the proof of Theorem 2.2. With a minor modification of the arguments used there we prove the following claims:

Claim 1. There exists a set $Z_1 \subseteq Z$ such that

(i) $Z_1 \neq J$,

(ii) $\forall A \subseteq Z_1 \forall g \in G \ (g[A] \subseteq Z_1 \rightarrow A \Delta g[A] \in J)$;

Claim 2. There exists a set $Z \subseteq Z_1$ and a family $F \subseteq G$ such that

(i) $Z \neq J$ and $|F| < \lambda$,

(ii) $\forall A \subseteq Z \forall g \in G \ (g[A] \cap \bigcup_{f \in F} Z = \emptyset \rightarrow A \in J)$.

This can be regarded as a satisfactory outline of the proof of Theorem 3.4.

References


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Locally connected curves viewed as inverse limits

by

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Abstract. Every locally connected curve is the limit of an inverse sequence of regular continua with monotone bonding surjections. Moreover, any space which is the limit of an inverse sequence of connected graphs with monotone bonding surjections is a rather small continuum.

1. Introduction. All spaces considered in this paper are assumed to be metric, all maps are continuous, $d$ always denotes a distance function, and 'continuum' means 'compact connected (metric) space'.

We will say that a space $X$ is:

(a) a graph provided $X$ is a one-dimensional (compact) polyhedron;

(b) a completely regular continuum provided $X$ is a continuum such that $\text{int}(Y) \neq \emptyset$ for each nondegenerate subcontinuum $Y$ of $X$;

(c) a regular continuum if $X$ is a continuum such that for any $\varepsilon > 0$ and each $x \in X$ there exists an open neighbourhood $U$ of $x$ in $X$ such that $\text{bd}(U)$ is finite and $\text{diam} U < \varepsilon$ (regular continua are often called 'rim-finite continua');

(d) a curve provided $X$ is a continuum of dimension 1.

Clearly, every connected graph is a completely regular continuum and every regular continuum is a locally connected curve. Moreover, each completely regular continuum is regular (see for example Proposition 3.2 below).

Recall that a (continuous) map $f: X \rightarrow Y$ is said to be monotone if $f^{-1}(y)$ is connected for each $y \in Y$.

It is well-known that every curve $X$ is the limit of some inverse sequence $(X_n, f_n)$ of (connected) graphs (see e.g. [2], Theorem 1.13.2, p. 145; it is not difficult to see that the sequence can be chosen in such a manner that all the bonding maps, $f_n: X_{n+1} \rightarrow X_n$, are surjections). If $X$ is locally connected, one can use the general method of S. Mardešić to produce an inverse sequence $(Y_n, g_n)$ of locally connected continua $Y_n$ with monotone bonding surjections $g_n: Y_{n+1} \rightarrow Y_n$ such that $X = \lim \text{inv}(Y_n, g_n)$ (where $Y_n, g_n$) is obtained as a 'modification' of $(X_n, f_n)$; see [8], p. 164 — the proof of Theorem 2). However, in general, almost nothing can be proved about $(Y_n, g_n)$. In particular, $Y_n$'s need not (and often they can not) be graphs; they are simply locally connected continua. The only essential information on
Y's is a consequence of the elementary fact that all the projections $h_i: X \to Y_i$ are monotone surjections ([1], Lemma 4.2, p. 241). For example: if $X$ is a connected graph (resp. a regular continuum, ...), then each $Y_i$ is also a connected graph (resp. a regular continuum, ...; see e.g. [7], p. 85). On the other hand, recall that if $X = (X_n, f_n)$ is a monotone bonding surjection, then $\liminv(X_n, f_n)$ is again a locally connected continuum ([1], Theorem 4.3, p. 241). Moreover, the limit of an inverse sequence of curves is again a curve ([22], Theorem 1.13.4, p. 149).

This consideration leads to some natural questions which motivate the research reported in this paper. In Chapter 2 we prove that every locally connected curve in the limit of some inverse sequence of regular continua with monotone bonding surjections. In Chapter 3, we show that limits of inverse sequences of connected graphs with monotone bonding surjections are 'very small' continua.

