the proof of Theorem 2.2. With a minor modification of the arguments used there we prove the following claims:

CLAIM 1. *There exists a set* $Z_1 \subset Z$ *such that*

(i) $Z_1 \notin J$,

(ii) $\forall A \subset Z_1\ \forall g \in G\ (g[A] \subset Z_1 \to A \Delta g[A] \in J)$:

CLAIM 2. *There exist a set* $Z \subset Z_1$ *and a family* $F \subset G$ *such that*

(i) $Z \notin J$ *and* $|F| < \lambda$,

(ii) $\forall A \subset Z\ \forall g \in G\ (g[A] \cap \bigcup_{f \in F} f[Z] = \emptyset \to A \in J)$.

This can be regarded as a satisfactory outline of the proof of Theorem 3.4.

References

[1] P. Erdös, R. D. Mauldin, *The nonexistence of certain invariant measures*, Proc. Amer. Math. Soc. 59 (1976), 321–322.

[2] A. B. Harazishvili, *O niekotorych tipach invariantnych mier*, Dokl. A. N. SSSR 222 (3) (1975), 538–540.

[3] P. J. Nyikos, *A provisional solution to the normal Moore space problem*, Proc. Amer. Math. Soc. 78 (1980), 429–435.

[4] A. Pelc, *Invariant measures and ideals on discrete groups*, Dissertationes Math. 255 (1986), 1–51.

[5] C. Ryll-Nardzewski, R. Telgarsky, *The nonexistence of universal invariant measures*, Proc. Amer. Math. Soc. 69 (1978), 240–242.

[6] P. Zakrzewski, *On universal semiregular invariant measures*, Journal of Symbolic Logic 53 (1988), 1170–1176.

[7] — *Paradoxical decompositions and invariant measures*, to appear.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
PKiN IX p.
00-901 Warsaw, Poland

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Locally connected curves viewed as inverse limits

by

Jacek Nikiel (Wrocław)

Abstract. Every locally connected curve is the limit of an inverse sequence of regular continua with monotone bonding surjections. Moreover, any space which is the limit of an inverse sequence of connected graphs with monotone bonding surjections is a rather small continuum.

1. Introduction. All spaces considered in this paper are assumed to be metric, all maps are continuous, d always denotes a distance function, and 'continuum' means 'compact connected (metric) space'.

We will say that a space X is:

(a) a *graph* provided X is a one-dimensional (compact) polyhedron;

(b) a *completely regular continuum* provided X is a continuum such that $\operatorname{int}(Y) \neq 0$ for each nondegenerate subcontinuum Y of X;

(c) a *regular continuum* if X is a continuum such that for any $\varepsilon > 0$ and each $x \in X$ there exists an open neighbourhood U of x in X such that $\operatorname{bd}(U)$ is finite and $\operatorname{diam} U < \varepsilon$ (regular continua are often called 'rim-finite continua');

(d) a *curve* provided X is a continuum of dimension 1.

Clearly, every connected graph is a completely regular continuum and every regular continuum is a locally connected curve. Moreover, each completely regular continuum is regular (see for example Proposition 3.2 below).

Recall that a (continuous) map $f\colon X \to Y$ is said to be *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$.

It is well-known that every curve X is the limit of some inverse sequence (X_n, f_n) of (connected) graphs (see e.g. [2], Theorem 1.13.2, p. 145; it is not difficult to see that the sequence can be chosen in such a manner that all the bonding maps, $f_n\colon X_{n+1} \to X_n$, are surjections). If X is locally connected, one can use the general method of S. Mardešić to produce an inverse sequence (Y_n, g_n) of locally connected continua Y_n with monotone bonding surjections $g_n\colon Y_{n+1} \to Y_n$ such that $X = \operatorname{liminv}(Y_n, g_n)$ ((Y_n, g_n) is obtained as a 'modification' of (X_n, f_n); see [8], p. 164 — the proof of Theorem 2). However, in general, almost nothing can be proved about (Y_n, g_n). In particular, Y_n's need not (and often they can not) be graphs; they are simply locally connected continua. The only essential information on

Y_n's is a consequence of the elementary fact that all the projections $h_n\colon X \to Y_n$ are monotone surjections ([1], Lemma 4.2, p. 241). For example: if X is a connected graph (resp. a regular continuum, ...) then each Y_n is also a connected graph (resp. a regular continuum, ...; see e.g. [7], p. 85). On the other hand, recall that if (X_n, f_n) is an inverse sequence of locally connected continua with monotone bonding surjections, then $\operatorname{liminv}(X_n, f_n)$ is again a locally connected continuum ([1], Theorem 4.3, p. 241). Moreover, the limit of an inverse sequence of curves is again a curve ([2], Theorem 1.13.4, p. 149).

