x,\operatorname{Inv}U)+\varrho(x,X\setminus U))$$

(we assume $\varrho(x,\emptyset)=1$) and the function

$$\gamma_U\colon U\ni x\to\inf\{G_U(f^n(x))/(n+1)\mid n\in Z^+, n\leqslant\omega_U(x)\}.$$

For every closed set $N\subseteq X$ define the function

$$F_N\colon X\ni x\to\min(1,\varrho(x,\operatorname{Inv}^-N\cup\operatorname{bd}N))$$

and the function

$$\varphi_N\colon N\ni x\to\sup\{(2n+1)F_N(f^n(x))/(n+1)\mid n\in Z^+, n\leqslant\omega_N(x)\}.$$

Lemma 3. *Assume K is a non-empty isolated invariant set of Rybakowski type and U is open such that $N:=\operatorname{cl}U$ is an isolating neighborhood of Rybakowski type for K. Then there exists $V\subseteq U$, an open neighborhood of K such that $(\varphi_N|_V,\gamma_U|_V)$ is a Lyapunov pair for K, $\gamma_U|_V$ is continuous and $\varphi_N|_V$ is u.s.c.*

Proof. In order to simplify the notation, we will drop the subscripts U, N in G_U, γ_U, F_N, φ_N throughout the proof.

First we shall show that γ and φ are u.s.c. The upper semi-continuity of γ follows directly from Remark 10. Assume φ is not u.s.c. at some $x_0\in N$. Then there exists a sequence $\{x_n\}_{n=1}^\infty\subseteq N$, $x_n\to x_0$ such that $\varphi(x_n)>\mu>\varphi(x_0)$ for some μ and all $n\in N$. Hence there exists also a sequence $\{i_n\}\subseteq Z^+$ such that

$$i_n\leqslant\omega_N(x_n)\quad\text{and}\quad(2i_n+1)(i_n+1)^{-1}F(f^{i_n}(x_n))>\mu>0.$$

It cannot be $i_n\to\infty$, since otherwise by Rybakowski condition, $\{f^{i_n}(x_n)\}$ admits a subsequence tending to some $z\in\operatorname{Inv}^-N$ and we get in the limit $0=2F(z)\geqslant\mu>0$, a contradiction. Thus, one can pick a constant subsequence from $\{i_n\}$, say $\{m\}$. It follows from the upper semi-continuity of ω_N that $m\leqslant\omega_N(x_0)$, so we get in the limit $(2m+1)(m+1)^{-1}F(f^m(x_0))\geqslant\mu>\varphi(x_0)$, again a contradiction.

Assume γ is not l.s.c. in every neighborhood of K. Then there exists a sequence $\{x_n\}_{n=1}^\infty\subseteq U$ such that γ is not lower semi-continuous at x_n and $\varrho(x_n,K)\to 0$. Since K is compact, we may assume that $x_n\to x_0$ for some $x_0\in K$. Thus, by l.s.c. of ω_U, $\lim\omega_U(x_n)\geqslant\omega_U(x_0)=\infty$. It must be also $\omega_U(x_n)<\infty$, since otherwise $\gamma(x_n)=0$ and then γ is continuous at x_n. The lower semidiscontinuity of γ at x_n means that there exists a sequence $\{x_{n,m}\}_{m=1}^\infty$ tending to x_n with $m\to\infty$ and a constant $\mu_n<\gamma(x_n)$ such that $\gamma(x_{n,m})<\mu_n$ for all $m\in Z^+$. Thus there exists also a sequence of integers $\{k_{n,m}\}_{m=1}^\infty$ such that

$$(k_{n,m}+1)^{-1}G(f^{k_{n,m}}(x_{n,m}))<\mu_n<\gamma(x_n)\quad\text{and}\quad k_{n,m}\leqslant\omega_U(x_{n,m}).\tag{17}$$

Without loss of generality we may assume that for each $n\in N$ there exists $l_n:=\lim_{m\to\infty}k_{n,m}\in Z^+\cup\{\infty\}$. We have $l_n>\omega_U(x_n)$, because otherwise passing in (17) to the limit as $m\to\infty$, we get

$$(l_n+1)^{-1}G(f^{l_n}(x_n))\leqslant\mu_n<\gamma(x_n),$$

which contradicts $l_n\leqslant\omega_U(x_n)$.

