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Configurations of points in sets of positive measure and in Baire sets of second category

by

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Abstract. Let $E_1, \ldots, E_r$ be subsets of a topological group G. If G is locally compact Hausdorff and every E_i has positive Haar measure, or if G is simply a topological group and the E_i's are Baire sets of second category, then there exist non-empty open subsets $V_1, \ldots, V_r$ of G such that any configuration $x_1, \ldots, x_r$, with $x_i \in V_i$ for all i, admits a translation by some element t of G such that $x_i t \in E_i$ holds for all i. We prove this and related facts which generalize some classical results.

Introduction. A well-known result of Steinhaus [4] says that if A and B are sets of positive Lebesgue measure in the real line then the difference set $A-B$ has non-empty interior. This was a strengthening of an earlier result of Sierpiński. Steinhaus also proved stronger and more general results about configurations of points lying in linear sets of positive measure. In Section 1 of this paper we prove results which greatly generalize and unify the results of Steinhaus, and the approach followed leads to considerably simpler proofs. Our results also easily imply the following result (see Bick [2]): if E is a set of positive lebesgue measure in $\mathbf{R}^n$ and C is a finite set of points in $\mathbf{R}^n$ then E has a subset C^* which is similar to C, in the sense that C can be transformed into C^* by applying a rigid motion followed by a 'radial' scaling down with respect to some point. In particular, the set E contains the vertices of some equilateral triangle, of some square, etc. Analogous to Steinhaus' result is the result of Piccard (quoted in Oxtoby [3]) which says that if A and B are Baire sets of second category in $\mathbf{R}$ then $A-B$ has non-empty interior. This was generalized to topological groups by Bhaskara Rao and Bhaskara Rao [1]. We present, in Section 2, a version which relates to the existence of configurations of points.

1. Configurations in sets of positive measure. Let X be a topological space $\mathscr{B}$ a σ-algebra of subsets of X, μ a non-negative, finitely additive and countably subadditive set function on $\mathscr{B}$ with $\mu(\emptyset) = 0$. Let $\mathscr{S}$ be the set of elements of $\mathscr{B}$ on which μ is finite. Equip $\mathscr{S}$ with the pseudo-metric $d(A, B) = \mu(A \triangle B)$. Let $\mathscr{K}$ be the set of all compact sets belonging to $\mathscr{S}$ and which satisfy

$$\mu(K) = \inf\{\mu(V)\colon\ V \text{ open},\ V \in \mathscr{B} \text{ and } V \supset K\}.$$

Let G be a topological space, $G\times X\to X$ a continuous map which has the following properties: (a) for any $E\in\mathscr{S}$, and any $g\in G$ we have $gE\in\mathscr{S}$ and $\mu(gE)=\mu(E)$ and (b) if $g\in G$ and $E\in\mathscr{K}$ then $gE\in\mathscr{K}$. Thus we have a map $\varphi\colon G\times\mathscr{S}\to\mathscr{S}\colon (g,E)\to gE$, which takes $G\times\mathscr{K}$ into $\mathscr{K}$.

LEMMA 1. *φ is continuous on $G\times\overline{\mathscr{K}}$.*

Proof. Let $g,h\in G$ and $E,F\in\overline{\mathscr{K}}\subset\mathscr{S}$. Given $K\in\mathscr{K}$ and open $U\supset gK$ with $U\in\mathscr{S}$, there is a neighborhood Ω of g such that $U\supset\Omega K$. Then for any $h\in\Omega$:

$$d(gE,hF)\leqslant d(gE,hE)+d(E,F)\leqslant 2d(E,K)+d(E,F)+d(gK,hK)$$
$$\leqslant 2d(E,K)+d(E,F)+2d(gK,U).$$

It is easy to see from this that φ is continuous on $G\times\overline{\mathscr{K}}$. ∎

If Y is a topological space then we denote by ΠY the countable product of Y with itself, equipped with the "box-product" topdogy. We can form the product map $\Phi\colon \Pi(G\times\mathscr{S})\to\Pi\mathscr{S}$ defined by $\Phi(\{x_\alpha\})=\{\varphi(x_\alpha)\}$. Clearly Φ is continuous. The map $m\colon \Pi\mathscr{S}\to[0,\infty)\colon (E_i)\to\mu(\cap_{k\geqslant 1}E_k)$ is continuous since:

$$|\mu(\cap_{k\geqslant 1}E_k)-\mu(\cap_{k\geqslant 1}F_k)|\leqslant\Sigma_{k\geqslant 1}\mu(E_k\triangle F_k).$$