This consideration leads to some natural questions which motivate the research reported in this paper. In Chapter 2 we prove that every locally connected curve is the limit of some inverse sequence of regular continua with monotone bonding surjections. In Chapter 3, we show that limits of inverse sequences of connected graphs with monotone bonding surjections are 'very small' continua.

Let X be a space and A a family of subsets of X. We write

$$\operatorname{mesh} A = \sup\{\operatorname{diam} Y\colon\ Y \in A\}.$$

A is said to be a *null-family* if for any $\varepsilon > 0$ the collection $\{Y \in A\colon \operatorname{diam} Y > \varepsilon\}$ is finite.

Now, we give an example of a 'large' locally connected curve X, homeomorphic to the limit of an inverse sequence of regular continua with monotone bonding surjections. Appropriately generalized ideas of the construction given in Example 1.1 below will give the proof of the main Theorem 2.2.

1.1. EXAMPLE. Let C denote (in this section only) the Cantor ternary set constructed as usual in $[0, 1]$. For each nonnegative integer n, let A_n be the unique family such that:

(a) A_n consists of exactly 2^n elements which are pairwise disjoint closed intervals;

(b) the length of each member of A_n is equal to $\frac{1}{3^n}$; and

(c) $C \subset \bigcup A_n$.

Then the family $A = \{I \cap C\colon I \in A_n, n = 0, 1, \ldots\}$ is a basis for C and A consists of (nonempty) closed-open subsets of C. Moreover, $C = \bigcap_{n=0}^{\infty} \bigcup A_n$.

For each $n = 0, 1, \ldots$, put $D_n = \left\{I \times \left\{\frac{k}{2^n}\right\}\colon I \in A_n, k = 0, 1, \ldots, 2^n\right\}$ and $E_n = \bigcup \bigcup_{k=n}^{\infty} D_k$.

Let $X = (C \times [0, 1]) \cup E_0$. Clearly, X is a locally connected curve in the plane. Moreover, X is not regular; in fact, X is rather 'large' — because it contains an uncountable family $\{\{c\} \times [0, 1]\colon c \in C\}$ of pairwise disjoint nondegenerate subcontinua.

For $n = 0, 1, \ldots$, let F_n denote the decomposition of X into the components of E_n and points; put $X_n = X/F_n$ and let $g_n\colon X \to X_n$ be the projection. Note that

each F_n is upper semicontinuous (because the family of components of E_n is a null-family). Since F_{n+1} is a refinement of F_n, there is a (unique) map $f_n\colon X_{n+1} \to X_n$ such that $g_n = f_n \circ g_{n+1}$, for $n = 0, 1, \ldots$ Observe that each f_n is a (continuous) monotone surjection. Hence, there is an induced surjection $g\colon X \to \operatorname{liminv}(X_n, f_n)$. Since the maps g_n, $n = 0, 1, \ldots$, separate points of X, g is one-to-one. Thus g is a homeomorphism. It is not difficult to check that each X_n is a (planable) regular continuum.

2. Inverse limits of regular continua.

2.1. LEMMA. *If A is a closed zero-dimensional subset of a locally connected continuum X, then for any $\varepsilon > 0$ there exists a finite collection K of pairwise disjoint subcontinua of X such that* $\operatorname{mesh} K < \varepsilon$ *and* $A \subset \bigcup K$.

Proof. Since A is compact zero-dimensional, there exists a (finite) family L of pairwise disjoint nonempty closed-open subsets of A such that $\operatorname{mesh} L < \varepsilon/3$ and $A = \bigcup L$. Let $r = \min\{d(B, C)\colon B, C \in L, B \neq C\}$. For each $x \in X$ let U_x be an open neighbourhood of x such that U_x is connected and $\operatorname{diam} U_x < \min\{r/2, \varepsilon/3\}$. Since A is compact, there are $x_1, \ldots, x_n \in A$ such that $A \subset U_{x_1} \cup \ldots \cup U_{x_n}$. It suffices to let K be the family of all components of $\operatorname{cl}(U_{x_1}) \cup \ldots \cup \operatorname{cl}(U_{x_n})$. In fact, then each $D \in K$ is contained in some set

$$E_B = \bigcup \{\operatorname{cl}(U_{x_i})\colon \operatorname{cl}(U_{x_i}) \cap B \neq \emptyset\}, \quad B \in L.$$

Clearly, $\operatorname{diam} E_B < \varepsilon$ and

$$E_B \cap E_C = \emptyset \quad \text{if } B, C \in L, B \neq C.$$

2.2. THEOREM. *If X is a locally connected curve, then there exists an inverse sequence (X_n, f_n) such that*:

(i) *each X_n is a regular continuum*;

(ii) *each $f_n\colon X_{n+1} \to X_n$ is a monotone surjection*;

(iii) $X = \operatorname{liminv}(X_n, f_n)$.

