On algebras with bases of different cardinalities

by

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Abstract. In the beginning of the sixties, E. Marczewski posed the following problem: if A is an algebra having two bases of different cardinalities, is it true that for all $n \geq 2$ the number p_n of essentially n-ary polynomials in A is greater than zero? In this paper we answer this question showing that actually a much stronger fact is true: for all $n \geq 2$ the number p_n is infinite. .

0. Introduction. Given an algebra A, let $p_n = p_n(A)$ denote the number of essentially n-ary polynomials in A (for $n \geq 1$), and let $p_0 = p_0(A)$ be the number of constants. In connection with a special attention focused on algebras with bases of different cardinalities (cf. [7, 10, 13, 17]) E. Marczewski put foreward the following conjecture (see [16], cf. [14], **P** 527, **P** 528):

(M) if an algebra A has two bases of different cardinalities, then $p_n(A) > 0$ for all $n \geq 2$.

The conjecture can be also formulated in terms of equational logic or in terms of composition of functions, and is still of some interest in connection with investigation of p_n-sequences and spectra of equational theories (see [16] and § 3 below, cf. also [2, 8, 11, 18, 19]). Some partial results connected with this conjecture were obtained in [16, 3, 4]. The condition $p_n > 0$ for other classes of algebras was considered also in [6, 20].

In this paper we use the following four properties of algebras A with bases of different cardinalities:

(P1) the cardinality $|A|$ is greater than 1, i.e. A is nontrivial;

(P2) for some $n > m > 0$ there exist polynomials $g_i(x_1, \ldots, x_m)$, $f_j(x_1, \ldots, x_n)$ over A ($i = 1, \ldots, n$, $j = 1, \ldots, m$) satisfying identically

$$(1)\qquad \begin{aligned} g_i(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) &= x_i, \\ f_j(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)) &= x_j, \end{aligned}$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$ (see [7] or [9], § 31);

(JT) every algebra with the properties P1, P2 (every nontrivial model of identities (1)) is infinite (this is a result of Jònsson and Tarski [10]);

(GR) if an algebra has property P2 with some $n > m > 0$, then it has also property P2 with the same m and some arbitrarily large n (this is by the result of Goetz and Ryll–Nardzewski [7]).

Using some more recent techniques of investigation of composition of functions we prove

THEOREM 1. *If an algebra A has two bases of different cardinalities, then $p_n(A) \geqslant \aleph_0$ for all $n \geqslant 2$. Moreover, one of the following holds*: (i) $p_0(A) = 0$ *and* $p_1(A) = 1$, *or* (ii) $p_0(A) = 0$ *and* $p_1(A) \geqslant \aleph_0$, *or* (iii) $p_0(A) = 1$ *and* $p_1(A) \geqslant \aleph_0$, *or else* (iv) *both* $p_0(A)$, $p_1(A) \geqslant \aleph_0$.

Examples showing that each of the conditions (i)–(iv) can actually hold are provided. An application in equational logic and an interesting problem arising in this connection are presented in § 3.

Our terminology is standard. As a general references we recommend [9, 14] and [19].

1. Proof of Theorem. Our proof splits into three cases: Let C be the subalgebra of A consisting of the constants of A. Clearly, C has property P2, as well. If follows, by JT, that either $|C| \leqslant 1$ or $|C| \geqslant \aleph_0$. Hence, we have the following three possibilities for $p_0(A) = |C|$: $p_0(A) = 0$, $p_0(A) = 1$ or $p_0(A) \geqslant \aleph_0$. In each of these cases a different approach is applied.

Throughout the paper, by A we denote an algebra satisfying P1 and P2, and by g_i, f_i — polynomials of A satisfying (1), with some m fixed, and n suitably large.

Case 1. $p_0(A) = 0$. In this case $p_1(A) = |P^{(1)}(A)|$, where $P^{(1)}(A)$ is the algebra of unary polynomials of A (cf. [9], p. 38). Clearly, $P^{(1)}(A)$ is a model for identities (1), and therefore, by JT, either $p_1(A) \leqslant 1$ or $p_1(A) \geqslant \aleph_0$. On the other hand, since $|A| > 1$, the unary projection is an essentially unary polynomial of A, i.e. $p_1(A) \geqslant 1$. This means that condition (i) or (ii) of Theorem 1 is satisfied.