Thus we may assume that $k_{n,m}\geqslant\omega_U(x_n)+1$ for all n. We know from Remark 9 that

$$f^{\omega_U(x_n)+1}(x_n)\in\operatorname{bd}U.\tag{18}$$

For each $n\in N$ choose $m=m(n)$ such that $y_n:=x_{n,m}$ satisfies

$$\varrho(f^{\omega_U(x_n)+1}(x_n),f^{\omega_U(x_n)+1}(y_n))\leqslant 1/n.\tag{19}$$

Put $k_n:=k_{n,m(n)}$ and $j_n:=[\omega_U(x_n)/2]$. We get from (17) and the definition of γ that

$$(k_n+1)^{-1}G(f^{k_n}(y_n))<(j_n+1)^{-1}G(f^{j_n}(x_n)).\tag{20}$$

Obviously $j_n\to\infty$ and the Rybakowski condition implies that the sequence $f^{j_n}(x_n)$ is precompact. Hence we can assume that $f^{j_n}(x_n)\to w\in\operatorname{Inv}^-N$. Since $\omega_N(f^{j_n}(x_n))\geqslant j_n\to\infty$, we get from the u.s.c. of ω_N that $\omega_N(w)=\infty$, i.e. $w\in\operatorname{Inv}^+N$. Thus $w\in K$ and consequently

$$\lim_{n\to\infty}G(f^{j_n}(x_n))=G(w)=0.\tag{21}$$

The sequence $\{f^{\omega_U(x_n)+1}(y_n)\}\subseteq U$ is also precompact, thus we can assume it tends to some $z\in\operatorname{Inv}^-N$. By (18) and (19) $z\in\operatorname{bd}U$. If the sequence $\{k_n-\omega_U(x_n)\}$ is unbounded then, by the second inequality of (21), $\omega_N(z)=\infty$ and $z\in K\cap\operatorname{bd}U$, a contradiction. Thus assume the sequence is bounded. Then without loss of generality we can assume that $k_n=\omega_U(x_n)+p+1$ for some constant $p\in Z^+$, because $k_n\geqslant\omega_U(x_n)+1$. From (20) we get

$$G(f^{k_n}(y_n))<(k_n+1)(j_n+1)^{-1}G(f^{j_n}(x_n))\leqslant(2j_n+p+3)(j_n+1)^{-1}G(f^{j_n}(x_n))$$

hence (21) implies that $\lim G(f^{k_n}(y_n))=0$. There is also

$$\omega_N(z)\geqslant p\quad\text{and}\quad G(f^p(z))=G(f^p(\lim f^{\omega_U(x_n)+1}(y_n)))=\lim G(f^{k_n}(y_n))=0.$$

Thus $f^p(z)\in K$ and $z\in K$, which contradicts $z\in\operatorname{bd}U$. Hence γ is continuous in some vicinity of K.

Choose V an open neighborhood of K which is small enough to ensure that $\gamma|_V$ is continuous and $\operatorname{cl}V\subseteq U$. In order to show that $(\varphi|_V,\gamma|_V)$ satisfies the assertion of the lemma we have to prove that it is a Lyapunov pair.

In order to show (5) assume $\gamma(x) > 0$ and $f(x) \in V$ for some $x \in V$. Then $\omega_U(x) < \infty$ and $\omega_U(f(x)) < \infty$. Thus there exists $n \in \{0, 1, \ldots, \omega_U(f(x))\}$ such that

$$\gamma(f(x)) = G(f^{(n+1)}(x))/(1+n) > G(f^{(n+1)}(x))/(1+(n+1)) \geqslant \gamma(x),$$

because $G(f^{n+1}(x)) > 0$. This shows that $\gamma|_V$ grows along trajectories of f on V.

