Added in proof. Problems 3.11 and 3.12 are already answered in the recent paper: R. D. Buskirk, J. Nikiel and E. D. Tymchatyn, *Totally regular curves as inverse limits*, preprint. Namely:

1) each totally regular continuum is the limit of an inverse sequence of connected graphs with monotone bonding surjections;

2) there exist totally (even: completely) regular continua with connected subsets which are not arcwise connected.

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Separating collections

by

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Abstract. A collection $\mathscr{X}$ of sets is said to be a *separating collection* if it satisfies the following: whenever f and g are two functions defined on a set $X \in \mathscr{X}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathscr{X}$, $Y \subseteq X$ such that $f''Y \cap g''Y = 0$. We characterize separating collections in terms of a weak version of the partition relation $\mathscr{X} \to (\mathscr{X}, 3)^2$ and we show this new partition relation holds for every countable indecomposable ordinal (although it can fail for indecomposable ordinals of cardinality ω_1). We also characterize one-one separating collections (i.e., those where we only consider functions f and g that are one-to-one) and derive from this some known and new results.

1. Introduction. If I is a collection of subsets of an infinite cardinal $\varkappa$, then I is said to be an *ideal on* $\varkappa$ if I is closed under subset formation and finite unions (i.e., if $X, Y \in I$ and $Z \subseteq X \cup Y$, then $Z \in I$). A subset of $\varkappa$ not in I is said to have *positive I-measure* and the collection of such sets is denoted by I^+; a subset of $\varkappa$ whose complement belongs to I is said to have *I-measure one* and the collection of such sets is denoted by I^*. The following definitions generalize some ideal theoretic notions from [MPT].

DEFINITION 1.1. A collection $\mathscr{X}$ of infinite sets is said to be:

(i) a *separating collection* if for every pair of functions f and g defined on a set $X \in \mathscr{X}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathscr{X}$ such that $Y \subseteq X$ and $f''Y \cap g''Y = 0$.

(ii) a *one-one separating collection* if for every pair of one-to-one functions f and g defined on a set $X \in \mathscr{X}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathscr{X}$ such that $Y \subseteq X$ and $f''Y \cap g''Y = 0$.

If I is an ideal on $\varkappa$ and I^+ is a separating collection (or a one-one separating collection), then we will refer to I as a *separating ideal* (or a *one-one separating ideal*). A uniform ultrafilter U on $\varkappa$ that is a separating collection is referred to as a *separating ultrafilter*. The following easy proposition shows that although Definition 1.1 suggests that "separating" is a property that pertains to very general collections of sets, it really is a notion that belongs in the context of ideals.

PROPOSITION 1.2. *Suppose that $\mathscr{X}$ is a separating collection and let $I_{\mathscr{X}}$ be given by*

$$Y \in I_{\mathscr{X}} \quad \text{if } \mathscr{P}(Y) \cap \mathscr{X} = 0 .$$

Then $I_{\mathscr{X}}$ is an ideal.

Proof. $I_{\mathscr{X}}$ is clearly closed under subset formation. To see that it is also closed under finite unions, assume that $Y_1, Y_2 \in I_{\mathscr{X}}$ but $Y_1 \cup Y_2 \notin I_{\mathscr{X}}$. Then we can choose $Z \subseteq Y_1 \cup Y_2$ so that $Z \in \mathscr{X}$. Let $Z_1 = Z \cap (Y_1 - Y_2)$ and let $Z_2 = Z \cap (Y_2 - Y_1)$. Notice that since $Z \nsubseteq Y_1$ and $Z \nsubseteq Y_2$ we have that both Z_1 and Z_2 are non-empty. Of course $Z_1 \subseteq Y_1$ and $Z_2 \subseteq Y_2$. Define f and g on Z so that $f''Z_1 = \{0\}$, $g''Z_1 = \{1\}$, $f''Z_2 = \{1\}$ and $g''Z_2 = \{0\}$. Since $\mathscr{X}$ is a separating collection we can choose $X \subseteq Z$ so that $X \in \mathscr{X}$ and $f''X \cap g''X = 0$. Then $X \subseteq Z_1$ or $X \subseteq Z_2$ and so $X \subseteq Y_1$ or $X \subseteq Y_2$. Hence either $Y_1 \notin I_{\mathscr{X}}$ or $Y_2 \notin I_{\mathscr{X}}$, and this is the desired contradiction. ■

It was shown in [MPT] that every weakly selective ideal on $\varkappa$ is a separating ideal and every $\varkappa$-complete ideal on $\varkappa$ is separating iff $\varkappa$ is not strongly inaccessible. Pelletier [P] has shown that if U is a separating ultrafilter, then ultrapowers modulo U are somewhat small (and hence, for example, if GCH holds and $\varkappa$ is the successor of an uncountable regular cardinal, then U is non-regular). This result was used in [KT$_1$] to show that if GCH holds and U is a uniform ultrafilter on $\varkappa = \mu^+$ where μ is regular, then $U \nrightarrow (U, 3)^2$.