Thus we have proved:

PROPOSITION. *The composite map $m\Phi\colon \Pi(G\times\mathscr{S})\to[0,\infty)$, taking $(g_k,E_k)_{k\geqslant 1}$ to $\mu(\cap_{k\geqslant 1}g_kE_k)$, is continuous. In particular, if, in the above setting, $\mu(\cap_{k\geqslant 1}g_kE_k)>0$, then there exists, for $k\geqslant 1$, a neighborhood V_k of g_k such that for any sequence $(x_k)_{k\geqslant 1}$ with $x_k\in V_k$ for all k, we have $\mu(x_kE_k)>0$.*

Specializing more, we have the following corollaries. Recall that if μ is a measure on a σ-algebra $\mathscr{B}$ of subsets of a set X then, for any $E\subset X$, $\mu*(E)$ is defined to be $\sup\{\mu(K)\colon K\in\mathscr{B}$ and $K\subset E\}$.

COROLLARY 1. *Let μ be a Borel measure on a Hausdorff space X such that every compact subset is regular, let G be a topological space, $G\times X\to X\colon (g,x)\to gx$ a continuous map, e an element of G having the property that $ex=x$ for all $x\in X$. Suppose that for any Borel set $E\subset X$, and any $g\in G$, gE is also a Borel set and $\mu(gE)=\mu(E)$, Let $E\subset X$ and $0<\mu*(E)$. Then there exists a sequence (V_n) of neighborhoods of e in G such that for any sequence (x_n) satisfying $x_n\in V_n$, the following holds:*

$$\mu*(\cap_{n\geqslant 1}x_nE)>0.$$

Proof. Follows from the proposition on setting all E_k's equal to some compact subset $K\subset E$, with $\mu(K)>0$. ∎

COROLLARY 2. *Let μ be left Haar measure on a locally compact Hausdorff topological group G, E a subset of G with $\mu*(E)>0$. Then there exists a sequence (V_n) of neighborhoods of e such that for any sequence (x_n) with $x_n\in V_n$, we have $\mu_*(\{t\in G\colon x_nt\in V_n$ for all $n\geqslant 1\})>0$. In particular, if $\mu*(E)>0$, then there exists a sequence (V_n) of neighborhoods of e such that for any sequence (x_n) with $x_n\in V_n$, a configuration (x_nt) lies in E.*

Proof. Immediate consequence of Corollary 1, noticing that

$$\cap_{n\geqslant 1}x_n^{-1}E=\{t\in G\colon x_nt\in E \text{ for all } n\}. \blacksquare$$

Corollary 2 gives the following fact: if E is a set of positive Lebesgue measure in $\boldsymbol{R}^n$ then there is a sequence of neighborhoods V_k of 0 in $\boldsymbol{R}^n$ such that for any sequence (x_k) satisfying $x_k\in V_k$ for each k, there is some $t\in\boldsymbol{R}^n$ such that each $x_k+t\in E$. Therefore, if $C=\{x_1,\ldots,x\}$ is any finite set of points in $\boldsymbol{R}^n$, we get $\lambda x_i\in V_i$ by taking $\lambda\in\boldsymbol{R}$ small enough and then translate by $t\in\boldsymbol{R}^n$ to obtain $\lambda x_i+t\in E$. This gives a simple proof of the result mentioned in the introduction.

For corollary 3 we need a lemma:

LEMMA 2. *Let μ be (left) Haar measure on a locally compact Hausdorff topological group G and $E_1,\ldots,E_n$, $n\geqslant 2$, subsets of G with $\mu*(E_i)>0$ for each i. Then there exist $g_2,\ldots,g_n\in G$ such that $\mu*(E_1\cap g_2E_2\cap\ldots\cap g_nE_n)>0$.*

Proof. We prove the case $n=2$, the general result following by induction. Let K_1 and K_2 be compact subsets of G. We have:

$$\mu(K_1)\mu(K_2)=(\mu\times\mu)\{(x,y)\colon x\in K_1 \text{ and } x^{-1}y\in K_2\}$$
$$=\int_G\mu(K_1\cap yK_2^{-1})\,d\mu(y).$$

($\mu\times\mu$ is the regular Borel product measure.)

Setting $K_1=K_2$ shows that $\mu(K_2^{-1})>0$ if $\mu(K_2)>0$. Returning to the general situation (where K_1 need not equal K_2) and taking K_2^{-1} in place of K_2 we see that if $\mu(K_1)>0$ and $\mu(K_2)>0$ then, for some $y\in G$, $\mu(K_1\cap yK_2)>0$.