To prove the first statement of Theorem 1, we modify the proof given in [16] applying an idea used in [4].

Denote by I the idempotent reduct of A, i.e. the algebra generated by all polynomials $f(x_1, \ldots, x_n)$ of A, $n \geqslant 2$, satisfying $f(x, \ldots, x) = x$ for every $x \in A$. Polynomials of I are just idempotent polynomials of A, and therefore $p_n(A) \geqslant p_n(I)$ for all n. On the other hand, by the general result of [12], for idempotent algebras $p_2 \geqslant \aleph_0$ implies $p_n \geqslant \aleph_0$ for all $n \geqslant 2$. Hence, to prove the first statement of Theorem 1 it is enough to show that the number of essentially binary idempotent polynomials in A, $p_2(I) \geqslant \aleph_0$.

To this end define

$$g_i^*(x) = g_i(x, x, \ldots, x) \quad (i = 1, 2, \ldots, n)$$

and for any n-tuple $\alpha = \langle i_1, \ldots, i_n \rangle$ with $i_k \in \{1, 2\}$

$$f_j^\alpha(x_1, x_2) = f_j(g_1^*(x_{i_1}), \ldots, g_n^*(x_{i_n})) \quad (j = 1, 2, \ldots, m). \tag{2}$$

Clearly, f_j^α are binary polynomials of A, and moreover, by virtue of (1), all these polynomials are idempotent, i.e. $f_j^\alpha(x, x) = x$.

Now, let B be the set of all binary idempotent polynomials of A. If we denote $d = p_2(I)$, then $|B| = d+2$, since B contains essentially binary polynomials of I and two projections, in addition. We define a mapping φ of the set S of all n-tuples $\alpha = \langle i_1, \ldots, i_n \rangle$ with $i_k \in \{1, 2\}$ into the set T of all m-tuples $\langle h_1, \ldots, h_m \rangle$ with $h_k \in B$ by the formula

$$\varphi(\alpha) = \langle f_1^\alpha(x_1, x_2), \ldots, f_m^\alpha(x_1, x_2) \rangle.$$

Observe that if $\varphi(\alpha) = \varphi(\beta)$ for some $\alpha = \langle i_1, \ldots, i_n \rangle$ and $\beta = \langle j_1, \ldots, j_n \rangle$, i.e., $f_j^\alpha(x_1, x_2) = f_j^\beta(x_1, x_2)$ for all $j = 1, \ldots, m$, then using (1) we have

$$\begin{aligned} g_k^*(x_{i_k}) &= g_k(f_1^\alpha(x_1, x_2), \ldots, f_m^\alpha(x_1, x_2)) \\ &= g_k(f_1^\beta(x_1, x_2), \ldots, f_m^\beta(x_1, x_2)) = g_k^*(x_{j_k}) \end{aligned} \tag{3}$$

for all $k = 1, \ldots, n$.

Since, by assumption, *there is no constant polynomial in* A (and $|A| > 1$), it follows that $i_k = j_k$ for all $k = 1, \ldots, n$, i.e., $\alpha = \beta$. This means that the mapping φ is one-to-one. Comparing now the cardinalities of the sets S and T we conclude that

$$2^n \leqslant (d+2)^m.$$

Since m and d are fixed, while n can be chosen arbitrarily large, it follows that $d \geqslant \aleph_0$. This completes the proof in the case when $p_0(A) = 0$.

Case 2. $p_0(A) = 1$. In this case $|P^{(1)}(A)| = p_1(A)+1$, and similarly to Case 1, we infer by JT that $p_1(A) \geqslant \aleph_0$. Hence, condition (iii) of Theorem 1 is satisfied.

