Sacks reals and Martin's axiom

by

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Abstract. It is shown that adding a Sacks real does not necessarily add a Souslin tree, in fact $2^{\aleph_0} = \aleph_2 + \mathrm{MA}$ can hold in the extension. On the other hand, if the ground model satisfies CH then the extension satisfies $\diamondsuit_{\omega_1}$.

If x is a generic real over V, how much of $\mathrm{MA}_{\aleph_1}$ can hold in $V[x]$? If x is a Cohen or a random real, then $\mathrm{MA}_{\aleph_1}$ fails in $V[x]$ ([6]). If x is Cohen, then in fact there is a Souslin tree in $V[x]$ ([8]). For x random, however, no Souslin trees exist in $V[x]$ assuming V satisfies $\mathrm{MA}_{\aleph_1}$([5]).

The results in this paper are about the case where x is a Sacks real (an $\mathscr{S}$-generic real, where $\mathscr{S}$ is the set of perfect downwards closed subtrees of $(2)^{<\omega}$ ([7]). We prove that if V satisfies CH, then $V[x]$ satisfies $\diamondsuit_{\omega_1}$ (Section 1). In Section 3 it is proved that if V satisfies a strengthening of Martin's axiom, then $V[x]$ satisfies $\mathrm{MA}_{\aleph_1}$. In the proof Sacks amoeba forcing $\mathscr{A}$ is used – the original use of it was Shelah's proof that consistently $2^{\aleph_0} > \aleph_1$ and forcing with $\mathscr{S}$ does not collapse cardinals. The strengthening of Martin's axiom that we use $(\mathrm{MA}_{\aleph_1}(\mathscr{S} * \mathrm{ccc}, \mathscr{A} * \mathrm{ccc}))$ is a consequence, for example, of PFA ([9]). In Section 4 we outline that $\mathrm{MA}_{\aleph_1}(\mathscr{S} * \mathrm{ccc}, \mathscr{A} * \mathrm{ccc})$ is consistent relative to ZFC.

Notation. For $s, t \in \mathscr{S}$, s extends t $(s \leqslant t)$ if and only if $s \subseteq t$. The nth level of t is t_n, and for $x \in t$, $t_x = \{y \in t\colon y <_t x \text{ or } x \leqslant_t y\}$. Let $s \sim t$ mean that s is compatible with t. $G_{\mathscr{S}}$ is a generic subset of $\mathscr{S}$.

Section 1.

THEOREM. *If* CH *holds in* V, *then* $V^{\mathscr{S}}$ *satisfies* $\diamondsuit_{\omega_1}$.

Proof. Suppose $f \in (\omega)^{\omega}$ and $f(n) > n$, all n. Then say $s \in \mathscr{S}$ is f-thin if for every n, $\operatorname{Card} s_{f(n)} \leqslant \operatorname{Card} s_n + 1$. And call $t \in \mathscr{S}$ f-thick if there are infinitely many n such that for each $x \in t_n$, $\operatorname{Card}\{y \in t_{f(n)}\colon x \leqslant_t y\} \geqslant 4$.

LEMMA 1. *If t is f-thick, then every maximal antichain in $\{u\colon u \leqslant t\}$ consisting of f-thin trees is uncountable.*

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Proof. Suppose $\{s^i: i<\omega\}$ is a set of f-thin trees. To define an $s \leqslant t$, $s \nsim$ each s^i. Suppose $s_n \subseteq t_n$ has been defined and $i<\omega$. Then by f-thickness of t there is an $m \geqslant n$ and an $s_{f(n)} \subseteq \{x \in t_{f(m)}: \exists y \in s_n\, y <_t x\}$ such that for each $z \in s_n$,

$$\operatorname{Card}\{x \in s_{f(m)}: z <_t x\} \geqslant 2,$$

with $s_{f(m)} \cap (s^i)_{f(m)} = \emptyset$. In ω-many such steps all $i<\omega$ are handled, thus giving s.

LEMMA 2. (CH) *For each $f \in \omega^\omega (f(n)>n$ all $n)$, there's an antichain $A_f \subseteq \mathscr{S}$ consisting of f-thin trees such that any f-thick tree is compatible with $2^{\aleph_0}$ many members of A_f.*

Proof. Let $\langle t_\alpha: \alpha<\omega_1\rangle$ be the f-thick trees, enumerated with $\aleph_1$-repetitions. Let $A_f = \{s_\alpha: \alpha<\omega_1\}$ where, by Lemma 1, s_α may be chosen to be an f-thin extension of t_α, incompatible with each s_β $(\beta<\alpha)$.

Proof of the Theorem. For $f, g \in \omega^\omega$ let $f<g$ mean that g eventually dominates f. By CH pick a scale $\{f_\alpha: \alpha<\omega_1\}$ in ω^ω. For $\alpha<\omega_1$, let Z_α be the set of canonical terms for subsets of α which are labeled f_α-thick trees; a member of Z_α is a $\langle t, W, V\rangle$, where t is f_α-thick, $W: \alpha \to \omega$ is $1-1$, and $V: \bigcup_{n \in \operatorname{range} W} t_n \to 2$. Thus, for $\beta<\alpha$, and assuming $t \in G_{\mathscr{S}}$, β is in the $G_{\mathscr{S}}$-denotation of $\langle t, W, V\rangle$ just in case $V(x) = 1$, where x is the member of $t_{W(\beta)}$ determined by $G_{\mathscr{S}}$.