Proof. Since $\dim X = 1$, there exists a sequence $P_1, P_2, \ldots$ such that, for each n, P_n is a covering of X, which consists of finitely many open sets, $\operatorname{mesh} P_n < 1/n$ and $A_n = \bigcup \{\operatorname{bd}(U)\colon U \in P_n\}$ is zero-dimensional.

Put $\varepsilon_1 = 1$ and let $K_1 = L_{1,1}$ be a finite collection of pairwise disjoint subcontinua of X such that $\operatorname{mesh} K_1 < \varepsilon_1$ and $A_1 \subset \bigcup K_1$ (see Lemma 2.1).

Suppose that for a positive integer n, a positive real number ε_n and families $L_{n,k}$, $1 \leqslant k \leqslant n$, are already defined such that:

(1) each $L_{n,k}$ is a finite collection of pairwise disjoint subcontinua of X,

(2) $$A_k \cup \ldots \cup A_n \subset \bigcup L_{n,k}, \quad \text{for each } k,$$

and

(3) if $1 \leqslant k < l \leqslant n$ and $B \in L_{n,l}$, then there is a $C \in L_{n,k}$ such that $B \subset C$.

For any k, $1 \leqslant k \leqslant n$, let $r_k^n = \min\{d(B, C)\colon B, C \in L_{n,k}, B \neq C\}$. By (1) $r_k^n > 0$. Put $\varepsilon_{n+1} = \frac{1}{4}\min\{\varepsilon_n, r_1^n, \ldots, r_n^n\}$ and let K_{n+1} be a finite collection of pairwise disjoint subcontinua of X such that $\operatorname{mesh} K_{n+1} < \varepsilon_{n+1}$ and $A_{n+1} \subset \bigcup K_{n+1}$ (see Lemma 2.1). Put $L_{n+1,n+1} = K_{n+1}$ and let $L_{n+1,k}$ be the family of all components of the set $\bigcup (K_{n+1} \cup L_{n,k})$, for $k = 1, \ldots, n$. Obviously, conditions (1)–(3) are satisfied, with n replaced by $n+1$.

Observe that, by the choice of r_k^{n-1} and ε_n,

(4) if $B \in L_{n,k}$, $1 \leqslant k < n$, then either $B \in K_n$ and $B \cap \bigcup L_{n-1,k} = \emptyset$, or there is a $B' \in L_{n-1,k}$ such that $B = B' \cup \bigcup \{C \in K_n\colon C \cap B' \neq \emptyset\}$.

Moreover,

$$\varepsilon_{n+1} \leqslant \tfrac{1}{4}\varepsilon_n . \tag{5}$$

For each positive integer k and any sequence $B_m, B_{m+1}, \ldots$ with the properties:

$$k \leqslant m; \quad B_m \cap \bigcup L_{m-1,k} = \emptyset \quad \text{for } k < m, \tag{6}$$

and

$$B_n \in L_{n,k} \quad \text{and} \quad B_n \subset B_{n+1} \quad \text{for} \quad n = m, m+1, \ldots, \tag{7}$$

let $C_k(B_m, B_{m+1}, \ldots) = \operatorname{cl}(B_m \cup B_{m+1} \cup \ldots)$. Hence

(8) each $C_k(B_m, B_{m+1}, \ldots)$ is a continuum.

For any positive integer k let $M_k = \{C_k(B_m, B_{m+1}, \ldots)\colon$ the sequence $B_m, B_{m+1}, \ldots$ satisfies conditions (6) and (7)$\}$. We show that

(9) if $C_k(B_m, B_{m+1}, \ldots)$, $C_k(B'_n, B'_{n+1}, \ldots) \in M_k$ and either $m \neq n$ or $m = n$ and $B_m \neq B'_m$, then $C_k(B_m, B_{m+1}, \ldots) \cap C_k(B'_n, B'_{n+1}, \ldots) = \emptyset$

and

(10) $\operatorname{diam} C_k(B_m, B_{m+1}, \ldots) < \dfrac{5}{3\cdot 4^{m-1}} \leqslant \dfrac{5}{3\cdot 4^{k-1}}$.