To prove the first statement denote the unique constant in A by 0 and consider the following unary polynomials of A:

$$s_j^h(x) = f_j(h(x), 0, 0, \ldots, 0),$$

Where $h(x)$ runs over the set $P^{(1)}(A)$, $j = 1, \ldots, m$. Since

$$g_1(s_1^h(x), \ldots, s_m^h(x)) = h(x)$$

and there are infinitely many distinct $h(x)$ in $P^{(1)}(A)$, there must be also infinitely many distinct polynomials in one of the sets

$$S_j = \{s_j^h(x) \colon h \in P^{(1)}(A)\}$$

for some $j = 1, \ldots, m$. Moreover, since $p_0(A) = 1$, all these polynomials, except at most one, are essentially unary.

Using analogous arguments for any i we conclude that for every $i = 1, \ldots, n$ there exists some $j = 1, \ldots, m$ such that the following condition $C(i,j)$ holds:

$C(i,j)$: there are infinitely many $h(x)$ in $P^{(1)}(A)$ such that the functions $f_j(0, \ldots, 0, h(x), 0, \ldots, 0)$ with $h(x)$ standing on ith place are essentially unary and pairwise distinct.

If follows that there is some $j = 1, \ldots, m$ and distinct indicies $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ such that $k \geqslant n/m$ and $C(i,j)$ holds for all $i = i_1, \ldots, i_k$. For simplicity, we assume that $\{i_1, \ldots, i_k\} = \{1, \ldots, k\}$ and for any $r \leqslant k$ we consider polynomials

$$p_j^h(x_1, \ldots, x_r) = f_j(h(x_1), x_2, \ldots, x_r, 0, 0, \ldots, 0)$$

with those $h(x)$ which yield infinitely many distinct essentially unary functions $f_j(h(x), 0, \ldots, 0)$.

All these polynomials p_j^h are essentially r-ary. Indeed, each of them depends on x_1, since by assumption, $f_1(h(x_1), 0, \ldots, 0)$ does. Furthermore, if $2 \leqslant r$, then substituting in $p_j^h(x_1, \ldots, x_r)$, $x_1 = 0$ and $x_3 = \ldots = x_r = 0$, we obtain $f_j(0, x_2, 0, \ldots, 0)$; *here we use the fact that* $p_1(A) = 1$. This expression depends on x_2 by the assumption that $C(2,j)$ holds $(2 \leqslant r \leqslant k)$, and therefore $p_j^h(x_1, \ldots, x_r)$ depends on x_2, as well. By similar arguments, $p_j^h(x_1, \ldots, x_r)$ depends on the remaining variables $x_3, \ldots, x_r$.

On the other hand, the polynomials $p_j^h(x_1, \ldots, x_r)$ in question are pairwise distinct. This is so, because the substitution $x_2 = \ldots = x_r = 0$ yields functions $f_j(h(x_1), 0, 0, \ldots, 0)$, which are pairwise distinct by assumption.

Thus, we have proved that there are infinitely many essentially r-ary polynomials in A, for any $r \leqslant k$.

Since $k \geqslant n/m$, and n can be arbitrarily large, if follows finally that $p_r(A) \geqslant \aleph_0$ for any $r \geqslant 2$, thus completing the proof in Case 2.

Case 3. $p_0(A) \geqslant \aleph_0$. First we have to show that $p_1(A) \geqslant \aleph_0$, which is no longer a consequence of JT in this case. To prove this we modify the proof of Case 1.

Let $0 \in A$ be a fixed constant of A. For any subset α of $\{1, 2, \ldots, n\}$ we denote by $f_j^\alpha(x)$ the function $f_j(y_1, \ldots, y_n)$ where $y_k = x$ for $k \in \alpha$ and $y_k = 0$, otherwise $(j = 1, \ldots, m)$. Each $f_j^\alpha(x)$ is of course a unary polynomial of A. Define a mapping φ of the power set of $\{1, 2, \ldots, n\}$ into a set of m-tuples by the formula

$$\varphi(\alpha) = \langle f_1^\alpha(x), \ldots, f_m^\alpha(x) \rangle .$$

Similarly to Case 1 we prove that φ is one-to-one. (Here, assuming that for some $\alpha \neq \beta$, $\varphi(\alpha) = \varphi(\beta)$, we obtain as in Case 1, $0 = x$, contradicting the fact that $|A| > 1$. Observing that if $f_j^\alpha(x)$ does not depend on x, then

$$f_j^\alpha(x) = f_j(0, 0, \ldots, 0) = C_j \text{ (for any } \alpha),$$

and writing $d = p_1(A)$ we obtain, as in Case 1, the inequality

$$2^n \leqslant (d+1)^m .$$

This yields again $d \geqslant \aleph_0$, as required.