The lemma is now clear. ∎

COROLLARY 3. *Let μ be Haar measure on a locally compact Hausdorff topological group G and, for each $K\geqslant 1$, let E_k be a subset of G with $\mu*(E_k)>0$. Then for each $n\geqslant 1$, there exists a family of non-empty open sets V_k, $1\leqslant k\leqslant n$, such that for any sequence $(x_k)_{1\leqslant k\leqslant n}$, with $x_k\in V_k$ for all k, we have*

$$\mu*(\cap_{1\leqslant k\leqslant n}x_kE_k)>0.$$

We can choose an infinite sequence (x_k) such that $\cap_{k\geqslant 1}x_kE_k\neq\emptyset$.

Proof. The first part is immediate from Lemma 2 and the proposition (applied to suitably chosen compact subsets of the E_k's). For the second part, we may assume, without loss of generality, that each E_k is compact. Applying the lemma we can choose a sequence $(x_k)_{k\geqslant 2}$ such that

$$\mu(E_1\cap x_2E_2\cap\ldots\cap x_nE_n)>0 \quad \text{for every } n\geqslant 2.$$

The result now follows by the finite-intersection property for compact sets. ∎

Corollary 3 generalizes Steinhaus's results.

2. Configurations in second category Baire sets. Let X be a topological space. The Banach Category theorem (see Oxtoby [3]) says that the union of any family of open sets of first category is of first category. A set $B \subset X$ is called a *Baire set of second category* if B is of the form $U \triangle N$, where U is open and N is of first category, and B is of second category. Note that this implies that no topological group of second category contains a non-empty open subset of first category.

We need the following analog of our previous lemma:

LEMMA. *If $U_1, \ldots, U_n$ are non-empty open subsets of a topological group G then there exist non-empty open sets $V_1, \ldots, V_n$ such that for any sequence of points $(v_i)_{1 \leqslant i \leqslant n}$ with $v_i \in V_i$, we have $\bigcap_{1 \leqslant i \leqslant n} v_i U_i \neq \emptyset$.*

Proof. Choose $g_1, \ldots, g_n \in G$ such that $e \in U = g_1 U_1 \cap \ldots \cap g_n U_n$, where e is the identity element of G. Choose an open neighborhood V of e with $V^{-1} V \subset U$. The lemma now follows on setting $V_k = V g_k$ for each k. For if $(v_i)_{1 \leqslant i \leqslant n}$ satisfies $v_i \in V_i$ for each i, then $\bigcap_{1 \leqslant i \leqslant n} v_i U_i \supset \bigcap_{1 \leqslant i \leqslant n} v_i g_i^{-1} U \supset V$, since each $v_i g_i^{-1} \in V$ so that $V \subset v_i g_i^{-1} U$. ■

The following result is analogous to Corollary 3 of Section 1:

PROPOSITION. *Let G be a topological group and $E_1, \ldots, E_n$ Baire sets of second category in G. Then there exist non-empty open sets $V_1, \ldots, V_n$ such that for any $g_1, \ldots, g_n$ with each $g_i \in V_i$, we have $g_1 E_1 \cap \ldots \cap g_n E_n \neq \emptyset$.*

Proof. Let, for $1 \leqslant i \leqslant n$, $E_i = U_i \triangle N_i$ where U_i is open in G and N_i is of first category in G. Choose $V_1, \ldots, V_n$ as in the lemma. Then for any $g_1 \in V_1, \ldots, g_n \in V_n$, we have: $g_1 U_1 \cap \ldots \cap g_n U_n \neq \emptyset$.

Now:

$$(*) \qquad g_1 E_1 \cap \ldots \cap g_n E_n \supset [g_1 U_1 \cap \ldots \cap g_n U_n] - [g_1 N_1 \cup \ldots \cup g_n N_n].$$

By the observation made earlier (following the statement of the Banach Category theorem above) the right side of $(*)$ is non-empty. This completes the proof. ■

It would be satisfying to have a complete characterization, in some sense, of sets which contain a suitable class of configurations of points. The following question may be asked: is it true that if a subset E of $\mathbf{R}^n$ has the property that there exists a sequence (V_n) of neighborhoods of 0 such that any sequence (x_n) with $x_n \in V_n$, for all n, has a translate lying in E, then E must have positive Lebesgue measure?

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