To prove (9) first consider the case $m \neq n$. We may assume that $k \leqslant n < m$. By (6), $B_m \cap B'_{m-1} = \emptyset$ — because $B_m \in L_{m,k}$ and $B'_{m-1} \in L_{m-1,k}$. By (4), $B_m \in K_m$ and $B_m \cap B'_m = \emptyset$. By the choice of r_k^m, $d(B_m, B'_m) \geqslant r_k^m \geqslant 4\cdot\varepsilon_{m+1}$. Since $\operatorname{diam} B < \varepsilon_{m+1}$, for $B \in K_{m+1}$, it follows (by (4)) that $d(B_{m+1}, B'_{m+1}) > r_k^m - 2\cdot\varepsilon_{m+1} \geqslant 2\cdot\varepsilon_{m+1}$. Now, an easy inductive proof shows that, for each positive integer i,

$$\begin{aligned} d(B_{m+i}, B'_{m+i}) &> r_k^m - 2\cdot\varepsilon_{m+1} - 2\cdot\varepsilon_{m+2} - \ldots - 2\cdot\varepsilon_{m+i} \\ &\geqslant 2\cdot\varepsilon_{m+1} - 2\cdot\varepsilon_{m+2} - \ldots - 2\cdot\varepsilon_{m+i} \\ &\geqslant 2\cdot\varepsilon_{m+1} - 2\cdot\varepsilon_{m+1}\left(\frac{1}{4} + \ldots + \frac{1}{4^i}\right) > \frac{4}{3}\varepsilon_{m+1} . \end{aligned}$$

Therefore,

$$d(C_k(B_m, B_{m+1}, \ldots), C_k(B'_n, B'_{n+1}, \ldots)) \geqslant \tfrac{4}{3}\varepsilon_{m+1} > 0 .$$

Suppose that $m = n$ and $B_m \neq B'_m$. Then

$$d(B_m, B'_m) \geqslant r_k^m \geqslant 4\cdot\varepsilon_{m+1}$$

and, just as above,

$$d(C_k(B_m, B_{m+1}, \ldots), C_k(B'_m, B'_{m+1}, \ldots)) \geqslant \tfrac{4}{3}\varepsilon_{m+1} > 0 .$$

Now, we estimate $\operatorname{diam} C_k(B_m, B_{m+1}, \ldots)$. Observe that $B_m \in K_m$ (because of (4) and (6)). Hence $\operatorname{diam} B_m < \varepsilon_m$. An inductive proof shows that, by (4) and (5),

$$\begin{aligned} &\operatorname{diam}(B_m \cup B_{m+1} \cup \ldots \cup B_{m+i}) \\ &\quad < \varepsilon_m + 2\cdot\varepsilon_{m+1} + 2\cdot\varepsilon_{m+2} + \ldots + 2\cdot\varepsilon_{m+i} \leqslant \varepsilon_m + 2\cdot\varepsilon_m\left(\frac{1}{4} + \ldots + \frac{1}{4^i}\right) \\ &\quad < \frac{5}{3}\varepsilon_m \leqslant \frac{5}{3\cdot 4^{m-1}} . \end{aligned}$$

This establishes (10).

By (8) and (9), we see that

(11) each M_k consists of pairwise disjoint continua.

Moreover, by (10) and the fact that each $L_{m,m}$ is finite (see (1)), we infer that

(12) each M_k is a null-family.

For each k let $N_k = M_k \cup \{\{x\}\colon x \in X - \bigcup M_k\}$. By (11) and (12), N_k is a monotone upper semi-continuous decomposition of X. Let X_k denote the quotient space, $X_k = X/N_k$, and let $g_k\colon X \to X_k$ be the quotient map, for $k = 1, 2, \ldots$ Clearly,

(13) each g_k is a monotone surjection.