To prove the first statement, we start from the observation that *in this case* we can assume that every element of A is a constant in A. Indeed, if not, we can consider the subalgebra C of constants of A, which clearly has both properties P1 and P2, and $p_n(C) \leqslant p_n(A)$ for all n. (We remark that it is enough to combine this observation with the results of [4] and [5] to get the verification of Marczewski's conjecture, $p_n(A) > 0$.) To prove that also in this case $p_n(A) \geqslant \aleph_0$ we need one more construction.

For any $i = 1, \ldots, n$ we find a polynomial $h_i(x_1, x_2)$ with the properties

(H1) $h_i(x_1, x_2)$ depends on x_2,

(H2) $h_i(x, x) = g_i(x, x, \ldots, x)$

(note that $g_i(x, x, \ldots, x)$ may be a constant).

To this end denote by k_i the least integer $k > 0$ such that $g_i(x_{i_1}, \ldots, x_{i_m})$ is not a constant function for some $i_1, \ldots, i_m \in \{1, \ldots, k\}$. Note that $g_i(x_1, \ldots, x_m)$ is not constant, in view of (1). Note also that the corresponding function $g_i(x_{i_1}, \ldots, x_{i_m})$ for $k = k_i > 1$ depends on all k variables, since otherwise some substitution $x_1 = x_{i_j}$ yields a nonconstant function, contradicting the fact that k is minimal. Denote this function by $g(x_1, \ldots, x_k)$, $k = k_i$.

Now, if $k = 1$, then $g_i(x, x, \ldots, x)$ is a nonconstant function, and $h_i(x_1, x_2) = g_i(x_2, x_2, \ldots, x_2)$ is as required.

If $k \geqslant 2$, then $g_i(x, x, \ldots, x) = C_i$ for some $C_i \in A$. Also, any identification of variables in $g(x_1, x_2, \ldots, x_k) = g(x_{i_1}, \ldots, x_{i_m})$ yields clearly the same C_i, i.e., for any $x_1, \ldots, x_k \in A$ we have

(G) $g(x_1, \ldots, x_k) = C_i$ whenever $x_1, \ldots, x_k$ are not pairwise distinct.

On the other hand, since $g(x_1, \ldots, x_k)$ depends, in particular, on x_1, there are some $a_2, \ldots, a_k$ in A such that $g(x_1, a_2, \ldots, a_k)$ depends on x_1. If follows that

$$h_i(x_1, x_2) = g(x_1, x_2, a_3, \ldots, a_k)$$

depends on x_1, as well, i.e., $h_i(x_1, x_2)$ is nonconstant. Hence, in view of (G), $h_i(x_1, x_2)$ depends on x_2, too. Also, by (G), $h_i(x, x) = C_i = g_i(x, x, \ldots, x)$, and since by assumption any element of A is a constant, $h_i(x_1, x_2)$ is a polynomial of A, as required.

Now, as in Case 1, we are able to show that there are infinitely many binary idempotent polynomials in A. We just modify slightly the proof given in Case 1.

For any m-tuple $\alpha = \langle i_1, \ldots, i_n \rangle$ with $i_k \in \{1, 2\}$ we define (instead of (2))

$$f_j^\alpha(x_1, x_2) = f_j(h_1(x_1, x_{i_1}), \ldots, h_n(x_1, x_{i_n})) \quad (j = 1, \ldots, m) .$$

Each polynomial $f_j^\alpha(x_1, x_2)$ is idempotent in view of H2 and (1). Instead of (3) we now have the identity

$$h_k(x_1, x_{i_k}) = h_k(x_1, x_{j_k}) ,$$

which, in view of H1, yields $i_k = j_k$.

The remaining part of the proof is the same as in Case 1.