Our goal in the present paper is, first of all, to provide a characterization of separating collections by means of a weakened version of the partition relation $\mathscr{X} \to (\mathscr{X}, 3)^2$. This is done in Section 2. In Section 3 we use this to show that the partition relation holds for every indecomposable countable ordinal. Section 4 contains a characterization of one-one separating collections in terms of a property closely related to that of Q-point ideal (or Q-point ultrafilter) and we use this to generalize some of the known results referred to above.

We are grateful to Fred Galvin and Donald Pelletier for some enlightening correspondence concerning the problems considered here.

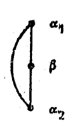
2. The characterization theorem. For our purposes, a graph G will be considered to be an ordered pair (V, E) where V is an arbitrary set (whose elements will be called vertices) and E is a collection of two element subsets of V (and if $\{x, y\} \in E$ we will say that x and y are *adjacent*.) A subset of V in which no two vertices are adjacent is called an *independent set*. The main result of this section is based on the following.

DEFINITION 2.1. Suppose that $G = (V, E)$ is a graph whose vertex set V is well-ordered by $<$. Suppose $k \geq 1$ and C is a set of $2k+1$ vertices. Then C will be called an *oscillating cycle* (of length $2k+1$) if the elements of C can be arranged in a sequence $\langle \beta, \alpha_1, \ldots, \alpha_{2k}, \beta \rangle$ of length $2k+2$ where:

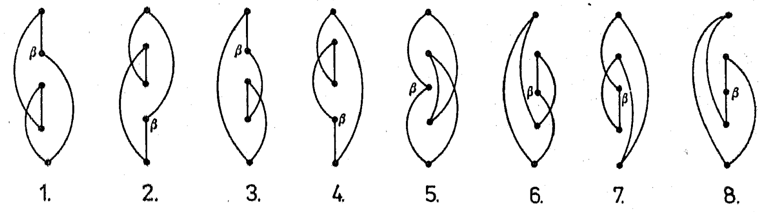
(i) the points $\beta, \alpha_1, \ldots, \alpha_{2k}$ are distinct, and

(ii) $\beta < \alpha_1 > \alpha_2 < \alpha_3 > \alpha_4 < \ldots > \alpha_{2k} < \beta$.

Notice that every 3-cycle (i.e. triangle) is an oscillating cycle:



It is also not difficult to convince oneself that there are eight oscillating cycles of length 5:



(Notice that (1, 2), (3, 4), and (6, 7) are pairs where one is obtained by flipping the other upside down.)

With Definition 2.1 at hand we can now state the desired characterization theorem.

THEOREM 2.2. *Suppose that λ is an ordinal and that $\mathscr{X}$ is a collection of subsets of λ. Then the following are equivalent:*

(1) $\mathscr{X} \to (\mathscr{X}, \text{oscillating cycle})^2$; *i.e., if $X \in \mathscr{X}$ and G is a graph with vertex set X, then either G contains an independent set Y with $Y \in \mathscr{X}$ or else G contains an oscillating cycle.*

(2) $\mathscr{X}$ *is a separating collection; i.e., if $X \in \mathscr{X}$ and f and g are functions defined on X so that for every $\beta \in X$ we have $f(\beta) \neq g(\beta)$, then there is a set $Y \subseteq X$ such that $Y \in \mathscr{X}$ and $f''Y \cap g''Y = 0$.*

The proof of Theorem 2.2 requires the following definition and lemma.

DEFINITION 2.3. Suppose that $G = (V, E)$ is again a graph whose vertex set V is well-ordered by $<$, and suppose that α and β are vertices with $\alpha < \beta$. Then the sequence $\langle \beta, \alpha_1, \alpha_2, \ldots, \alpha_{2k-1}, \alpha \rangle$ where $k \geq 1$ will be called an *oscillating path from β to α* iff $\beta < \alpha_1 > \alpha_2 < \alpha_3 > \ldots < \alpha_{2k-1} > \alpha$.

Notice that we do not demand that the vertices $\beta, \alpha_1, \ldots, \alpha_{2k-1}, \alpha$ be distinct (as we did in Definition 2.1). Also note that we only speak of such a path from β to α when $\beta > \alpha$ (i.e., oscillating paths are "downhill").