Let $X$ be a space and $A$ a family of subsets of $X$. We write

$$\text{mesh } A = \sup \{ \text{diam } Y : Y \in A \}.$$ 

$A$ is said to be a null-family if for any $\varepsilon > 0$ the collection $\{ Y \in A : \text{diam } Y > \varepsilon \}$ is finite.

Now, we give an example of a 'large' locally connected curve $X$, homeomorphic to the limit of an inverse sequence of regular continua, with monotone bonding surjections. Appropriately generalized ideas of the construction given in Example 1.1 below will give the proof of the main Theorem 2.2.

1.1. Example. Let $C$ denote (in this section only) the Cantor ternary set constructed as usual in [0,1]. For each nonnegative integer $n$, let $A_n$ be the unique family such that:

(a) $A_n$ consists of exactly $2^n$ elements which are pairwise disjoint closed intervals;

(b) the length of each member of $A_n$ is equal to $\frac{1}{2^n}$; and

(c) $C = \bigcup A_n$.

Then the family $A = \{ A_n : n = 0, 1, \ldots \}$ is a basis for $C$ and $A$ consists of (nonempty) closed-open subsets of $C$. Moreover, $C = \bigcap_{n=0}^{\infty} A_n$.

For each $n = 0, 1, \ldots$, put $D_n = \{ (k, 2^n) : k = 0, 1, \ldots, 2^n \}$ and $E_n = \bigcup_{k=0}^{2^n} D_n$.

Let $Y = (C \times [0,1]) \cup E_0$. Clearly, $Y$ is a locally connected curve in the plane. Moreover, $Y$ is not regular; in fact, $Y$ is rather 'large' — because it contains an uncountable family $\{ (c) \times [0,1] : c \in C \}$ of pairwise disjoint nondegenerate subcontinua.

For $n = 0, 1, \ldots$, let $F_n$ denote the decomposition of $X$ into the components of $E_n$ and points; put $X_n = X/F_n$ and let $g_n: X \to X_n$ be the projection. Note that each $F_n$ is upper semi-continuous (because the family of components of $E_n$ is a nullfamily). Since $F_{n+1}$ is a refinement of $F_n$, there is a (unique) map $f_{n+1}: X_{n+1} \to X_n$ such that $g_n \circ f_{n+1} = g_{n+1}$, for $n = 0, 1, \ldots$. Observe that each $f_n$ is a (continuous) monotone surjection. Hence, there is an induced surjection $g: X \to \liminv(X_n, f_n)$.

Since the maps $g_n, n = 0, 1, \ldots$, separate points of $X$, $g$ is one-to-one. Thus $g$ is a homeomorphism. It is not difficult to check that each $X_n$ is a (planar) regular continuum.

2. Inverse limits of regular continua.

2.1. Lemma. If $A$ is a closed $d$-dimensional subset of a locally connected curve $X$, then for any $\varepsilon > 0$ there exists a finite collection $K$ of pairwise disjoint subcontinua $X$ such that $\text{mesh } K < \varepsilon$ and $A \subseteq K$.

Proof. Since $A$ is compact zero-dimension, there exists a (finite) family $L$ of pairwise disjoint nonempty closed-open subsets of $A$ such that $\text{mesh } L < \varepsilon / 3$ and $A = \bigcup L$. Let $\varepsilon = \min \{ d(B, C) : B, C \in L, B \neq C \}$. For each $x \in X$ let $U_x$ be an open neighborhood of $x$ such that $U_x$ is connected and $\text{diam } U_x < \min \{ \varepsilon / 2, \varepsilon / 4 \}$. Since $A$ is compact, there are $x_1, \ldots, x_n \in A$ such that $A \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. It suffices to let $K$ be the family of all components of $\text{cl}(U_{x_1}) \cup \cdots \cup \text{cl}(U_{x_n})$. In fact, then each $D \in K$ is contained in some set $E_D \subseteq \bigcup \{ \text{cl}(U_x) : \text{cl}(U_x) \cap B = \emptyset \}$, $B \subseteq L$.