The proof of Theorem 1 is thus completed.

2. Examples. Examples below show that each of the conditions (i)–(iv) of Theorem 1 is satisfied by a certain algebra with bases of different cardinalities.

(i) Examples of algebras satisfying $p_0 = 0$ and $p_1 = 1$ are given in [16].

(ii) For an infinite set X consider a one-to-one mapping Φ of X^m to X^n $(m < n)$ such that $\Phi(\langle 0, \ldots, 0\rangle) = \langle 0, \ldots, 0\rangle \in X^n$, $\Phi(\langle 1, \ldots, 1\rangle) = \langle 1, \ldots, 1\rangle \in X^n$ and $\Phi(\langle 2, \ldots, 2\rangle) = \langle 0, 1, \ldots, 1\rangle \in X^n$ for some fixed elements $0, 1, 2 \in X$. Suppose that Φ is given by

$$y_i = g_i(x_1, \ldots, x_m) \quad (i = 1, 2, \ldots, n),$$

$$x_j = f_j(y_1, \ldots, y_n) \quad (j = 1, 2, \ldots, m)$$

and take $F = \{g_1, \ldots, g_n, f_1, \ldots, f_m\}$. Then the algebra $X = \langle X, F\rangle$ has no constants, since for any polynomial $p(x_1, \ldots, x_k)$ of X, $p(0, \ldots, 0) = 0$, while $p(1, \ldots, 1) = 1$. On the other hand, $p_1(X) > 1$, since $g(x) = g_1(x, \ldots, x)$ is a unary polynomial of X other than the unary projection ($g(2) = 0$).

Now, the algebra $P^{(n)}(X)$ of n-ary polynomials of X has bases with m and n elements, and obviously $p_0 = 0$ and $p_1 > 1$ (cf. [9], p. 38). By virtue of Theorem 1, $p_1 \geqslant \aleph_0$, i.e. condition (ii) of Theorem 1 is satisfied.

(iii) Suppose now that Φ is one-to-one mapping of X^m to X^n such that $\Phi(\langle x, \ldots, x\rangle) = \langle 0, x, \ldots, x\rangle \in X^n$ for any $x \in X$, and F is defined as previously. Then $g_1(x, \ldots, x) = 0$ identically and therefore 0 is a constant in $X = \langle X, F\rangle$. Since $p(0, \ldots, 0) = 0$ for any polynomial p of X, there is no other constant in X, i.e. $p_0 = 1$. As in Example (ii), $P^{(n)}(X)$ has bases with m and n elements, and $p_0 = 1$, which by Theorem 1 means that condition (iii) holds.

(iv) Finally, an algebra $P^{(n)}(X)$ with bases of different cardinalities and $p_0 > 1$ is constructed as in the previous examples starting from a mapping Φ of X^m to X^n which satisfies $\Phi(\langle x, \ldots, x\rangle) = \langle 0, 1, x, \ldots, x\rangle$ $(n > 2)$. Then 0 and 1 are constants in $X = \langle X, F\rangle$, and by virtue of Theorem 1, $p_0 \geqslant \aleph_0$ for $P^{(n)}(X)$.

3. Application in equational logic. Examining our proof of Theorem 1 one can see that we have in fact proved the following

THEOREM 2. *In every nontrivial model A of the identities* (1) *with any $n > m > 0$, $p_k(A) \geqslant \aleph_0$ for all $k \geqslant 2$. Moreover, one of conditions* (i)–(iv) *of Theorem* 1 *holds.*

This theorem can be viewed as generalization of the result of Jònsson and Tarski (JT). Turning to more recent terminology, given a variety V of algebras, the spectrum of V, $\mathbf{spec}(V)$, is the set of the cardinalities of finite members of V; in symbols

$$\mathbf{spec}(V) = \{n \in \omega\colon (\exists A \in V)|A| = n\}.$$

JT states that the variety given by (1) satisfies the condition $\mathbf{spec}(V) = \{1\}$. This peculiar condition has been also considered by other authors (see e.g. [1, 15, 18 (p. 382), 19 (p. 38)]); the varieties given by (1) with $n > m > 0$ are the most typical examples of nontrivial varieties satisfying this condition. In this connection the following question arises.