LEMMA 2.4. *Suppose that $G = (V, E)$ is a graph whose vertex set V is well-ordered by $<$ and assume that G contains no oscillating cycles. Suppose that $\alpha, \beta \in V$ with $\alpha < \beta$ and that $\langle \beta, \alpha_1, \alpha_2, \ldots, \alpha_{2k-1}, \alpha \rangle$ is an oscillating path from β to α. Then β is not adjacent to α.*

Proof of Lemma 2.4. Assume there exists such an oscillating path from β to α where β *is* adjacent to α and assume that among all such we have chosen β, α and $P = \langle \beta, \alpha_1, \alpha_2, \ldots, \alpha_{2k-1}, \alpha \rangle$ so that $k \geq 1$ is as small as possible. If the points $\beta, \alpha_1, \ldots, \alpha_{2k-1}, \alpha$ are distinct then the sequence $\langle \beta, \alpha_1, \ldots, \alpha_{2k-1}, \alpha, \beta \rangle$ shows that $C = \{\beta, \alpha_1, \ldots, \alpha_{2k-1}, \alpha\}$ is an oscillating cycle (of length $2k+1$) as desired. So suppose that some vertex is repeated in P and let i be the least index so that

$\alpha_i = \alpha_j$ for some index j where we relabel β as α_0 and relabel α as α_{2k}. We will consider the different possibilities for i and j and in each case exhibit an oscillating path P' that will contradict the minimality of P. We leave it to the reader to check that the various paths P' indeed work. (This involves verifying that each P' starts with α_n for some even n, ends with α_m for some even m, and that the last vertex in P' is both less than and adjacent to the first vertex in P'.)

Case 1. $i = 0$ (i.e., β is repeated; $\beta = \alpha_j$).

1.1. j is even: $P' = \langle \beta, \alpha_{j+1}, \ldots, \alpha_{2k-1}, \alpha \rangle$.

1.2. j is odd: $P' = \langle \beta, \alpha_1, \ldots, \alpha_{j-2}, \alpha_{j-1} \rangle$.

Case 2. $0 < i < 2k$ and $j < 2k$.

2.1 i is even and j is odd: $P' = \langle a_i, \ldots, \alpha_{j-1} \rangle$.

2.2. i and j have same parity: $P' = \langle \beta, \alpha_1, \ldots, \alpha_i, \alpha_{j+1}, \ldots, \alpha \rangle$.

2.3. i is odd and j is even: $P' = \langle a_j, \alpha_{j-1}, \ldots, \alpha_{i+1} \rangle$.

Case 3. $0 < i < 2k$ and $j = 2k$ (i.e., α is repeated; $a_i = \alpha$).

3.1. i is even: $P' = \langle \beta, \alpha_1, \ldots, \alpha_i \rangle$.

3.2. i is odd: $P' = \langle \alpha, \alpha_{2k-1}, \ldots, \alpha_{i+1} \rangle$.

This completes the proof of Lemma 2.4. ■

Proof of Theorem 2.2. (1)→(2): Suppose that $X \in \mathscr{X}$ and f, g are functions defined on X so that for every $\beta \in X$ we have $f(\beta) \neq g(\beta)$. Let G be the graph on X obtained by making α adjacent to β iff $\alpha < \beta$ and $f(\alpha) = g(\beta)$. Suppose for the moment that $C = \langle \beta, \alpha_1, \ldots, \alpha_{2k}, \beta \rangle$ is an oscillating cycle in G (so $\beta < \alpha_1 > \alpha_2 < \alpha_3 > \alpha_4 < \ldots > \alpha_{2k} < \beta$). Then

$$f(\beta) = g(\alpha_1) = f(\alpha_2) = g(\alpha_3) = f(\alpha_4) = \ldots = f(\alpha_{2k}) = g(\beta).$$

This contradicts our assumption that $f(\beta) \neq g(\beta)$ and so G contains no oscillating cycles. Hence G contains an independent set $Y \subseteq X$ so that $Y \in \mathscr{X}$. Notice that if $\alpha, \beta \in Y$ with $\alpha < \beta$ then $f(\alpha) \neq g(\beta)$, but we still cannot guarantee that $g(\alpha) \neq f(\beta)$ in this situation. What is required is that we repeat the procedure starting with Y in place of X and with the roles of f and g reversed. The resulting independent set $Z \subseteq Y$ with $Z \in \mathscr{X}$ will then have the property that $f''Z \cap g''Z = 0$ as desired.

(2)→(1): Suppose that $\mathscr{X}$ is a separating collection and let G be a graph with vertex set $X \in \mathscr{X}$. For notational simplicity, assume that X is the ordinal γ. We will assume that G contains no oscillating cycles and produce an independent set $Y \in \mathscr{X}$.