Assign to each $s \in A_{f_\alpha}$ an $H_\alpha(s) = \langle t, W, V\rangle \in Z_\alpha$ such that $t \sim s$. By Lemma 2, we may make this assignment such that H_α is onto Z_α. Now in $V^{\mathscr{S}}$ define the αth member $D_\alpha \subseteq \alpha$ of the $\diamondsuit$-sequence as follows. Let s be the member of $G_{\mathscr{S}} \cap A_{f_\alpha}$, assuming there is one. Then $H_\alpha(s)$ is of the form $\langle t, W, V\rangle$. Suppose $s \cap t \in G_{\mathscr{S}}$. Then let D_α be the subset of α determined by the term $\langle s \cap t, W, V \restriction s \cap t\rangle$. If the above conditions aren't satisfied, let $D_\alpha = \emptyset$.

Now suppose $u \in \mathscr{S}$ forces that $\dot{X} \subseteq \omega_1$ and $\dot{X} \cap \alpha \neq \dot{D}_\alpha$, all limit $\alpha<\omega_1$. For $s, t \in \mathscr{S}$ let $s \leqslant_n t$ if $s \leqslant t$ and $s_n = t_n$. Note that, since $\{f_\beta: \beta<\omega_1\}$ is a scale, every $t \in \mathscr{S}$ is f_β-thick for eventually all $\beta<\omega_1$. Take a countable elementary submodel of the forcing, giving an $\alpha<\omega_1$, a countable $\mathscr{C} \subseteq \{t \in \mathscr{S}: t \leqslant u\}$, with $u \in \mathscr{C}$, so that for each $t \in \mathscr{C}$ there is a $\beta<\alpha$ such that t is f_β-thick. Moreover, if $t \in \mathscr{C}$, $n<\omega$, $\beta<\alpha$, then there is an $s \in \mathscr{C}$ with $s \leqslant_n t$ such that for each $x \in s_n$, s_x decides whether or not $\beta \in \dot{X}$.

Let $\alpha = \{\gamma_n: n<\omega\}$. Construct a sequence $u^0 \geqslant_{n_1} u^1 \geqslant_{n_2} u^2 \geqslant_{n_3} \ldots$ such that $u^0 = u$, each $u^i \in \mathscr{C}$, and $n_1<n_2<n_3<\ldots$ Suppose u^k and n_k have been chosen. Then u^k is f_{β_k}-thick, some $\beta_k<\alpha$, and $f_{\beta_k}<f_\alpha$, so there is an $m \geqslant n_k$, with $f_{\beta_k}(m)<f_\alpha(m)$, such that for each $y \in (u^k)_m$, $(\operatorname{Card}\{z \in u^k)_{f_{\beta_k}(m)}: y <_{u^k} z\} \geqslant 4$. Let $n_{k+1} = f_{\beta_k}(m)$ and let $u^{k+1} \in \mathscr{C}$ be such that $u^{k+1} \leqslant_{n_{k+1}} u^k$ and for each $x \in (u^{k+1})_{n_{k+1}}$, $(u^{k+1})_x$ decides whether $\gamma_{k+1} \in \dot{X}$.

Let $t = \bigcap_i u^i$. Then by construction t is an f_α-thick member of $\mathscr{S}$ extending u, and, letting $W(\gamma_k) = n$, there is a function V such that $\langle t, W, V\rangle \in Z_\alpha$ is a term which is forced by t to equal $\dot{x} \cap \alpha$. So pick $s \in A_{f_\alpha}$ such that $H_\alpha(s) = \langle t, W, V\rangle$. Then $s \cap t$ is a condition, extending u, which forces that $\dot{x} \cap \alpha = \dot{D}_\alpha$.

Under the hypothesis, weaker than the continuum hypothesis, of the existence of a scale on $(\omega)^\omega$ of length $\cdot\,\omega_1$, does $V^{\mathscr{S}}$ satisfy CH or even $\diamondsuit_{\omega_1}$? That V can satisfy $\neg$CH and $V^{\mathscr{S}}$ satisfy CH was proved by Baumgartner ([2]), where the model V has a scale. An alternate model V having those properties, except not having a scale, is obtained by adding $2^{\aleph_0}$ many Cohen reals to a ground model of $\neg$CH: in such a V there are antichains $A_\alpha \subseteq \mathscr{S}$ and functions $F_\alpha: A_\alpha \to 2^{\aleph_0}$, for each $\alpha<\omega_1$, such that for each $t \in \mathscr{S}$ there is a $\beta<\omega_1$ so that for all $\alpha \geqslant \beta$, $\{F_\alpha(s): s \leqslant t, s \in A_\alpha\} = 2^{\aleph_0}$.