Assume $\varphi(x) > 0$ and $f(x) \in V$ for some $x \in V$. Let

$$\{i_n\}_{n=1}^{\infty} \subseteq \{0, 1, \ldots, \omega_N(f(x))\}$$

be a sequence such that

$$(2i_n+1)(i_n+1)^{-1}F\big(f^{i_n}(f(x))\big) > \varphi(f(x)) - 1/n.$$

As previously the Rybakowski condition excludes the case $i_n \to \infty$ and we can assume that $\{i_n\}$ is constant, i.e. for some $k \in \{0, 1, \ldots, \omega_N(f(x))\}$

$$\varphi(f(x)) = (2k+1)(k+1)^{-1}F\big(f(f(x))\big) < (2(k+1)+1)(k+2)^{-1}F(f^{k+1}(x)) \leqslant \varphi(x).$$

This shows that $\varphi|_V$ decreases along trajectories of f in V.

To show (6) assume that $x \in K$. Then $\omega_U(x) = \infty$, hence $\gamma(x) \leqslant 1/(n+1)$ for all $n \in N$, i.e. $\gamma(x) = 0$. Also for all $n \in N$ $f^n(x) \in \mathrm{Inv}^- N$, thus $\varphi(x) = 0$.

In order to prove (7) fix an open neighborhood W of K and assume that for all $n \in N$

$$\mathrm{cl}\, H(1/n, \varphi|_V, \gamma|_V) \not\subseteq W.$$

Then there exists a sequence $\{y_n\} \subseteq \setminus W$ such that

$$y_n \in \mathrm{cl}\, H(1/n, \varphi|_V, \gamma|_V) \subseteq \mathrm{cl}\, V.$$

Pick $x_n \in H(1/n, \varphi|_V, \gamma|_V)$ such that $\varrho(x_n, y_n) \leqslant 1/n$. Then $\varrho(x_n, \setminus W) \to 0$, $\varrho(x_n, \mathrm{cl}\, V) \to 0$, $\gamma(x_n) \to 0$, $\varphi(x_n) \to 0$. In particular $F(x_n) \to 0$, i.e.

$$\varrho(x_n, \mathrm{Inv}^- N \cup \mathrm{bd}\, N) \to 0.$$

Thus we can assume that $x_n \to x \in \mathrm{cl}\, V \cap (\mathrm{Inv}^- N \cup \mathrm{bd}\, N) \setminus W \subseteq \mathrm{Inv}^- N \setminus W$. It cannot be $x \in \mathrm{Inv}^+ N$, because otherwise $x \in K \setminus W = \emptyset$. Thus $\omega_U(x) < \infty$, $\{\omega_U(x_n)\}$ is bounded and, taking a subsequence if necessary, we can find $k \in Z^+$ such that $\gamma(x_n) = G(f^k(x_n))/(k+1)$ for all $n \in Z^+$. Hence $G(f^k(x)) = (k+1)\lim \gamma(x_n) = 0$, which shows that $x \in K \setminus W$, a contradiction. This finishes the proof of (7). ∎

Proof of Theorem 2. Choose an open neighborhood U of K such that $N := \mathrm{cl}\, U$ is an isolating neighborhood for K of Rybakowski type and put $\gamma := \gamma_U$, $\varphi := \varphi_N$. By Lemma 3 we can find an open neighborhood V of K such that $(\varphi|_V, \gamma|_V)$ is a Lyapunov pair for K and $\gamma|_V$ is continuous, $\varphi|_V$ is u.s.c. We may also assume that $f(V) \subseteq V$.

By (7) we can find an $\varepsilon > 0$ such that $\mathrm{cl}\, H(\varepsilon, \varphi|_V, \gamma|_V) \subseteq V$. Put $U' := H(\varepsilon, \varphi|_V, \gamma|_V)$ and $N' := \mathrm{cl}\, U'$, $\gamma' := \gamma_{U'}$, $\varphi' := \varphi_{N'}$. Since $\gamma|_V$ and $\varphi|_V$ are u.s.c. U' is an open neighborhood of K. It is obviously an isolating neighborhood of Rybakowski type.

Again by Lemma 3 we can find an open neighborhood $V' \subseteq U'$ of K such that $(\varphi'|_{V'}, \gamma'|_{V'})$ is a Lyapunov pair for K with $\gamma'|_{V'}$ continuous and $\varphi'|_{V'}$ u.s.c.