We show that

(14) X_k is regular for $k = 1, 2, \ldots$

Let x be a point of X_k and U an open neighbourhood of x in X_k. There is a positive real number ε such that

$$g_k^{-1}(x) \subset \{y \in X\colon d(y, g_k^{-1}(x)) < \varepsilon\} \subset g_k^{-1}(U) .$$

Let n be an integer such that $n \geqslant k$ and $\operatorname{mesh} P_n < \dfrac{1}{n} \leqslant \varepsilon$. Put

$$Q = \bigcup \{A\colon A \in P_n \text{ and } A \cap g_k^{-1}(x) \neq \emptyset\} .$$

Then $g_k^{-1}(x) \subset \operatorname{int}(Q) = Q \subset g_k^{-1}(U)$. Since P_n is finite, $\operatorname{bd}(Q) \subset A_n$. By (2), there are finitely many sets $C_1, \ldots, C_l \in M_k$ such that $\operatorname{bd}(Q) \subset C_1 \cup \ldots \cup C_l$ and $\operatorname{bd}(Q) \cap C_i \neq \emptyset$ for $i = 1, \ldots, l$. Put

$$W = Q - (C_1 \cup \ldots \cup C_l) .$$

Obviously, W is open in X. Observe that, by (11), if $C \in M_k$ and $C \cap Q \neq \emptyset$, then either $C \subset Q$ or $C = C_i$ for some $i \in \{1, \ldots, l\}$. Hence $W = \bigcup\{A \in N_k\colon A \subset Q\}$.

Since $g_k^{-1}(x) \subset Q$, it follows that $g_k^{-1}(x) \subset W$ and $g_k^{-1}(x) \cap C_i = \emptyset$ for $i = 1, \ldots, l$. Put $V = g_k(W)$. Then $x \in V \subset U$, V is open in X_k and $\mathrm{bd}(V) \subset \{g_k(C_1), \ldots, g_k(C_l)\}$ is finite.

Note that, for each $k = 1, 2, \ldots$, there is a unique map f_k: $X_{k+1} \to X_k$ such that

$$g_k = f_k \circ g_{k+1} . \tag{15}$$

In fact, (3) implies that for each $C \in M_{k+1}$ there is a (unique) $D \in M_k$ such that $C \subset D$. Obviously, by (13),

(16) each f_k is a (continuous) monotone surjection.

According to (15), there is the induced map g: $X \to \lim\mathrm{inv}(X_k, f_k)$. Since all g_k's are surjections, g is also a surjection. By (10), the maps $g_1, g_2, \ldots$ separate points of X. Hence g is one-to-one. In view of (14) and (16), the proof of Theorem 2.2 is complete.

2.3. Remark. (i) Recall that the limit of an inverse sequence of curves is again a curve ([2], Theorem 1.13.4, p. 149) and the limit of an inverse sequence of locally connected continua with monotone bonding surjections is again a locally connected continuum ([1], Theorem 4.3, p. 241). Thus Theorem 2.2 gives a characterization of locally connected curves.

(ii) It can be shown that if a continuum X of Theorem 2.2 is planable, then all X_n's can be chosen to be planable regular continua.

3. Totally regular continua and their inverse limits. We will say that X is a *totally regular continuum* if X is a continuum such that, for any countable subset Q of X, each $x \in X$, and each $\varepsilon > 0$, there exists an open neighbourhood U of x in X such that $\mathrm{diam}\, U < \varepsilon$, $\mathrm{bd}(U)$ is finite and $\mathrm{bd}(U) \cap Q = \emptyset$.

3.1. Lemma ([4], Lemma 2, p. 605; see also [5], Theorem 1.3, p. 202). *If X is a nondegenerate completely regular continuum, then there exist subsets C and P_n, $n = 1, 2, \ldots$ of X such that:*

(a) *C is homeomorphic to the Cantor set;*

(b) *for each $n = 1, 2, \ldots, P_n$ is an arc with end-points a_n, b_n;*

(c) *$P_n - \{a_n, b_n\}$ is open in X, for $n = 1, 2, \ldots$;*

(d) *$\{P_1, P_2, \ldots\}$ is a null-family;*

(e) *$P_n \cap C = \{a_n, b_n\}$, for $n = 1, 2, \ldots$;*

(f) *$P_m \cap P_n = \emptyset$, for any positive integers m, n such that $m \neq n$;*

(g) *$X = C \cup P_1 \cup P_2 \cup \ldots$*

3.2. Proposition. (i) *Every completely regular continuum is a totally regular continuum.*

(ii) *Every totally regular continuum is a regular continuum.*

Proof. (i) is an easy corollary to Lemma 3.1, and (ii) is obvious.

3.3. Proposition. *A subcontinuum of a totally regular continuum is also a totally regular continuum.*

3.4. Example. Note that the triangular Sierpiński curve (see e.g. [6], p. 276) X has the following properties: (a) X is a regular continuum — in fact, for each $x \in X$, the order of x in X does not exceed 4; however, (b) X is not a totally regular continuum.