Let $\mathscr{F}$ and $\mathscr{G}$ be disjoint sets of cardinality $|\gamma|^+$. We will define functions $f, g\colon \gamma \to \mathscr{F} \cup \mathscr{G}$ simultaneously by induction so that if $f(\alpha)$ and $g(\alpha)$ have been defined for every $\alpha < \beta$, then we obtain $f(\beta)$ and $g(\beta)$ as follows:

$g(\beta)$:

$G(1)$: If there is some $\alpha < \beta$ so that α is adjacent to β, then we choose such an α and set $g(\beta) = f(\alpha)$.

$G(2)$: Otherwise, we let $g(\beta)$ be any element of $\mathscr{G}$.

$f(\beta)$:

$F(1)$: If there is some $\alpha < \beta$ and an oscillating path from β to α, then we choose such an α and set $f(\beta) = f(\alpha)$.

$F(2)$: Otherwise, we let $f(\beta)$ be any element of $\mathscr{F}$ such that $f(\beta)$ is distinct from $f(\alpha)$ for every $\alpha < \beta$.

Notice that we always have $f(\beta) \in \mathscr{F}$, while $g(\beta) \in \mathscr{F}$ if clause $G(1)$ is used and $g(\beta) \in \mathscr{G}$ if clause $G(2)$ is used.

Claim 1. *For all β, if $f(\beta)$ is defined by clause $F(1)$, then the value of $f(\beta)$ is independent of which $\alpha < \beta$ we choose (subject to the condition that there is an oscillating path from β to α).*

Proof. Suppose the claim fails and let β be the least counterexample. Then there exist $\alpha' < \alpha < \beta$, an oscillating path $P = \langle \beta, \alpha_1, \ldots, \alpha_{2k-1}, \alpha \rangle$ from β to α, and an oscillating path $P' = \langle \beta, \alpha'_1, \ldots, \alpha'_{2m-1}, \alpha' \rangle$ from β to α' so that $f(\alpha) \neq f(\alpha')$. Let $P'' = \langle \alpha, \alpha_{2k-1}, \ldots, \alpha_1, \beta, \alpha'_1, \ldots, \alpha'_{2m-1}, \alpha' \rangle$. Then P'' is an oscillating path from α to α'. Hence, $f(\alpha')$ was defined by clause $F(1)$ and since β was the least counterexample to Claim 1 and $\alpha' < \beta$ we have that $f(\alpha') = f(\alpha)$; contradiction.

Claim 2. *For all β, if $g(\beta)$ is defined by clause $G(1)$, then the value of $g(\beta)$ is independent of which $\alpha < \beta$ we choose.*

Proof. Suppose that $\alpha' < \alpha < \beta$ and that both α' and α are adjacent to β. Then $P = \langle \alpha, \beta, \alpha' \rangle$ is an oscillating path from α to α' and so (by Claim 1 and clause $F(1)$ at stage α) we have $f(\alpha') = f(\alpha)$. Thus, at stage β, $g(\beta)$ has the same value whether we choose α' or α.

Claim 3. *For all β, if $\alpha < \beta$ and $f(\beta) = f(\alpha)$, then there is an oscillating path from β to α.*

Proof. We proceed by induction on β. Suppose that $f(\beta) = f(\alpha)$ where $\alpha < \beta$. Then $f(\beta)$ must have been defined by clause $F(1)$. Hence, there is some $\alpha' < \beta$ and an oscillating path $P = \langle \beta, \alpha_1, \ldots, \alpha_{2k-1}, \alpha' \rangle$ from β to α' and we have $f(\beta) = f(\alpha')$. Since $f(\beta) = f(\alpha)$ also, we have $f(\alpha) = f(\alpha')$. Assume that $\alpha' < \alpha$; the argument for $\alpha < \alpha'$ is similar. Since $\alpha < \beta$ our inductive hypothesis guarantees that there is an oscillating path $P' = \langle \alpha, \alpha'_1, \ldots, \alpha'_{2m-1}, \alpha' \rangle$ from α to α'. Let

$$P'' = \langle \beta, \alpha_1, \ldots, \alpha_{2k-1}, \alpha', \alpha'_{2m-1}, \ldots, \alpha'_1, \alpha \rangle.$$

Then P'' is an oscillating path from β to α as desired.