Section 2. *Sacks amoeba forcing* is the partial ordering $\mathscr{A}$ consisting of all pairs (t, n) where $t \in \mathscr{S}$ and $n \in \omega$, given by $(t, n) \leqslant (s, m)$ iff $t \subseteq s$, $n \geqslant m$ and $t_m = s_m$. Forcing with $\mathscr{A}$ gives rise to a perfect set of Sacks reals.

Suppose $\dot{x} \in V^{\mathscr{A}}$ is a name for an object in V. A condition (t, n) *determines* $\dot{x}$ if there is some y in V such that $(t, n) \Vdash \dot{x} = y$. Say that (t, n) *weakly determines* $\dot{x}$ if whenever $(s, m) \leqslant (t, n)$ and y have the property that $(s, m) \Vdash \dot{x} = y$, then $(t', m) \Vdash \dot{x} = y$ where t' consists of all the elements of t which are compatible with some element of s of length m. One easily checks that given $(t, n) \in \mathscr{A}$ and a name $\dot{x} \in V^{\mathscr{A}}$ for an element of V there is a $(s, n) \leqslant (t, n)$ which weakly determines $\dot{x}$. In particular, the collection of conditions which weakly determine $\dot{x}$ are dense.

Corresponding definitions can be made for Sacks forcing. Suppose $\dot{x} \in V^{\mathscr{S}}$ is a name for an element of V. Say that t *determines* $\dot{x}$ if $t \Vdash \dot{x} = y$ for some y in V. t *weakly determines* $\dot{x}$ at n if for every element z of t of length n, t_z determines $\dot{x}$. Clearly, for any t and n there is a $(t', n) \leqslant (t, n)$ (in $\mathscr{A}$) such that t' weakly determines $\dot{x}$ at n.

The following lemma is useful in showing that $\mathscr{A}$ is proper and will be used later in Section 3.

LEMMA. *Assume M is an inner model of* ZFC, *$\mathscr{A}^M$ is Sacks amoeba forcing in the sense of M and $M^{\mathscr{A}^M} \models \dot{x} \in M$. If (t, n) weakly determines $\dot{x}$ in M, then the collection of $(s, m) \in \mathscr{A}^M$ which determine $\dot{x}$ is predense below (t, n) in $\mathscr{A}$, i.e. if $(r, k) \in \mathscr{A}$ and $(r, k) \leqslant (t, n)$, then there is some $(s, m) \in \mathscr{A}^M$ which determines $\dot{x}$ such that $(s, m) \leqslant (t, n)$ and (s, m) is compatible with (r, k).*

Proof. By absoluteness. Since (t, n) weakly determines $\dot{x}$ in M, the statement "for all $(r, k) \leqslant (t, n)$ there is an (s, m) which determines $\dot{x}$ and is compatible with (r, k)" can be coded as a Π^1_1 statement.

The corresponding lemma for Sacks forcing is obvious.

Section 3. The purpose of this section is to show that $\mathrm{MA}_{\aleph_1}$ can be preserved when adding a Sacks real. For a partial ordering P let $P * \mathrm{ccc}$ be the class of partial orderings $P * \dot{Q}$ where $V^P \models$ "$\dot{Q}$ is c.c.c.".

THEOREM. $\mathrm{MA}_{\aleph_1}(\mathscr{S} * \mathrm{ccc}, \mathscr{A} * \mathrm{ccc})$ *implies that* $\mathrm{MA}_{\aleph_1}$ *holds in* $V^{\mathscr{S}}$.

For the proof of this theorem the following two lemmas are needed. Note that $\mathrm{MA}_{\aleph_1}(P * \mathrm{ccc})$ implies $\mathrm{MA}_{\aleph_1}(P)$.

Lemma 1. *Assume* $MA_{\aleph_1}(P*\text{ccc})$. *If* $V^P \models$ "$\mathring{Q}$ *is c.c.c.*" *and* $V^P \models$ "$\mathring{q}_\alpha \in \mathring{Q}$" *for* $\alpha \in \omega_1$, *then there is a subset* X *of* ω_1 *of size* $\aleph_1$ *such that* $V^P \Vdash$ "$\mathring{q}_\alpha$ *is incompatible with* $\mathring{q}_\beta$" *for all* $\alpha, \beta \in X$.

Proof. By the remark preceding the lemma, $MA_{\aleph_1}(P)$ holds. Therefore there is $\mathring{q}$ such that $V^P \models$ "every element of $\mathring{Q}$ below $\mathring{q}$ is compatible with $\aleph_1$ many of the $\mathring{q}_\alpha(\alpha \in \omega_1)$". By $MA_{\aleph_1}(P*\text{ccc})$ there is a filter G on $P*\mathring{Q}$ such that for cofinally many $\alpha \in \omega_1$ there is a $p \in P$ with $(p, \mathring{q}_\alpha) \in G$. Let X be the collection of such α.

Lemma 2. *Assume* $MA_{\aleph_1}(\mathscr{A})$. *Then* $V^{\mathscr{A}} \models$ "*every subset of* V *of size* $\aleph_1$ *has a subset of size* $\aleph_1$ *in* V".