Another example of a regular but not totally regular curve is the Knaster dyadic continuum

$$Y = ([0, 1] \times \{0\}) \cup \{(x, y) \in R^2\colon \left(x - \frac{2k-1}{2^n}\right)^2 + y^2 = \frac{1}{4^n},\ y \geqslant 0,$$
$$k = 1, 2, \ldots, 2^{n-1}, n = 1, 2, \ldots\} .$$

Moreover, Y has the following interesting property: each connected subset of Y is arcwise connected (see [9], Example on p. 232).

3.5. Lemma. *If Y is a hereditarily locally connected continuum, Z a continuum and h: $Y \to Z$ a monotone map, then the set $P = \{z \in Z\colon h^{-1}(z)$ is nondegenerate$\}$ is countable.*

Proof. Indeed, since Y is hereditarily locally connected, the collection $\{h^{-1}(z)\colon z \in P\}$ of subcontinua of Y is easily seen to be a null-family.

3.6. Theorem. *If (X_n, f_n) is an inverse sequence such that all the spaces X_n are totally regular continua and all the bonding maps f_n: $X_{n+1} \to X_n$ are monotone surjections, then $X = \lim\mathrm{inv}(X_n, f_n)$ is a totally regular continuum.*

Proof. By [1], Theorem 4.3, p. 241, X is a locally connected continuum. Let Q be a countable subset of X. For $n = 1, 2, \ldots$ let g_n: $X \to X_n$ denote the projection. Then g_n is a continuous and monotone surjection, and $g_n = f_n \circ g_{n+1}$, for $n = 1, 2, \ldots$ For each n let $P_n = \{y \in X_n\colon g_n^{-1}(y)$ is nondegenerate$\}$.

Suppose that P_n is uncountable for some positive integer n. Hence, there is an $\varepsilon > 0$ such that the set $A = \{y \in P_n\colon \mathrm{diam} g_n^{-1}(y) \geqslant \varepsilon\}$ is uncountable. There exists an integer m such that $m > n$ and $\mathrm{diam} g_m^{-1}(z) < \varepsilon$, for each $z \in X_m$. Put $f = f_n \circ f_{n+1} \circ \ldots \circ f_{m-1}$. Hence f: $X_m \to X_n$ is a monotone surjection. By Lemma 3.5, $B = \{y \in X_n\colon f^{-1}(y)$ is nondegenerate$\}$ is countable (because every regular continuum is hereditarily locally connected). Hence the set $A - B$ is uncountable. Let $y \in A - B$. Since $y \notin B$, $f^{-1}(y) = \{z\}$ for some $z \in X_m$. Recall that $\mathrm{diam} g_m^{-1}(z) < \varepsilon$. However, $g_m^{-1}(z) = (f \circ g_m)^{-1}(y) = g_n^{-1}(y)$. Since $y \in A$, we get $\mathrm{diam} g_n^{-1}(y) \geqslant \varepsilon$, a contradiction. We have proved that the sets P_n, $n = 1, 2, \ldots$, are countable.

Let $x \in X$ and U be an open neighbourhood of x in X. There exist a positive integer k and an open subset W of X_k such that $x \in g_k^{-1}(W) \subset U$. Let $P = g_k(Q) \cup P_k$; so P is a countable subset of X_k. Since W is a neighbourhood of $g_k(x)$ and X_k is totally regular, there exists an open set W' such that $g_k(x) \in W' \subset W$, $\mathrm{bd}(W') = \{y_1, \ldots, y_l\}$ is finite and $\mathrm{bd}(W') \cap P = \emptyset$. Put $V = g_k^{-1}(W')$. Hence V is open and $x \in V \subset U$. Note that $\mathrm{bd}(V) \subset g_k^{-1}(\mathrm{bd}(W')) = g_k^{-1}(y_1) \cup \ldots \cup g_k^{-1}(y_l)$. Since $y_1, \ldots, y_l \notin P_k$, the sets $g_k^{-1}(y_1), \ldots, g_k^{-1}(y_l)$ are singletons. Therefore $\mathrm{bd}(V)$ is finite. Since $y_1, \ldots, y_l \notin g_k(Q)$, it follows that $\mathrm{bd}(V) \cap Q = \emptyset$.

3.7. COROLLARY. *If X is the inverse limit of an inverse sequence of connected graphs (completely regular continua) with monotone bonding surjections, then X is a totally regular continuum.*

3.8. THEOREM. *If X is a completely regular continuum, then there exists an inverse sequence (X_n, f_n) such that:*

(i) *each X_n is a connected graph;*

(ii) *each $f_n\colon X_{n+1}\to X_n$ is a monotone surjection;*

(iii) $X = \lim\operatorname{inv}(X_n, f_n)$.