Clearly, $\text{diam } E_D < \varepsilon$ and

$E_D \cap E_D = \emptyset$ if $B, C \subseteq L, B \neq C$.

2.2. Theorem. If $X$ is a locally connected curve, then there exists an inverse sequence $(X_n, f_n)$ such that:

(i) each $X_n$ is a regular continuum;

(ii) each $f_n: X_{n+1} \to X_n$ is a monotone surjection;

(iii) $X = \liminv(X_n, f_n)$.

Proof. Since $\dim X = 1$, there exists a sequence $P_1, P_2, \ldots$ such that for each $n$, $P_n$ is a covering of $X$ which consists of finitely many open sets, $\text{mesh } P_n < 1/n$ and $A_n = \bigcup \{ \text{bd } U_n : U \subseteq P_n \}$ is zero-dimensional.

Let $U_n = (c) \times [0,1]$ and $X_n = U_n$ be a finite collection of pairwise disjoint subcontinua of $X$ such that $\text{mesh } K_{X_n} < \varepsilon_n$ and $A_n \subseteq K_{X_n}$ (see Lemma 2.1).

Suppose that for a positive integer $n$, a positive real number $\varepsilon_n$, and families $L_{n,k}, 1 \leq k \leq n$, are already defined such that:

(i) each $L_{n,k}$ is a finite collection of pairwise disjoint subcontinua of $X$,

(ii) $A_{k} \cup \cdots \cup A_{k} \subseteq L_{n,k}$, for each $k$,

and

(iii) if $1 \leq k < l \leq n$ and $B \subseteq L_{n,k}$, then there is a $C \subseteq L_{n,k}$ such that $B \subseteq C$. 

For any $k$, $1 \leq k \leq n$, let $r_k^* = \min \{d(B, C) : B, C \in \mathcal{L}_{k+1}, B \neq C \}$. By (1) $r_k^* > 0$. Put $\varepsilon_{k+1} = \frac{1}{4} \min \{\varepsilon_k, r_1^*, ..., r_k^* \}$ and let $K_{k+1}$ be a finite collection of pairwise disjoint subcontinua of $X$ such that $\text{mesh} K_{k+1} < \varepsilon_k$ and $A_{k+1} \subset \bigcup K_{k+1}$ (see Lemma 2.1). Put $L_{k+1} = K_{k+1}$ and let $L_{k+1, k}$ be the family of all components of the set $\bigcup \{K_{k+1} \cup L_{k, k} \}$, for $k = 1, ..., n$. Obviously, conditions (1)-(3) are satisfied, with $n$ replaced by $n+1$.

Observe that, by the choice of $r_k^*$ and $\varepsilon_k$,

(4) if $B \in L_{k, k}$, $1 \leq k < n$, then either $B \in K_k$ and $B \cap \bigcup L_{k, k} = \emptyset$, or there is a $B' \in L_{k-1, k}$ such that $B = B' \cup \bigcup \{C \in K_k : C \cap B' \neq \emptyset \}$.

Moreover,

(5) $\varepsilon_{k+1} < \frac{1}{4} \varepsilon_k$.

For each positive integer $k$ and any sequence $B_1, B_2, ..., B_{n+1}$, with the properties:

(6) $k \leq m; \quad B_n \cap \bigcup L_{k, k} = \emptyset$ for $k < m$,

and

(7) $B_n \in L_{k, k}$ and $B_n \notin B_{k+1}$ for $n = m, m+1, ..., n$,

let $C_k(B_1, B_2, ..., B_n) = \text{cl}(B_n \cup B_{n+1} \cup ...)$, $c_k(B_1, B_2, ..., B_n) = \text{cl}(B_n \cap B_{n+1} \cap ...)$, $C_k(B_1, B_2, ..., B_n)$ is a continuum.

For any positive integer $k$ let $\mathcal{M}_k = \{C_k(B_1, B_2, ..., B_n) : B_1, B_2, ..., B_n \in \mathcal{M}_k \}$; the sequence $B_n, B_{n+1}, ...$ satisfies conditions (6) and (7).