Proof. Suppose $(t, n) \Vdash$ "$\mathring{X}$ is a subset of V of size $\aleph_1$". Choose $\mathring{x}_\alpha \in V^{\mathscr{A}}$ for $\alpha \in \omega_1$, such that $(t, n) \Vdash$ "$\mathring{x}_\alpha(\alpha \in \omega_1)$ are distinct elements of $\mathring{X}$". By $MA_{\aleph_1}(\mathscr{A})$ there is a filter G on A containing (t, n) such that

(i) for $\alpha \in \omega_1$ there are y_α and (t_α, m_α) with $(t_\alpha, m_\alpha) \Vdash \mathring{x}_\alpha = y_\alpha$;

(ii) $t_\infty = \bigcap_{\alpha \in \omega_1} t_\alpha$ is in $\mathscr{S}$.

There is a subset I of ω_1 of size $\aleph_1$ on which m_α is constant. Let this constant value be m and let Y be the set of y_α for $\alpha \in I$. Then (t_∞, m) is compatible with (t, n) and $(t_\infty, m) \Vdash$ "Y is a subset of $\mathring{X}$ of size $\aleph_1$".

A similar argument shows that assuming $MA_{\aleph_1}(\mathscr{A})$, the conclusion of the lemma holds with $\mathscr{A}$ replaced by $\mathscr{S}$. This fact will not be needed in what follows.

Proof of Theorem. Assume $MA_{\aleph_1}(\mathscr{S}*\text{ccc}, \mathscr{A}*\text{ccc})$.

Suppose $V^{\mathscr{S}} \models$ "$\mathring{Q}$ is a c.c.c partial ordering" and $V^{\mathscr{S}} \models$ "$\mathring{D}$ is a dense open subset of $\mathring{Q}$" for $\alpha \in \omega_1$. Let T be the set of all terms $\mathring{q}$ in $V^{\mathscr{S}}$ such that $V^{\mathscr{S}} \Vdash$ "$\mathring{q}$ is an element of $\mathring{Q}$". Define $\mathring{Q}/G_{\mathscr{A}}$ to be the partial ordering in $V^{\mathscr{A}}$ which is defined by $V^{\mathscr{A}} \models$ "for $q_1, q_2 \in T$, $q_1 \leqslant q_2$ in $\mathring{Q}/G_{\mathscr{A}}$ iff $t \Vdash$ '$q_1 \leqslant q_2$ in $\mathring{Q}$' for some $(t, n) \in G_{\mathscr{A}}$".

Claim. $V^{\mathscr{A}} \models$ "$\mathring{Q}/G_{\mathscr{A}}$ *is ccc*".

Assume to the contrary that $(t, n) \Vdash$ "$\mathring{A}$ is an uncountable antichain in $\mathring{Q}/G_{\mathscr{A}}$". Choose $\mathring{a}_\alpha$ for $\alpha \in \omega_1$ such that $(t, n) \Vdash \mathring{a}_\alpha \in \mathring{A}$ and $(t, n) \Vdash \mathring{a}_\alpha \neq \mathring{a}_\beta$ if $\alpha \neq \beta$. Since $MA_{\aleph_1}(\mathscr{A})$ holds, Lemma 2 implies there is no loss of generality in assuming there is $\mathring{q}_\alpha \in T$ for $\alpha \in \omega_1$ such that $(t, n) \Vdash \mathring{a}_\alpha = \mathring{q}_\alpha$. If z is an element of t of length n, Lemma 1 implies there is an uncountable subset X of ω_1 such that for all $\alpha, \beta \in X$ $t_z \Vdash$ "$\mathring{q}_\alpha$ is incompatible with $\mathring{q}_\beta$". Even more, by applying Lemma 1 successively to each element of t of length n one obtains an uncountable subset X of ω_1 such that for all $\alpha, \beta \in X$ and all z in t of length n, $t_z \Vdash$ "$\mathring{q}_\alpha$ is incompatible with $\mathring{q}_\beta$". Fix distinct α and β in X and for each z in t of length n choose $t'_z \leqslant t_z$ such that $t'_z \Vdash$ "$\mathring{q}_\alpha$ and $\mathring{q}_\beta$ are compatible". Let t' be the union of the t'_z. Clearly $(t', n) \leqslant (t, n)$. Moreover, $(t', n) \Vdash$ "$\mathring{q}_\alpha$ and $\mathring{q}_\beta$ are compatible (in $\mathring{Q}/G_{\mathscr{A}}$)" which is a contradiction. This proves the claim.