Claim 4. *For all β we have $f(\beta) \neq g(\beta)$.*

Proof. Suppose for contradiction that $f(\beta) = g(\beta)$. If at stage β we had used clause $G(2)$, then we would have $g(\beta) \in \mathscr{G}$ while we always have $f(\beta) \in \mathscr{F}$. Hence, at stage β we used clause $G(1)$ and so there is some $\alpha < \beta$ with α adjacent to β and $g(\beta) = f(\alpha)$. But since $g(\beta) = f(\beta)$ also, we have $f(\alpha) = f(\beta)$ and so by Claim 3 we know there is an oscillating path from β to α. But now Lemma 2.3 guarantees that β is not adjacent to α; contradiction.

Since we are assuming that $\mathscr{X}$ is a separating collection, Claim 4 guarantees that there is a set $Y \in \mathscr{X}$ so that $Y \subseteq X$ and $f''Y \cap g''Y = 0$. Hence, to complete the proof of Theorem 2.2 it remains only to verify the following.

CLAIM 5. *Y is an independent set in the graph G.*

Proof. Suppose $\alpha, \beta \in Y$ with $\alpha < \beta$ and α is adjacent to β. Then at stage β, $g(\beta)$ was defined by clause $G(1)$ and Claim 2 guarantees that $g(\beta) = f(\alpha)$. But then $g(\beta) \in g''Y \cap f''Y$; contradiction. ■

Since every oscillating cycle is *a fortiori* an odd cycle, it follows from Theorem 2.2 that if $\mathscr{X}$ is a separating collection then $\mathscr{X} \to (\mathscr{X}, \text{odd cycle})^2$. The converse, however, fails. The point is that a graph G fails to contain an odd cycle iff G is bipartite (see [BCL pg. 23]). From this it easily follows that $\mathscr{X} \to (\mathscr{X}, \text{odd cycle})^2$ iff the collection of sets not in $\mathscr{X}$ is closed under finite unions. Combining this observation with the first sentence of this paragraph yields another proof of Proposition 1.2.

§ 3. Countable ordinals.

An ordinal α is *indecomposable* if whenever a set A of order-type α is decomposed $A = B \cup C$ into two sets, then either B has order-type α or C has order-type α. For an ordinal α we let I_α be the following collection:

$$I_\alpha = \{X \subseteq \alpha\colon \text{order-type of } X \text{ is less than } \alpha\}.$$

Notice that I_α is an ideal iff α is indecomposable. It is well-known [MR] that the indecomposable ordinals are precisely the powers of ω. It also follows from the comments in the last paragraph of the previous section that α is an indecomposable ordinal iff $\alpha \to (\alpha, \text{odd cycle})^2$. Our goal in this section is to prove a preservation theorem for separating collections that yields as a corollary the following.

THEOREM 3.1. *If α is an indecomposable countable ordinal then I_α is a separating ideal and (hence) $\alpha \to (\alpha, \text{oscillating cycle})^2$.*

In order to state the preservation theorem that will yield 3.1 as a corollary, we need to invoke some terminology and notation from the theory of ideals (see [BTW]). First, suppose the sets W_n for $n \in \omega$ are pairwise disjoint and that I_n is an ideal on W_n for each n. Let $W = \bigcup\{W_n\colon n \in \omega\}$ and let $I_\omega = \{X \subseteq \omega\colon X \text{ is finite}\}$. Let I be the ideal on W defined by

$$X \in I \quad \text{if } \{n \in \omega\colon X \cap W_n \in I_n\} \text{ is cofinite}.$$

I is usually called "the I_ω sum of the I_n's" and denoted $I_\omega \Sigma I_n$. Notice that $X \in I^+$ iff $X \cap W_n \in I_n^+$ for infinitely many n's.

If I is an ideal and $X \in I^+$ then the *restriction of I to X* is the ideal $I|X$ given by

$$Y \in I|X \quad \text{iff} \quad Y \cap X \in I.$$

If I is an ideal and f is a function then $f_*(I)$ is the ideal given by

$$Y \in f_*(I) \quad \text{iff} \quad f^{-1}(Y) \in I.$$

In this case we say that $f_*(I)$ is below I in the Rudin–Keisler ordering and write $f_*(I) \leqslant_{\mathrm{RK}} I$. If $f_*(I)$ is dual to an ultrafilter U, then we will often say "U is RK-below I" even though this is a slight abuse of terminology.

The preservation theorem we will establish is the following.

THEOREM 3.2. *Let $\mathscr{I}_1$ be the collection of ideals I satisfying $(*)$ and let $\mathscr{I}_2$ be the collection of all ideals I satisfying both $(*)$ and $(**)$:*

$(*)$ The only ultrafilters that are RK-below any restriction of I are principal.

$(**)$ I is a separating ideal.

Then both $\mathscr{I}_1$ and $\mathscr{I}_2$ are closed under I_ω-sums.