We have to show that $\varphi'|_{V'}$ is l.s.c. If $\omega_{N'}$ is continuous at $x \in V'$ then the lower semi-continuity of φ' at x follows from Remark 10. Thus assume $\omega_{N'}$ is not continuous. Then we can derive from Remark 9 that $k := \omega_{U'}(x) < \omega_{N'}(x)$. Then $y := f^{k+1}(x) \in N' \setminus U'$. In particular $y \in V \setminus U'$ means that $\gamma(y) \geqslant \varepsilon > 0$. Since $f(N') \subseteq f(V) \subseteq V$, we have also $f(y) \in V$, by (5), we get $\gamma(f(y)) > \gamma(y) \geqslant \varepsilon$, i.e.

$$f^{k+2}(x) = f(y) \notin \mathrm{cl}\, U' = N',$$

which shows that $\omega_{N'}(x) = k+1 = \omega_{U'}(x)+1 < \infty$. Since $y = f^{k+1}(x) \in \mathrm{bd}\, N'$, we get $F_{N'}(f^{k+1}(x)) = 0$. Consequently

$$\varphi'(x) = (2j+1)F_{N'}(f^j(x))/(j+1) \quad \text{for some } j \in \{0, 1, \ldots, k\}.$$

Fix $\mu > 0$. From the lower semi-continuity of $\omega_{U'}$ we get that

$$\varphi'(y) \geqslant (2j+1)F_{N'}(f^j(y))/(j+1) > (2j+1)F_{N'}(f^j(x))/(j+1) - \mu = \varphi'(x) - \mu$$

for y sufficiently close to x. This shows that φ' is continuous at x and completes the proof. ∎

References

[1] H. Amann and E. Zehnder, *Non-trivial solutions for a class of nonresonance problems and applications to non-linear differential equations*, Ann. Scuola Norm. Sup. Pisa 7 (1980), 539–603.

[2] K. Borsuk, *Theory of retracts*, PWN, Warszawa 1967.

[3] C. Bowszyc, *Fixed point theorems for the pairs of spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 16 (11) (1968) 845–850.

[4] R. Churchill, *Isolated invariant sets in compact metric spaces*, J. Differential Equations 12 (1972), 330–352.

[5] C. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. Math. 38, Amer. Math. Soc., Providence, R. I., 1978.

[6] C. C. Conley and R. W. Easton, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc. 158 (1971), 35–61.

[7] C. C. Conley and R. Gardner, *An application of the generalized Morse index to travelling wave solutions of a competitive Reaction-diffusion model*, Indiana Univ. Math. J. 33 (1984) 319–343.

[8] C. C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. 37 (1984), 207–253.

[9] E. N. Dancer, *Perturbation of zeros in the presence of symmetries*, J. Austral. Math. Soc. (Ser. A) 36 (1984), 106–125.

[10] A. Dold, *Fixed point index and fixed point theorem for euclidean neighborhood retracts*, Topology 4 (1965), 1–8.

[11] — *Lectures on Algebraic Topology*, Springer-Verlag, Berlin–Heidelberg–New York 1972.

[12] C. C. Fenske and H. O. Peitgen, *On fixed points of zero index in asymptotic fixed point theory*, Pacific J. Math. 66 (2) (1976), 391–410.

[13] C. C. Fenske, *Method of asymptotic fixed-point theory applied to semi-dynamical systems*, in: *Game Theory and Related Topics*, North-Holland, 1979, 157–164.