Now to show that $V^{\mathscr{A}} \models MA_{\aleph_1}$ suppose $V^{\mathscr{S}} \models$ "$\mathring{Q}$ is ccc" and $V^{\mathscr{S}} \models$ "$\mathring{D}_\alpha$ is a dense subset of $\mathring{Q}$" for $\alpha \in \omega_1$. Let T_α be the collection of all $\mathring{q}$ such that $V^{\mathscr{S}} \models \mathring{q} \in \mathring{D}_\alpha$. One easily checks that $V^{\mathscr{A}} \models$ "T_α is a dense subset of $\mathring{Q}/G_{\mathscr{A}}$". Define E_α to be the set of all $((t, n), \mathring{q}) \in \mathscr{A} * (\mathring{Q}/G_{\mathscr{A}})$ such that $\mathring{q} \in T_\alpha$, and for $i \in \omega$ let C_i be the set of all $((t, n), \mathring{p}) \in \mathscr{A} * (\mathring{Q}/G_{\mathscr{A}})$ such that $i < n$ and every element of t of length i has a branching node above it in t of length less than n. E_α and C_i are dense in $\mathscr{A} * (\mathring{Q}/G_{\mathscr{A}})$.

To conclude the proof of $V^{\mathscr{S}} \models MA_{\aleph_1}$ by showing $V^{\mathscr{S}} \models$ "there is a filter on $\mathring{Q}$ which intersects each $\mathring{D}_\alpha$", assume $t \in \mathscr{S}$. Let F be a filter on $\mathscr{A} * (\mathring{Q}/G_{\mathscr{A}})$ which contains $((t, 0), 1)$ and meets each E_α and C_i. Let t^* be the intersection of all the t' such that $((t', n), \mathring{p}) \in G$ for some n and $\mathring{p}$. t^* is in $\mathscr{S}$ since F meets each C_i. Let F^* be the collection of all $\mathring{q} \in T$ such that $((t', n), \mathring{q}) \in F$ for some t' and n. Define $\mathring{F}$ in $V^{\mathscr{S}}$ so that $V^{\mathscr{S}} \models \mathring{F}$ "consists of the interpretations of the elements of F^*". One easily checks that $t^* \leqslant t$ and $t^* \Vdash$ "$\mathring{F}$ is a filter on $\mathring{Q}$ which meets each $\mathring{D}_\alpha$".

Corollary. PFA *implies* $V^{\mathscr{S}} \Vdash MA_{\aleph_1}$.

Proof. The elements of $\mathscr{S} * \text{ccc}$ and $\mathscr{A} * \text{ccc}$ are all proper partial orderings so PFA implies $MA_{\aleph_1}(\mathscr{S}*\text{ccc}, \mathscr{A}*\text{ccc})$.

Section 4. The following theorem establishes the consistency of ZFC+$MA_{\aleph_1}$ $(\mathscr{S}*\text{ccc}, \mathscr{A}*\text{ccc})$ relative to that of ZFC.

Theorem. *There is a partial ordering which forces* $MA_{\aleph_1}(\mathscr{S}*\text{ccc}, \mathscr{A}*\text{ccc})$.

The rest of this section is devoted to proving this theorem. By a preliminary forcing we may assume the ground model satisfies CH and $\diamondsuit_{\omega_2}(\text{cof}\,\omega_1)$.

The argument is an iteration of length ω_2 with countable supports in which generic objects are added stage by stage for all possible elements of $\mathscr{S} * \text{ccc}$ and $\mathscr{A} * \text{ccc}$. Since the $\mathscr{S}$ and $\mathscr{A}$ of the final model will not be available at earlier stages, a reflection argument will be needed to see that the factors used provide the necessary filters to witness $MA_{\aleph_1}(\mathscr{S}*\text{ccc}, \mathscr{A}*\text{ccc})$. The $\diamondsuit_{\omega_2}(\text{cof}\,\omega_1)$-sequence will be used to guess the factors so it will be necessary to code names of partial orderings as sets of ordinals.

Let g: ON$\times$ON$\to$ON be Gödel's pairing function.

Lemma 1. *Assume* $\varkappa$ *is a regular cardinal and* $S_\alpha(\alpha \in E)$ *is a* $\diamondsuit_\varkappa(E)$-*sequence. If* P *is a partial ordering which is* $\varkappa$-*c.c. and* $D = \{d_\gamma\colon \gamma \in \varkappa\}$ *is dense in* P, *then* $V^P \models$ "$\mathring{R}_\alpha$ $(\alpha \in E)$ *is a* $\diamondsuit_\varkappa(E)$-*sequence*" *where* $R_\alpha \in V^P$ *is defined so that* $V^P \models$ "*for any* β, $\beta \in \mathring{R}_\alpha$ *iff there is some* $d_\gamma \in G_P$ *with* $g(\gamma, \beta) \in S_\alpha$".

Proof. Suppose $V^P \models$ "$\mathring{C}$ is a club subset of $\varkappa$ and $\mathring{X}$ is a subset of $\varkappa$". An $\alpha \in E$ can be found such that $V^P \models$ "$\mathring{X} \cap \alpha = \mathring{R}_\alpha$ and $\alpha \in \mathring{C}$" as follows.

Without loss of generality $V^P \models$ "if $\alpha \in \mathring{C}$ and $\beta < \alpha$, then $\beta \in \mathring{X}$ iff there is $\gamma < \alpha$ such that $d_\gamma \in G_P$ and $d_\gamma \Vdash \beta \in \mathring{X}$" and $V^P \models$ "each element of $\mathring{C}$ is closed under g". Let C be the set of $\alpha \in \varkappa$ such that $V^P \models \alpha \in \mathring{C}$. C is a club subset of $\varkappa$. There is an $\alpha \in C$ such that for all $\beta, \gamma \in \alpha$, $g(\beta, \gamma) \in S_\alpha$ iff $d_\gamma \Vdash \beta \in \mathring{X}$. By definition of $\mathring{R}_\alpha$, $V^P \models$ "$\mathring{R}_\alpha = \mathring{X} \cap \alpha$", giving the lemma.