The proof of Theorem 3.2 requires the following lemmas.

LEMMA 3.3. *For any ideal I, condition $(*)$ in Theorem 3.2 is equivalent to the following*

$(*')$: For every $X \in I^+$ and every function f with domain X, if f is not constant on a set in I^+ then there are sets $Y, Z \in I^+$ with $Y, Z \subseteq X$ and $f''Y \cap f''Z = 0$.

Proof. $(*) \to (*')$: Given f and X we know that $f_*(I|X)$ is either dual to a principal ultrafilter or not dual to an ultrafilter at all. The former case holds iff there is an object a so that $a \notin Y$ iff $f^{-1}(Y) \in I|X$ and this holds iff f is constant on $X-Z$ for some set $Z \in I$, and so f is constant on a set in I^+. The latter case holds iff there are disjoint sets $A, B \in f_*(I|X)^+$, so here we can let $Y = f^{-1}(A)$ and $Z = f^{-1}(B)$ and have $Y, Z \in I^+$ and $f''Y \cap f''Z = 0$.

$(*') \to (*)$: Suppose $f_*(I|X)$ is dual to the ultrafilter U. Then we cannot have disjoint sets $Y, Z \subseteq X$ with $f''Y \cap f''Z = 0$ and $Y, Z \in I^+$ so we must have $f''Y = \{a\}$ for some set $Y \subseteq X$ with $Y \in I^+$. But since $Y \in I^+$, $f''Y \in U$ and so $\{a\}$ shows that U is principal. ■

LEMMA 3.4. *Suppose that the set $\{W_n\colon n \in \omega\}$ are pairwise disjoint and that I_n is an ideal on W_n for each n. Let $I = I_\omega \Sigma I_n$. Assume that f is a function such that $\forall n \in \omega\ \forall X \in I_n^+$ either (i) or (ii) below holds:*

(i) $\exists Y, Z \subseteq X$ s.t. $Y, Z \in I_n^+$ and $f''Y \cap f''Z = 0$.

(ii) $\exists Y \subseteq X$ s.t. $Y \in I_n^+$ and f is constant on Y.

Let g be any function defined on $W = \bigcup\{W_n\colon n \in \omega\}$ so that g is not constant on any set of positive I-measure. (We allow the possibility that $g = f$.) Then (iii) below holds:

(iii) $\forall n \in \omega\ \forall X \in I_n^+\ \forall A \in I^+\ \exists X' \subseteq X\ \exists A' \subseteq A$ s.t.

(a) $X' \in I_n^+$;

(b) $A' \in I^+$;

(c) $f''X' \cap g''A' = 0$.

Proof. Suppose that n, X and A are given as in (iii). Assume first that for this n and this X, (i) holds. Then either $g^{-1}(f''Y) \cap A \in I^+$ or $A - g^{-1}(f''Y) \in I^+$. If it is the former, let $X' = Z$ and $A' = g^{-1}(f''Y) \cap A$. If it is the latter, let $X' = Y$ and let $A' = A - g^{-1}(f''Y)$. Clearly (a), (b), and (c) hold for these choices of X'

and A'. Now assume that for this n and this X that (ii) holds. Let X' be the set $Y \in I_n^+$ with f constant on Y (say $f''X' = \{p\}$). Then $g^{-1}(\{p\}) \cap A \in I$ (since g is not constant on any set of positive I-measure) so we can take $A' = A - g^{-1}(\{p\})$. ■

LEMMA 3.5. *Assume that f, g and I are as in Lemma 3.4. Then $\forall A \in I^+ \ \exists B \subseteq A$ such that $B \in I^+$ and*

$$\forall n, m \in \omega \ [\text{if } n < m \text{ then } f''(B \cap W_n) \cap g''(B \cap W_m) = 0].$$

Proof. Simply apply Lemma 3.4 $\aleph_0$ times. ■

With these lemmas at our disposal, we can now return to the task at hand.

Proof of Theorem 3.2. We first show that $\mathscr{I}_1$ is closed under ω-sums. So suppose that $\{W_n: n \in \omega\}$ are pairwise disjoint and that for each $n \in \omega$ the ideal I_n on W_n satisfies (*) of Theorem 3.2. Let $I = I_\omega \Sigma I_n$. We want to show that I also satisfies (*). By Lemma 3.3 it suffices to show that I satisfies (*'), so suppose that $X \in I^+$ and f is a function with domain X that is not constant on any set of positive I-measure. Since each I_n satisfies (*) and hence (*') we can apply Lemma 3.5 with $g = f$. Let $\{X_{n_i}: i < \omega\}$ be such that $X_{n_i} \in I_{n_i}^+$, $n_0 < n_1 < \ldots$, and $f''X_{n_i} \cap f''X_{n_j} = 0$ whenever $i < j$. Let $Y = \bigcup \{X_{n_i}: i \text{ even}\}$ and $Z = \bigcup \{X_{n_i}: i \text{ odd}\}$. Then $Y, Z \in I^+$ and $f''Y \cap f''Z = 0$. This shows that I satisfies (*').