[14] G. Fournier, *Généralisations du théorème de Lefschetz pour des espaces non-compacts*, II. *Applications d'attraction compacte*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 23 (1975), 693–711.
[15] A. Floer, *A refinement of the Conley index and an application to the stability of hyperbolic invariant sets*, Ergodic Theory Dynamical Systems 7 (1987), 93–103.
[16] A. Granas, *The Leray–Schauder index and the fixed point theory for arbitrary* ANR's, Bull. Soc. Math. France 100 (1972), 209–228.
[17] Ph. Hartman, *Ordinary Differential Equations*, John Wiley and Sons Inc., New York–London-Sydney 1964.
[18] M. C. Irwin, *Smooth Dynamical Systems*, Academic Press, 1980.
[19] H. L. Kurland, *The Morse index of an isolated invariant set is a connected simple system*, J. Differential Equations 42 (1981), 234–259.
[20] Ch. K. Mc Cord, *On the Hopf index and the Conley index*, preprint.
[21] J. T. Montgomery, *Cohomology of isolated invariant sets under perturbations*, J. Differential Equations 13 (1973), 257–299.
[22] M. Mrozek, *The fixed point index of a translation operator of a semiflow*, Acta Math. Univ., Iag. 27 (1988), 13–22.
[23] — *Periodic and stationary trajectories of flows and ordinary differential equations*, Acta Math. Univ. Iag. 27 (1988), 29–37.
[24] — *Strongly isolated invariant sets and strongly isolating blocks*, Bull. Polish Acad. Sci. Math. 35 (1987), 19–27.
[25] — *Leray functor and the cohomological Conley index for discrete time dynamical systems*, Trans. Amer. Math. Soc., to appear.
[26] J. W. Robbin, D. Salamon, *Dynamical systems, shape theory and the Conley index*, Ergodic Theory Dynamical Systems 8* (1988), 375–393.
[27] K. P. Rybakowski, *On the homotopy index for infinite dimensional semiflows*, Trans. Amer. Math. Soc. 269 (1982), 351–382.
[28] — *The Morse index, repeller-attractor pairs and the connection index for semiflows on noncompact spaces*, J. Differential Equations 47 (1983), 66–98.
[29] — *On a relation between the Brouwer degree and the Conley index for gradient flows*, Bull. Soc. Math. Belg. (B) 37 (II) (1985), 87–96.
[30] D. Salamon, *Connected simple systems and the Conley index of isolated invariant sets*, Trans. Amer. Math. Soc. 291 (1985), 1–41.
[31] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817.
[32] R. Srzednicki, *On rest points of dynamical systems*, Fund. Math. 126 (1985), 69–81.
[33] T. Ważewski, *Sur un principe topologique pour l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. 20 (1947), 279–313.

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On a classification of pointwise compact sets of the first Baire class functions

by

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Abstract. The paper is concerned with compact separable subspaces of the space $B_1(\omega^\omega)$ of the first Baire class functions on irrationals endowed with the pointwise topology, i.e. Rosenthal compacta. We associate to each separable Rosenthal compactum K an ordinal number $\eta(K) \leqslant \omega_1$, which indicates the "Borel complexity" of the compactum. The index $\eta(K)$ is a topological invariant of the function space $C_p(K)$ endowed with the pointwise topology. We construct Rosenthal compacta of arbitrarily large countable index and we use them to give examples of open linear continuous maps raising the Borel class of linear spaces.

§ 1. Introduction. Our terminology follows [En], [Ku] and [Se]. We shall denote by R the real line; ω is the set of natural numbers and ω^ω is the Baire space, i.e. topologically the irrationals.

A map $f\colon X \to Y$, where X and Y are separable metrizable spaces, is of the first Baire class if $f^{-1}(U)$ is an F_σ-set for every open $U \subset Y$ (if Y is a separable Banach space, this means that f is a pointwise limit of a sequence of continuous maps from X into Y), cf. [Ku, § 31]. Given a separable metrizable space X, we denote by $B_1(X)$ the space of real-valued first Baire class functions on X equipped with the topology of pointwise convergence.

This paper is concerned mainly with compact spaces which can be embedded in the space $B_1(\omega^\omega)$ of the first Baire class functions on irrationals, i.e. with Rosenthal compacta, see [Go]. For fundamental facts about Rosenthal compacta we refer the reader to the papers by Bourgain, Fremlin, Talagrand [BFT], Godefroy [Go] and Negrepontis [Ne]. In the sequel we shall often use the deep result by Bourgain, Fremlin and Talagrand [BFT, Th. 3F] stating that Rosenthal compacta K are Fréchet topological spaces, i.e. for every set $A \subset K$ and a point $x \in \bar{A}$ there exists a sequence of points from A which converges to x.

Let us notice that if A is a metrizable space which is a continuous image of irrationals (i.e. A is an analytic space), then compact subspaces of $B_1(A)$ are Rosenthal compacta, as the map $f \to f \circ u$, where $u\colon \omega^\omega \to A$ is a continuous surjection, embeds $B_1(A)$ homeomorphically into $B_1(\omega^\omega)$.

Let K be a separable compact space; given a countable dense subset D of K