So we will view a $\diamondsuit_{\omega_2}(\text{cof}\,\omega_1)$ sequence as a name for a $\diamondsuit_{\omega_2}(\text{cof}\,\omega_1)$ sequence. We also need to code an ordering in $\mathscr{S} * \text{ccc}$ or $\mathscr{A} * \text{ccc}$ by a set of ordinals.

Fix a coding of triples (p, α, β), where $p \in \mathscr{S} \cap \mathscr{A}$ and $\alpha, \beta \in \omega_1$, by subsets of ω_1. More specifically, this is a function which maps $\mathscr{P}(\omega_1)$ onto $(\mathscr{S} \cup \mathscr{A}) \times \omega_1 \times \omega_1$ which is absolute between models of ZFC.

Suppose $V^{\mathscr{S}} \models$ "$\mathring{Q}$ is a partial ordering of ω_1". A subset X of an ordinal $\alpha = \omega_1 \cdot \beta$ is said to *code* $\mathscr{S} * \mathring{Q}$ if for all $t \in \mathscr{S}$ and $\alpha, \beta \in \omega_1$

$$t \Vdash \text{“}\alpha \leqslant \beta \text{ in } \mathring{Q}\text{”}$$

iff

$$\text{one of the } \beta \text{ many } \omega_1\text{-blocks of } X \text{ codes } (t, \alpha, \beta).$$

Note that whether X codes $\mathscr{S} * \mathring{Q}$ apparently depends on α.

Similarly, define when a subset of an ordinal of the form $\omega_1 \cdot \beta$ codes $\mathscr{A} * \mathring{Q}$ for some $\mathring{Q}$ with $V^{\mathscr{A}} \models$ "$\mathring{Q}$ is a partial ordering of ω_1".

Fix a $\diamondsuit_{\omega_2}(\mathrm{cof}\,\omega_1)$-sequence $S\ (\alpha \in E)$ and assume CH for the rest of this section.

Define an iteration $P_\alpha (\alpha \leqslant \omega_2)$ with countable supports with factors $\mathring{Q}_\alpha (\alpha < \omega_2)$ along with an enumeration δ_α of a dense subset of P_α for $\alpha \leqslant \omega_2$ such that

(1) If $\alpha < \omega_2$, then the domain of δ_α is an ordinal less than ω_2.

(2) If $\alpha < \beta \leqslant \omega_2$, then δ_β extends δ_α.

(3) If $\lambda \leqslant \omega_2$ has uncountable cofinality, then δ_γ is the union of the δ_α with $\alpha < \lambda$.

(4) Assume $\alpha < \omega_2$ has cofinality ω_1. Let $\mathring{R}_\alpha \in V^{P_\alpha}$ be such that $V^{P_\alpha} \models$ "$\mathring{R}_\alpha$ is the subset of α satisfying $\beta \in \mathring{R}_\alpha$ iff there is some $\delta_\alpha(\xi) \in G_{P_\alpha}$ with $g(\xi, \beta) \in S_\alpha$". $\mathring{Q}_\alpha$ is given by

(a) $V^{P_\alpha} \models$ "if $\mathring{R}_\alpha$ codes an element of $\mathscr{S} * \mathrm{ccc}$ or $\mathscr{A} * \mathrm{ccc}$, then $\mathring{Q}_\alpha$ is this ordering".

(b) $V^{P_\alpha} \models$ "if $\mathring{R}_\alpha$ codes some $\mathscr{S} * Q$ not in $\mathscr{S} * \mathrm{ccc}$, then $\mathring{Q}_\alpha$ is $\mathscr{A}$ restricted to $(t, 0)$ where t forces that Q is not c.c.c.".

(c) $V^{P_\alpha} \models$ "if $\mathring{R}_\alpha$ codes some $\mathscr{A} * Q$ not in $\mathscr{A} * \mathrm{ccc}$, then $\mathring{Q}_\alpha$ is $\mathscr{A}$ restricted to (t, n) where (t, n) forces that Q is not c.c.c.".

(d) $V^{P_\alpha} \models$ "if $\mathring{R}_\alpha$ does not code some $\mathscr{S} * Q$ or $\mathscr{A} * Q$, then $\mathring{Q}_\alpha$ is trivial".

(5) If $\alpha < \omega_2$ does not have cofinality ω_1, $\mathring{Q}_\alpha$ is trivial.

FACTS. Assume CH. If $P_\alpha (\alpha \leqslant \lambda)$ is any iteration with countable supports such that the αth factor $\mathring{Q}_\alpha$ satisfies $V^{P_\alpha} \models$ "$\mathring{Q}_\alpha$ is proper and has size $\aleph_1$" for $\alpha < \lambda$, then

(1) P_λ preserves ω_1.