We now show that $\mathscr{I}_2$ is closed under ω-sums. Suppose that W_n, I_n and I are as before and assume that each I_n satisfies both (*) and (**). It remains to show that I satisfies (**); i.e., that I is separating. Suppose that f and g are such that $f(x) \neq g(x)$ for any x and, without loss of generality, assume that both f and g have domain W. If either f or g is constant on a set of positive I-measure, then we are clearly done. So assume otherwise. Now, for any $n \in \omega$ we known that I_n satisfies (*) and hence (*'). Thus the hypotheses (i) and (ii) of Lemmas 3.4 and 3.5 hold so we can apply Lemma 3.5 for this f and g. Applying Lemma 3.5 once more with the roles of f and g reversed yields a set $B \in I^+$ so that if $n \neq m$ then

$$f''(B \cap W_n) \cap g''(B \cap W_m) = 0.$$

To finish the proof we consider each n such that $B \cap W_n \in I_n^+$ and use the fact that I_n is separating to get $B_n \subseteq B$ with $B_n \in I_n^+$ and $f''B_n \cap g''B_n = 0$. If $B' = \bigcup \{B_n: n \in \omega\}$, then $B' \in I^+$ and $f''B' \cap g''B' = 0$ as desired. ■

Of course, Theorem 3.1 now follows trivially by induction from Theorem 3.2 (using the fact that the indecomposable ordinals are the powers of ω).

A natural question that arises is whether or not I is a separating ideal for every (even uncountable) indecomposable ordinal. This turns out not to be the case. For example, Galvin has shown (unpublished) that if CH holds, $\mathrm{cf}(\alpha) = \omega_1$, $\omega_1^{\omega+2} \leqslant \alpha < \omega_2$ and α is indecomposable then I_α is *not* a separating ideal (and, in fact, (**) fails for I_α as well).

§ 4. One-one separating collections. The notions of P-point and Q-point ultrafilters are well known and have long been studied. In [GW] and [BTW], these and some related notions were considered in the context of $\varkappa$-complete ideals on $\varkappa$, and those investigations led to the investigation (first in [GW]) of a property called *Ulamness* that turned out to be both very natural and quite central to the study of structural properties of ideals. The definition, generalized here to ideals that are not necessarily $\varkappa$-complete, runs as follows: An ideal I is an *Ulam ideal* iff every at most two-to-one function defined on a set of I-measure one is one-to-one on a set of I-measure one. Our goal in this section is to prove the following characterization theorem and then use it to derive a result from [MPT] and to affirm a conjecture of Pelletier.

THEOREM 4.1. *For any collection $\mathscr{X}$ of sets, the following are equivalent:*

(i) *$\mathscr{X}$ is an Ulam collection: i.e., if $X \in \mathscr{X}$ and h is an at most two-to-one function defined on X, then there exists a set $Y \in X$ such that $Y \subseteq X$ and h is one-to-one on Y.*

(ii) *$\mathscr{X}$ is a one-to-one separating collection: i.e., if $X \in \mathscr{X}$ and f and g are one-to-one functions with domain X so that $f(x) \neq g(x)$ for every $x \in X$, then there exists a set $Y \in \mathscr{X}$ such that $Y \subseteq X$ and $f''Y \cap g''Y = 0$.*

Proof. (i)$\to$(ii). Suppose that $\mathscr{X}$ is an Ulam collection and that f, g and X are as in (ii). Let $\{x_\alpha: \alpha < \varkappa\}$ be an enumeration of X in order-type $\varkappa = |X|$. For each subset Y of X we will inductively construct a partition $\mathrm{Part}(Y) = \{A_\alpha^Y: \alpha < \varkappa\}$ of Y into sets of size at most two as follows. Suppose $\alpha < \varkappa$ and A_γ^Y has been constructed for every $\gamma < \alpha$. We consider three cases.

Case 1: Either $x_\alpha \notin Y$ or $x_\alpha \in A_\gamma^Y$ for some $\gamma < \alpha$.
In this case we set $A_\gamma^Y = 0$.

Case 2: $x_\alpha \in Y$ and $\forall \gamma < \alpha \ [x_\alpha \notin A_\gamma^Y]$ and
$\forall \beta < \alpha \ [x_\beta \in Y \Rightarrow g(x_\beta) \neq f(x_\alpha)]$.
In this case we set $A_\alpha^Y = \{x_\alpha\}$.