We show that

(9) if $c_k(B_1, B_2, ..., B_n) = \mathcal{M}_k$ and either $m \neq n$ or $m = n$ and $B_n \neq B_n^*$, then $c_k(B_1, B_2, ..., B_n) \cap c_k(B_1, B_2, ..., B_n) = \emptyset$ and

(10) $\text{diam} c_k(B_1, B_2, ..., B_n) \leq \frac{5}{3} \cdot 4^{n-k} \leq \frac{5}{3} \cdot 4^{n-1}$.

To prove (9) first consider the case $m \neq n$. We may assume that $k \leq m < n$. By (6), $B_n \cap L_{k, k} = \emptyset$ — by choice of $B_n \in L_{k, k}$ and $B_n \notin L_{k, k-1}$. By (4), $B_n \in K_k$ and $B_n \cap B_n^* = \emptyset$. By the choice of $r_k^*, d(B_n, B_n^*) \geq r_k^* > 4 \cdot 4^{n-k+1}$. Since $d(B, B^*) \geq \varepsilon_{k+1}$, for $B \in K_{k+1}$, it follows by (4) that $d(B_{n+1}, B_{n+1}^*) \geq r_{k+1}^* > 4 \cdot 4^{n-k+1}$. Now, an easy inductive proof shows that, for each positive integer $i$,

$$d(B_n, B_{n+i}) > r_k^* - 2 \cdot 4^{n-k} - 2 \cdot 4^{n-k+1} - ... - 2 \cdot 4^{n-i} > 2 \cdot 4^{n-k} - 2 \cdot 4^{n-k+1} - ... - 2 \cdot 4^{n-i}$$

Therefore,

$$d(B_n, B_{n+i}) > 2 \cdot 4^{n-k} - 2 \cdot 4^{n-k+1} - ... - 2 \cdot 4^{n-i}$$

Hence,

$$d(C_k(B_1, B_2, ..., B_n), C_k(B_1, B_2, ..., B_n)) \geq \frac{5}{3} \varepsilon_{k+1} > 0. $$
Locally connected curves

3.4. EXAMPLE. Note that the triangular Sierpiński curve (see e.g. [6], p. 276) X has the following properties: (a) X is a regular continuum — in fact, for each \( x \in X \), the order of \( x \) in \( X \) does not exceed 4; however, (b) \( X \) is not a totally regular continuum.

Another example of a regular but not totally regular curve is the Knaster dyadic continuum

\[
Y = (0, 1) \times \{0\} \cup \{(x, y) \in R^2 : \left( x - \frac{2k-1}{2^n} \right)^2 + y^2 = \frac{1}{4^k}, \ y > 0, \ k = 1, 2, ..., 2^{-n}, n = 1, 2, \ldots \}.
\]

Moreover, \( Y \) has the following interesting property: each connected subset of \( Y \) is arcwise connected (see [9], Example on p. 232).

3.5. LEMMA. If \( Y \) is a hereditarily locally connected continuum, \( Z \) a continuum and \( h : Y \to Z \) a monotone map, then the set \( P = \{z \in Z : h^{-1}(z) \text{ is nondegenerate}\} \) is countable.

Proof. Indeed, since \( Y \) is hereditarily locally connected, the collection \( \{h^{-1}(z) : z \in P\} \) of subcontinua of \( Y \) is easily seen to be a null-family.

3.6. THEOREM. If \( \{X_n, f_n\} \) is an inverse sequence such that all the spaces \( X_n \) are totally regular continua and all the bonding maps \( f_n : X_{n-1} \to X_n \) are monotone surjections, then \( X = \liminf\{X_n, f_n\} \) is a totally regular continuum.

Proof. By [1], Theorem 4.3, p. 241, \( X \) is a locally connected continuum. Let \( Q \) be a countable subset of \( X \). For \( n = 1, 2, \ldots \) let \( g_n : X \to X \) denote the projection. Then \( g_n \) is a continuous and monotone surjection, and \( g_n = f_{n-1} \circ g_{n-1} \), for \( n = 1, 2, \ldots \).