(2) If $\lambda < \omega_2$, then P_λ has a dense subset of size $\aleph_1$.

(3) If $\lambda \leqslant \omega_2$, then P_λ is $\aleph_2$-c.c.

Note that from (1) and (2) (and CH), if $\lambda < \omega_2$, then $V^{P_\lambda} \models \mathrm{CH}$. Also, if X is a subset of ω_1 which codes an element Q of $\mathscr{S} * \mathrm{ccc}$ or $\mathscr{A} * \mathrm{ccc}$, then Q has a dense subset of size $\aleph_1$.

Since elements of $\mathscr{S} * \mathrm{ccc}$ and $\mathscr{A} * \mathrm{ccc}$ are proper, the facts above imply that P_{ω_2} preserves both ω_1 and ω_2 and $V^{P_{\omega_2}} \models 2^{\aleph_0} = \aleph_2$. By the lemma $V^{P_{\omega_2}} \models$ "$\mathring{R}_\alpha (\alpha \in \mathrm{cof}\,\omega_1)$ is a $\diamondsuit_{\omega_2}(\mathrm{cof}\,\omega_1)$-sequence".

LEMMA 2. (ZFC) *Assume P is a partial ordering, $\varkappa$ is an infinite cardinal and let $\mathscr{C}$ be the collection of all $P * \mathring{Q}$ such that $V^P \models$ "$\mathring{Q}$ is a* c.c.c. *partial ordering of $\varkappa$". $\mathrm{MA}_\varkappa(\mathscr{C})$ implies $\mathrm{MA}_\varkappa(P * \mathrm{ccc})$.*

Proof. The proof is a modification of the corresponding statement for $\mathrm{MA}_\varkappa$. Assume $\mathrm{MA}_\varkappa(\mathscr{C})$. Note P preserves all cardinals $\leqslant \varkappa$.

Suppose $V^P \models$ "$\mathring{Q}$ is c.c.c" and D_α is a dense subset of $P * \mathring{Q}$ for $\alpha \in \varkappa$. Define $\mathring{E}_\alpha \in V^P$ for $\alpha \in \varkappa$ so that $V^P \models$ "$\mathring{E}_\alpha$ is the subset of $\mathring{Q}$ determined by $q \in \mathring{E}_\alpha$ iff there is some $(p, d) \in D_\alpha$ such that $p \in G_P$ and the interpretation of d is q" $V^P \models$ "$\mathring{E}_\alpha$ is dense in $\mathring{Q}$ for $\alpha \in \omega_1$". By a Lowenheim–Skolem argument in V^P there is some $\mathring{Q}_*$ such that $V^P \models$ "$\mathring{Q}_*$ is a c.c.c. subordering of $\mathring{Q}$ of size $\leqslant \varkappa$ and $\mathring{E}_\alpha$ is dense in $\mathring{Q}_*$ for $\alpha \in \varkappa$". Let $D'_\alpha = D_\alpha \cap P * \mathring{Q}_*$ for $\alpha \in \varkappa$. By $\mathrm{MA}_\varkappa(\mathscr{C})$ there is a filter G' on $P * \mathring{Q}_*$ which meets each D'_α. If G is the filter on $P * \mathring{Q}$ generated by G' then G meets each D_α.

Fix a generic filter G on P_{ω_2} for the rest of this section.

CLAIM. $V[G] \models \mathrm{MA}_{\aleph_1}(\mathscr{S} * \mathrm{ccc})$.

Proof. Work in $V[G]$. Let $G_\alpha = G \cap P_\alpha$ and $\mathscr{S}_\alpha = \mathscr{S} \cap V[G_\alpha]$, the version of Sacks forcing in $V[G_\alpha]$.

Suppose $V[G]^{\mathscr{S}} \models$ "$\mathring{Q}$ is a c.c.c partial ordering" and D_ξ is dense in $\mathscr{S} * \mathring{Q}$ for $\xi \in \omega_1$. There is no loss of generality in assuming $V[G]^{\mathscr{S}} \models$ "$\mathring{Q}$ has universe ω_1", by the previous lemma, and that the elements of the D_ξ are of the form (t, α) where $\alpha \in \omega_1$. Choose a subset X of ω_2 which codes $\mathscr{S} * \mathring{Q}$.

Since $\mathscr{S}_\lambda$ is the union of the $\mathscr{S}_\alpha$ with $\alpha < \lambda$ whenever λ has uncountable cofinality, there is a club set of λ's in ω_2 such that when λ in the set has cofinality ω_1,

(a) $X \cap \lambda$ is in $V[G_\lambda]$ and codes $\mathscr{S}_\lambda * \mathring{Q}_*$ in $V[G_\lambda]$ for some Q_*.

(b) for all $t \in \mathscr{S}_\lambda$ and $\alpha, \beta \in \omega_1$, $t \Vdash$ "$\alpha \leqslant \beta$ in $\mathring{Q}$" iff $t \Vdash$ "$\alpha \leqslant \beta$ in $\mathring{Q}_*$",

(c) for all $\xi \in \omega_1$, $D_\xi \cap \mathscr{S}_\lambda * \mathring{Q}_* (= D_\xi \cap \mathscr{S}_\lambda \times \omega_1)$ is in $V[G_\lambda]$ and is dense in $\mathscr{S}_\lambda * \mathring{Q}_*$.