Case 3: $x_\alpha \in Y$ and $\forall \gamma < \alpha \ [x_\alpha \notin A_\gamma^Y]$ and
$\exists \beta > \alpha \ [x_\beta \in Y \text{ and } g(x_\beta) = f(x_\alpha)]$.

Notice that if we are in Case 3, then there is exactly one $\beta > \alpha$ with $g(x_\beta) = f(x_\alpha)$ since g is one-to-one. So in Case 3 we choose this β and set $A_\alpha^Y = \{x_\alpha, x_\beta\}$.

Consider $\mathrm{Part}(X)$. Since $\mathscr{X}$ is an Ulam collection there is a set $Y \subseteq X$ with $Y \in \mathscr{X}$ and such that $|Y \cap A_\alpha^X| \leqslant 1$ for every $\alpha < \varkappa$. Notice that it may still be the case that we have $x_\alpha, x_\beta \in Y$ with $\alpha < \beta$ and $f(x_\alpha) = g(x_\beta)$. However, if this occurs then in the construction of A_α^X we must have been in Case 1 and so there was some $\gamma < \alpha$ with $x_\alpha \in A_\gamma^X$. But this means that $f(x_\gamma) = g(x_\alpha)$ and $A_\gamma^X = \{x_\gamma, x_\alpha\}$. Since $|Y \cap A_\gamma^X| \leqslant 1$ and $x_\alpha \in Y$ we have $x_\gamma \notin Y$. This, together with the fact that f is one-to-one, shows that we have $f(x_\xi) \neq g(x_\alpha)$ for any $x_\xi \in Y$ with $\xi < \alpha$. Hence, if we now consider $\mathrm{Part}(Y)$, then in the construction of A^Y we will be in Case 3 and so we will get $A_\alpha^Y = \{x_\alpha, x_\beta\}$. Thus, if we choose $Z \in \mathscr{X}$ so that $Z \subseteq Y$ and $|Z \cap A_\zeta^Y| \leqslant 1$ for every $\alpha < \varkappa$ then it will be true that for $x_\alpha, x_\beta \in Z$ with $\alpha < \beta$ we have $f(x_\alpha) \neq g(x_\beta)$. Now to complete the proof we need only repeat the above two step procedure with the roles of f and g reversed and starting with the set Z in place of the set X.

(ii) → (i). Suppose that $\mathscr{X}$ is a one-one separating collection, $X \in \mathscr{X}$ and h is a function that is at most two-to-one on X. Let g be the identity function on X and let f be any one-to-one function defined on X satisfying:

(a) $\forall x \in X\ [f(x) \neq x]$, and

(b) $\forall x, y \in X$ [if $x \neq y$ and $h(x) = h(y)$, then $f(x) = y$ and $f(y) = x$].

Since both f and g are one to one we get a set $Y \in \mathscr{X}$ so that $Y \subseteq X$ and $f''Y \cap g''Y = 0$. Suppose, for contradiction that we have $x, y \in Y$ with $x \neq y$ and $h(x) = h(y)$. Then $f(x) = y$ and $f(y) = x$. But then $x \in f''Y$ (since $x = f(y)$ and $y \in Y$) and $x \in g''Y$ (since $x = g(x)$ and $x \in Y$); contradiction. ■

In order to state the corollaries we want, a couple of definitions from [BTW] are needed. If I is an ideal on $\varkappa$ then I is said to be *selective* (*weakly selective*) iff every function defined on a set of I-measure one (positive I-measure) is either constant on a set of positive I-measure or one-to-one on a set of I-measure one (positive I-measure). It is easy to see that I^+ is an Ulam collection for every ideal I and I^* is an Ulam collection iff I is an Ulam ideal. Moreover, "separating" and "one-one separating" are equivalent for I^+ when I is weakly selective. Given these observations, the following are easy consequences of Theorem 4.1.

COROLLARY 4.2 [MPT]. *Every ideal is a one-one separating ideal and every weakly selective ideal is a separating ideal.*

COROLLARY 4.3. *I is an Ulam ideal iff I^* is a one-one separating collection.*

COROLLARY 4.4. *Suppose that I is a selective ideal on $\varkappa$ and $f, g: \varkappa \to \varkappa$ are such that neither is constant on a set of positive I-measure and for every $\alpha < \varkappa$, $f(\alpha) \neq g(\alpha)$. Then there is a set $X \in I^*$ so that $f''X \cap g''X = 0$.*

Corollary 4.4 confirms a conjecture of D. Pelletier.

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