For each \( n \) let \( P_n = \{ x \in X : g_n^{-1}(x) \text{ is nondegenerate}\} \) be a countable set. Suppose that \( P_n \) is uncountable for some positive integer \( n \). Hence, there is an \( e > 0 \) such that the set \( A = \{ y \in P_n : \text{diameter}(g_n^{-1}(y)) > e\} \) is uncountable. There exists an integer \( m > n \) such that \( m > n \) and \( \text{diameter}(g_m^{-1}(x)) < e \), for each \( x \in X_n \). Put \( f = f_n \circ f_{n-1} \circ \ldots \circ f_{m-1} \). Hence \( f : X_n \to X_m \) is a monotone surjection. By Lemma 3.5, \( B = \{ y \in X_m : f^{-1}(y) \text{ is nondegenerate}\} \) is countable (because every regular continuum is hereditarily locally connected). Hence the set \( A - B \) is uncountable. Let \( y \in A - B \). Since \( y \notin B \), \( f^{-1}(y) = \{ z \} \) for some \( z \in X_n \). Recall that \( \text{diameter}(g_n^{-1}(z)) < e \), however, \( g_m^{-1}(z) = f^{-1}(y) = g_n^{-1}(y) \). Since \( y \in A \), we get \( \text{diameter}(g_m^{-1}(y)) < e \), a contradiction. We have proved that the sets \( P_n \), \( n = 1, 2, \ldots \), are countable.

Let \( x \in X \) and \( U \) be an open neighbourhood of \( x \) in \( X \). There exist a positive integer \( k \) and an open subset \( W \) of \( X_k \) such that \( x \in g_k^{-1}(W) \subset U \). Put \( P = (g_k(\cup P_1) \cup P) \subset P_1 \); so \( P \) is a countable subset of \( X_k \). Since \( W \) is a neighbourhood of \( g_k(x) \) and \( X_k \) is totally regular, there exists an open set \( W' \) such that \( g_k(x) \in W' \subset W \). Let \( V = g_k^{-1}(W') \). Hence \( V \) is open and \( x \in V \subset U \). Note that \( \text{bd}(V) \subset g_k^{-1}(\text{bd}(W')) \subset W' \). Since \( W' \) is a neighbourhood of \( g_k(x) \) and \( X_k \) is totally regular, there exists an open set \( W'' \) such that \( \text{bd}(V) \subset W'' \subset W'. \)

3.3. PROPOSITION. A subcontinuum of a totally regular continuum is also a totally regular continuum.
3.7. Corollary. If $X$ is the inverse limit of an inverse sequence of connected graphs (completely regular continua) with monotone bonding surjections, then $X$ is a totally regular continuum.

3.8. Theorem. If $X$ is a completely regular continuum, then there exists an inverse sequence $(X_n, f_n)$ such that:

(i) each $X_n$ is a connected graph;

(ii) each $f_n: X_{n+1} \rightarrow X_n$ is a monotone surjection;

(iii) $X = \liminf(X_n, f_n)$.

Proof. Let $C$ and $P_n$, $n = 1, 2, ...$, be subsets of $X$ as in Lemma 3.1. For each positive integer $n$, let $A_n$ denote the family of all components of the set $Y_n = X - \bigcup_{k=1}^{n} (P_k - \{a_k, b_k\})$. Observe that $Y_n$ is locally connected and compact for $n = 1, 2, ...$. Hence, each family $A_n$ is finite. Let $B_n$ denote the decomposition of $X$ into points and the members of $A_n$. Let $X_n$ denote the quotient space, $X_n = X/B_n$, and let $g_n: X \rightarrow X_n$ be the quotient map, for $n = 1, 2, ...$. Observe that each $X_n$ is a connected graph and each $g_n$ is a monotone surjection. Moreover, for $n = 1, 2, ... B_{n+1}$ is a refinement of $B_n$. Hence there are (unique) monotone surjections $f_n: X_{n+1} \rightarrow X_n$ such that $g_n = f_n \circ g_{n+1}$, for $n = 1, 2, ...$. In order to prove that $X$ is homeomorphic to $\liminf(X_n, f_n)$, it suffices to show that:

(*) the maps $g_n$, $n = 1, 2, ...$, separate points of $X$.