Proof. Let C and P_n, $n = 1, 2, \ldots$, be subsets of X as in Lemma 3.1. For each positive integer n, let A_n denote the family of all components of the set $Y_n = X - \bigcup_{k=1}^{n} (P_k - \{a_k, b_k\})$. Observe that Y_n is locally connected and compact for $n = 1, 2, \ldots$ Hence, each family A_n is finite. Let B_n denote the decomposition of X into points and the members of A_n. Let X_n denote the quotient space, $X_n = X/B_n$, and let $g_n\colon X\to X_n$ be the quotient map, for $n = 1, 2, \ldots$ Observe that each X_n is a connected graph and each g_n is a monotone surjection. Moreover, for $n = 1, 2, \ldots$ B_{n+1} is a refinement of B_n. Hence there are (unique) monotone surjections $f_n\colon X_{n+1}\to X_n$ such that $g_n = f_n\circ g_{n+1}$, for $n = 1, 2, \ldots$ In order to prove that X is homeomorphic to $\lim\operatorname{inv}(X_n, f_n)$, it suffices to show that

(*) the maps g_n, $n = 1, 2, \ldots$, separate points of X.

By the properties of C and P_n's stated in Lemma 3.1, it follows that $C = \bigcap_{n=1}^{\infty} \bigcup A_n$. Hence $A = A_1 \cup A_2 \cup \ldots$ is a null-family. This implies that (*) holds. The proof is complete.

3.9. EXAMPLE. There exists a continuum X such that:

(i) X is the limit of an inverse sequence of (connected) graphs with monotone bonding surjections (then, by Corollary 3.7, X is a totally regular continuum; this can also be derived from the fact that X is a dendrite), and

(ii) X is not a completely regular continuum.

Let $\{r_1, r_2, \ldots\}$ be an enumeration of all rational numbers from the open interval $]0, 1[$. For each positive integer n, let $I_n = \{r_n\}\times\left[0, \frac{1}{n}\right]$ and $X_n = ([0, 1]\times\{0\})\cup I_1\cup\ldots\cup I_n$. Moreover, put $X = ([0, 1]\times\{0\})\cup I_1\cup I_2\cup\ldots = X_1\cup X_2\cup\ldots$ and, for $n = 1, 2, \ldots$, let $f_n\colon X_{n+1}\to X_n$ be defined by the formula

$$f_n(x) = \begin{cases} x & \text{if } x\in X_n, \\ r_{n+1} & \text{if } x\in I_{n+1}. \end{cases}$$

Observe that each X_n is a graph, each f_n is a monotone retraction, X is a dendrite which is not a completely regular continuum (because $[0, 1]\times\{0\}$ is a nowhere dense subcontinuum of X), and X is homeomorphic to $\lim\operatorname{inv}(X_n, f_n)$.

3.10. Remarks. (i) Observe that, for any completely regular continuum X,

the inverse sequence (X_n, f_n) which was constructed in the proof of Theorem 3.8 has the following additional property:

(+) there exist triangulations K_n of X_n such that $f_n\colon (X_{n+1}, K_{n+1})\to(X_n, K_n)$ is a simplicial map for $n = 1, 2, \ldots$

Furthermore, it is easy to give a straightforward proof that if a continuum X is the limit of an inverse sequence (X_n, f_n) of connected graphs with monotone bonding maps which fulfil the condition (+), then X is completely regular.

(ii) Recall that limits of inverse sequences of connected graphs with 'monotone simplicial bonding maps' have been considered in [3]; but: (X_n, f_n) was called there an inverse sequence of graphs with monotone simplicial bonding maps provided, for every n, there exist triangulations K_n and K'_n of X_n such that $f_n\colon (X_{n+1}, K'_{n+1})\to(X_n, K_n)$ is simplicial. It follows that each dendrite is the limit of the inverse sequence of finite dendrites with monotone simplicial bonding maps. Now, recall that there exist dendrites which are not completely regular continua (Example 3.9, above).

(iii) Recall that a continuum X is *strongly regular* [3]; see also Theorem 2 of [9] if there exists a sequence $S_1 S_2, \ldots$ of finite subsets of X such that, for each n, $X - S_n$ has finitely many components and each component of $X - S_n$ has diameter less than $1/n$. It follows that every strongly regular continuum is regular and, moreover, each totally regular continuum is strongly regular (see [9], Theorem 2, p. 230).