Fix such a λ so that the interpretation of $\mathring{R}_\lambda$ is $X \cap \lambda$.

We first suppose $V[G_\lambda] \models$ "$\mathscr{S}_\lambda * \mathring{Q}_*$ is in $\mathscr{S} * \mathrm{ccc}$". By condition (iv) on the iteration there is a $V[G_\lambda]$-generic filter F on $\mathscr{S}_\lambda * \mathring{Q}_*$ in $V[G]$. $F \cap \mathscr{S}_\lambda \times \omega_1$ generates a filter on $\mathscr{S} * \mathring{Q}$ which meets each D_ξ.

Suppose now that $V[G_\lambda] \models$ "$\mathscr{S}_\lambda * \mathring{Q}_*$ is not in $\mathscr{S} * \mathrm{ccc}$". By condition (4)(b) on the iteration there is a $V[G_\lambda]$-generic filter F on $\mathscr{A}_\lambda = \mathscr{A} \cap V[G_\lambda]$, the version of Sacks amoeba forcing in $V[G_\lambda]$, which contains a condition $(t, 0)$ where t forces (with respect to $\mathscr{S}_\lambda$ over $V[G_\lambda]$) that $\mathring{Q}_*$ is not c.c.c. Fix $\mathring{q}_\xi \in V[G_\lambda]^{\mathscr{S}_\lambda}$ for $\xi \in \omega_1$ such that $t \Vdash$ "$\mathring{q}_\xi \in Q'$ and $\mathring{q}_\xi$ is incompatible with $\mathring{q}_\eta$" if $\xi, \eta \in \omega_1$ are distinct. The idea now is that the $\mathring{q}_\xi$ should provide an antichain in $\mathring{Q}$ contradicting that it is c.c.c. However, $\mathring{q}_\xi$ comes from $V[G_\lambda]^{\mathscr{S}_\lambda}$ and even if it is interpreted as an element of $V[G]^{\mathscr{S}}$ there is no guarantee that $V[G]^{\mathscr{S}} \models$ "$\mathring{q}_\xi$ is in ω_1". This problem can be remedied by restricting to the condition t^* which is the intersection of all t' such that $(t', n) \in F$ for some n.

For $\xi \in \omega_1$ there is a condition $(t_\xi, n_\xi) \in F$ such that t_ξ weakly determines $\mathring{q}_\xi$ at n_ξ, i.e. for all z in t_ξ of length n_ξ there is an $\alpha \in \omega_1$ such that $(t_\xi)_z \Vdash \mathring{q}_\xi = \alpha$. Choose $\mathring{r}_\xi \in V[G]^{\mathscr{S}}$ such that $t^*_z \Vdash \mathring{r}_\xi = \alpha$ whenever $\alpha \in \omega_1$, $z \in t_\xi$ has length n_ξ, and

$(t_\xi)_z \Vdash \mathring{q}_\xi = \alpha$. One easily verifies that $t^* \Vdash$ "$\mathring{r}_\xi(\xi \in \omega_1)$ is an antichain in $\mathring{Q}$" contradicting $V[G]^{\mathscr{S}} \models$ "$\mathring{Q}$ is c.c.c." This completes the proof of the claim.

Likewise, we have that $V[G] \models MA_{\aleph_1}(\mathscr{A} * \text{ccc})$. The proof is analogous to that for $MA_{\aleph_1}(\mathscr{S} * \text{ccc})$ and is left to the reader.

As to the problem of getting $V[x]$, with $x \notin V$ a real, to satisfy stronger versions of Martin's axiom than $MA_{\aleph_1}$, Velickovic and Todorcevic have negative results. Velickovic derives that if $\omega_2^{V[x]} = \omega_2^V$, then PFA^+ and SPFA fail in $V[x]$. Namely, by Baumgartner ([1]), Foreman, Magidor and Shelah ([3]), and Shelah ([10]), each of PFA^+ and SPFA implies that for every stationary $S \subseteq [\omega_2]^{\aleph_0}$ there is an $\alpha < \omega_2$ such that $S \cap [\alpha]^{\aleph_0}$ is stationary in $[\alpha]^{\aleph_0}$. However, since $\omega_2^V = \omega_2^{V[x]}$, it follows from Gitik ([4]) that $[\omega_2]^{\aleph_0} \cap (V[x]-V)$ is a stationary subset of $[\omega_2]^{\aleph_0}$ in $V[x]$, which clearly doesn't reflect as above to any $\alpha < \omega_2$. See [11] for related results. Todorcevic has shown that if x is a Sacks real, then PFA fails in $V[x]$, in fact $V[x] \models$ not $MA_{\aleph_1}(2^{<\omega_1} * \text{ccc})$, where $2^{<\omega_1}$ is the usual poset for adding a subset of ω_1 with countable conditions.

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