PROBLEM. Is it true that for a variety V, $\mathbf{spec}(V) = 1$ if and only if for every nontrivial A in V and every $n \geqslant 2$, $p_n(A)$ is infinite?

The problem to find a counterexample to it, in somewhat different terminology, was asked by J. Dudek during the conference in Wien, 1984. The most natural candidate for examining is, of course, the well-known Austin's variety of groupoids [1] given by the single identity

$$\big((y^2y)x\big)\big((y^2(y^2y))z\big) = x.$$

But even in this particular case the problem seems to be not easy. This illustrates difficulties which can arise even in very special problems on p_n-sequences. On the other hand, note that the condition $\mathbf{spec}(V) = 1$ involves no identities and polynomials. Therefore, if the answer to our problem is positive, it is likely that new nonstandard techniques have to be applied to prove it (cf. [11], where results and methods of theory of finite groups were applied to solve other general problem on p_n-sequences).

Added in proof (December 19, 1989). The problem above has a solution in the negative. There exists a variety V of algebras in type $\langle 2,2\rangle$ such that $\mathbf{spec}(V) = \{1\}$, $p_2(V) = \aleph_0$ and $p_n(V) = 0$ for all $n > 2$. In Austin's variety $p_n(V) = \aleph_0$ for all $n \geqslant 2$. For these and related results see A. Kisielewicz, *Varieties of algebras with no nontrivial finite members*, in: *Lattices, Semigroups, and Universal Algebra*, J. Almeida et al. (eds.), Plenum Press, to appear.

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Configurations of points in sets of positive measure and in Baire sets of second category

by

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Abstract. Let $E_1, \ldots, E_r$ be subsets of a topological group G. If G is locally compact Hausdorff and every E_i has positive Haar measure, or if G is simply a topological group and the E_i's are Baire sets of second category, then there exist non-empty open subsets $V_1, \ldots, V_r$ of G such that any configuration $x_1, \ldots, x_r$, with $x_i \in V_i$ for all i, admits a translation by some element t of G such that $x_i t \in E_i$ holds for all i. We prove this and related facts which generalize some classical results.

Introduction. A well-known result of Steinhaus [4] says that if A and B are sets of positive Lebesgue measure in the real line then the difference set $A-B$ has non-empty interior. This was a strengthening of an earlier result of Sierpiński. Steinhaus also proved stronger and more general results about configurations of points lying in linear sets of positive measure. In Section 1 of this paper we prove results which greatly generalize and unify the results of Steinhaus, and the approach followed leads to considerably simpler proofs. Our results also easily imply the following result (see Bick [2]): if E is a set of positive lebesgue measure in $\boldsymbol{R}^n$ and C is a finite set of points in $\boldsymbol{R}^n$ then E has a subset C^* which is similar to C, in the sense that C can be transformed into C^* by applying a rigid motion followed by a 'radial' scaling down with respect to some point. In particular, the set E contains the vertices of some equilateral triangle, of some square, etc. Analogous to Steinhaus' result is the result of Piccard (quoted in Oxtoby [3]) which says that if A and B are Baire sets of second category in $\boldsymbol{R}$ then $A-B$ has non-empty interior. This was generalized to topological groups by Bhaskara Rao and Bhaskara Rao [1]. We present, in Section 2, a version which relates to the existence of configurations of points.

1. Configurations in sets of positive measure. Let X be a topological space $\mathscr{B}$ a σ-algebra of subsets of X, μ a non-negative, finitely additive and countably subadditive set function on $\mathscr{B}$ with $\mu(\emptyset) = 0$. Let $\mathscr{S}$ be the set of elements of $\mathscr{B}$ on which μ is finite. Equip $\mathscr{S}$ with the pseudo-metric $d(A, B) = \mu(A \triangle B)$. Let $\mathscr{K}$ be the set of all compact sets belonging to $\mathscr{S}$ and which satisfy

$$\mu(K) = \inf\{\mu(V)\colon\ V \text{ open},\ V \in \mathscr{B} \text{ and } V \supset K\}.$$