By the properties of $C$ and $P_n$'s stated in Lemma 3.1, it follows that $C = \bigcup_{n=1}^{\infty} A_n$. Hence $A = A_1 \cup A_2 \cup ...$ is a null-family. This implies that (*) holds. The proof is complete.

3.9. Example. There exists a continuum $X$ such that:

(i) $X$ is the limit of an inverse sequence of (connected) graphs with monotone bonding surjections (then, by Corollary 3.7, $X$ is a totally regular continuum; this can also be derived from the fact that $X$ is a dendrite), and

(ii) $X$ is not a completely regular continuum.

Let $(r_1, r_2, ...)$ be an enumeration of all rational numbers from the open interval $]0, 1[$. For each positive integer $n$, let $L_n = [r_1] \times \left[ \frac{0}{n}, \frac{1}{n} \right]$ and $X_n = ([0, 1] \times \{0\}) \cup L_1 \cup L_2 \cup \ldots$. Moreover, put $X = ([0, 1] \times \{0\}) \cup L_1 \cup L_2 \cup \ldots$. And, for $n = 1, 2, ...$, let $f_n: X_{n+1} \rightarrow X_n$ be defined by the formula:

$$f_n(x) = \begin{cases} x & \text{if } x \notin X_n, \\ f_{n+1}(x) & \text{if } x \in L_{n+1}. \end{cases}$$

Observe that each $X_n$ is a graph, each $f_n$ is a monotone retraction, $X$ is a dendrite which is not a completely regular continuum (because $[0, 1] \times \{0\}$ is a nowhere dense subcontinuum of $X$), and $X$ is homeomorphic to $\liminf(X_n, f_n)$.

3.10. Remarks. (i) Observe that, for any completely regular continuum $X$, the inverse sequence $(X_n, f_n)$ which was constructed in the proof of Theorem 3.8 has the following additional property:

(+) there exist triangulations $K_n$ of $X_n$ such that $f_n: (X_{n+1}, K_{n+1}) \rightarrow (X_n, K_n)$ is a simplicial map for $n = 1, 2, ...$

Furthermore, it is easy to give a straightforward proof that if a continuum $X$ is the limit of an inverse sequence $(X_n, f_n)$ of connected graphs with monotone bonding maps which fulfill the condition (+), then $X$ is completely regular.

(ii) Recall that limits of inverse sequences of connected graphs with `monotone simplicial bonding maps' have been considered in [3]; but: $(X_n, f_n)$ was called there an inverse sequence of graphs with monotone simplicial bonding maps provided, for every $n$, there exist triangulations $K_n$ and $K'_n$ of $X_n$ such that $f_n: (X_{n+1}, K'_{n+1}) \rightarrow (X_n, K_n)$ is simplicial. It follows that each dendrite is the limit of the inverse sequence of finite dendrites with monotone simplicial bonding maps. Now, recall that there exist dendrites which are not completely regular continua (Example 3.9, above).

(iii) Recall that a continuum $X$ is strongly regular [3]; see also Theorem 2 of [9] if there exists a sequence $S_1, S_2, ...$ of finite subsets of $X$ such that, for each $n$, $X - S_n$ has finitely many components and each component of $X - S_n$ has diameter less than $1/n$. It follows that every strongly regular continuum is regular and, moreover, each totally regular continuum is strongly regular (see [9], Theorem 2, p. 230).

In [3], p. 219, it was stated (without 'proof') that the limit of an inverse sequence of strongly regular continua with monotone bonding maps is again a strongly regular continuum. Unfortunately, this is not true. In fact, since any regular continuum without cutpoints is strongly regular [9], Corollary 2, p. 231), it follows that each space $X_n$, $n = 1, 2, ...$, of Example 1.1 is strongly regular. However, the inverse limit $X$ of Example 1.1 is not even a rational continuum.

(iv) If $X$ is a regular continuum (resp. a connected graph, a dendrite), $Y$ a continuum and $f: X \rightarrow Y$ a monotone surjection, then $Y$ is again a regular continuum (resp. a connected graph, a dendrite) — see for example [7], p. 85. In [9], Remark on p. 232, it was noted that the proof of Theorem 5 of [3], p. 225, can be modified to show that each regular continuum is the monotone image of some completely regular continuum.

3.11. Problem. Characterize continua which are the limits of an inverse sequence of (connected) graphs (resp. completely regular continua) with monotone bonding surjections. In particular, is every totally regular continuum such an inverse limit?

3.12. Problem. Let $Y$ be a connected subset of a totally regular continuum. Does it follow that $Y$ is arcwise connected?

3.13. Problem. Does there exist a universal totally regular continuum?

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1) each totally regular continuum is the limit of an inverse sequence of connected graphs with monotone bonding surjections;

2) there exist totally (even: completely) regular continua with connected subsets which are not arcwise connected.

References


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Separating collections

by

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Abstract. A collection $\mathcal{E}$ of sets is said to be a separating collection if it satisfies the following:

whenever $f$ and $g$ are two functions defined on a set $X \in \mathcal{E}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathcal{E}$, $Y \subseteq X$ such that $f''Y \cap g''Y = \emptyset$. We characterize separating collections in terms of a weak version of the partition relation $\mathcal{E} \rightarrow (\mathcal{E}, 3)^{+}$ and we show this new partition relation holds for every countable indecomposable ordinal (although it can fail for indecomposable ordinals of cardinality $\omega_1$). We also characterize one-one separating collections (i.e., those where we only consider functions $f$ and $g$ that are one-to-one) and derive from this some known and new results.

1. Introduction. If $I$ is a collection of subsets of an infinite cardinal $\kappa$, then $I$ is said to be an ideal on $\kappa$ if $I$ is closed under subset formation and finite unions (i.e., if $X, Y \in I$ and $Z \subseteq X \cup Y$, then $Z \in I$). A subset of $\kappa$ not in $I$ is said to have positive $I$-measure and the collection of such sets is denoted by $I^{+}$; a subset of $\kappa$ whose complement belongs to $I$ is said to have $I$-measure one and the collection of such sets is denoted by $I^{*}$. The following definitions generalize some ideal theoretic notions from [MPT].

DEFINITION 1.1. A collection $\mathcal{E}$ of infinite sets is said to be:

(i) a separating collection if for every pair of functions $f$ and $g$ defined on a set $X \in \mathcal{E}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathcal{E}$ such that $Y \subseteq X$ and $f''Y \cap g''Y = \emptyset$.

(ii) a one-one separating collection if for every pair of one-to-one functions $f$ and $g$ defined on a set $X \in \mathcal{E}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathcal{E}$ such that $Y \subseteq X$ and $f''Y \cap g''Y = \emptyset$.

If $I$ is an ideal on $\kappa$ and $I^{*}$ is a separating collection (or a one-one separating collection), then we will refer to $I$ as a separating ideal (or a one-one separating ideal). A uniform ultrafilter $U$ on $\kappa$ that is a separating collection is referred to as a separating ultrafilter. The following easy proposition shows that although Definition 1.1 suggests that "separating" is a property that pertains to very general collections of sets, it really is a notion that belongs in the context of ideals.

PROPOSITION 1.2. Suppose that $\mathcal{E}$ is a separating collection and let $I_{\mathcal{E}}$ be given by $Y \in I_{\mathcal{E}}$ if $\mathcal{E}(Y) \cap \mathcal{E} = \emptyset$.

Then $I_{\mathcal{E}}$ is an ideal.

$\mathcal{E}(Y)$ denotes the collection of all sets $X$ in $\mathcal{E}$ such that $X \subseteq Y$. This is a good place to stop for now.