In [3], p. 219, it was stated (without 'proof') that the limit of an inverse sequence of strongly regular continua with monotone bonding maps is again a strongly regular continuum. Unfortunately, this is not true. In fact, since any regular continuum without cutpoints is strongly regular ([9], Corollary 2, p. 231), it follows that each space X_n, $n = 1, 2, \ldots$, of Example 1.1 is strongly regular. However, the inverse limit X of Example 1.1 is not even a rational continuum.

(iv) If X is a regular continuum (resp. a connected graph, a dendrite), Y a continuum and $f\colon X\to Y$ a monotone surjection, then Y is again a regular continuum (resp. a connected graph, a dendrite) — see for example [7], p. 85. In [9], Remark on p. 232, it was noted that the proof of Theorem 5 of [3], p. 225, can be modified to show that each regular continuum is the monotone image of some completely regular continuum.

3.11. PROBLEM. Characterize continua which are the limits of an inverse sequence of (connected) graphs (resp. completely regular continua) with monotone bonding surjections. In particular, is every totally regular continuum such an inverse limit?

3.12. PROBLEM. Let Y be a connected subset of a totally regular continuum. Does it follow that Y is arcwise connected?

3.13. PROBLEM. Does there exist a universal totally regular continuum?

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Added in proof. Problems 3.11 and 3.12 are already answered in the recent paper: R. D. Buskirk, J. Nikiel and E. D. Tymchatyn, *Totally regular curves as inverse limits*, preprint. Namely:

1) each totally regular continuum is the limit of an inverse sequence of connected graphs with monotone bonding surjections;

2) there exist totally (even: completely) regular continua with connected subsets which are not arcwise connected.

References

[1] G. E. Capel, *Inverse limit spaces*, Duke Math. J. 21 (1954), 233–243.
[2] R. Engelking, *Dimension theory*, PWN — Polish Scientific Publishers and North-Holland, 1978.
[3] B. B. Epps, *A classification of continua and weakly confluent mappings*, Colloq. Math. 36 (1976), 217–227.
[4] S. D. Iliadis, *Universal continuum for the class of completely regular continua*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 28 (1980), 603–607.
[5] J. Krasinkiewicz, *On two theorems of Dyer*, Colloq. Math. 50 (1986), 201–208.
[6] K. Kuratowski, *Topology*, Vol. II, Academic Press and PWN — Polish Scientific Publishers, 1968.
[7] T. Maćkowiak, *Continuous mappings on continua*, Dissertationes Math. 158 (1979).
[8] S. Mardešić, *Locally connected, ordered and chainable continua*, Rad Jugoslav. Akad. Znan. Umjet. Odjel. Mat. Fiz. Tehn. Nauke 319 (1961), 147–166 (in Serbo-Croatian, English summary).
[9] E. D. Tymchatyn, *Weakly confluent mappings and a classification of continua*. Colloq. Math. 36 (1976), 229–233.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
Pl. Grunwaldzki 2/4
50-384 Wrocław
Poland

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Separating collections

by

Alan D. Taylor (Schenectady, NY)

Abstract. A collection $\mathscr{X}$ of sets is said to be a *separating collection* if it satisfies the following: whenever f and g are two functions defined on a set $X \in \mathscr{X}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathscr{X}$, $Y \subseteq X$ such that $f''Y \cap g''Y = 0$. We characterize separating collections in terms of a weak version of the partition relation $\mathscr{X} \to (\mathscr{X}, 3)^2$ and we show this new partition relation holds for every countable indecomposable ordinal (although it can fail for indecomposable ordinals of cardinality ω_1). We also characterize one-one separating collections (i.e., those where we only consider functions f and g that are one-to-one) and derive from this some known and new results.

1. Introduction. If I is a collection of subsets of an infinite cardinal $\varkappa$, then I is said to be an *ideal on* $\varkappa$ if I is closed under subset formation and finite unions (i.e., if $X, Y \in I$ and $Z \subseteq X \cup Y$, then $Z \in I$). A subset of $\varkappa$ not in I is said to have *positive I-measure* and the collection of such sets is denoted by I^+; a subset of $\varkappa$ whose complement belongs to I is said to have *I-measure one* and the collection of such sets is denoted by I^*. The following definitions generalize some ideal theoretic notions from [MPT].

Definition 1.1. A collection $\mathscr{X}$ of infinite sets is said to be:

(i) a *separating collection* if for every pair of functions f and g defined on a set $X \in \mathscr{X}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathscr{X}$ such that $Y \subseteq X$ and $f''Y \cap g''Y = 0$.

(ii) a *one-one separating collection* if for every pair of one-to-one functions f and g defined on a set $X \in \mathscr{X}$ so that $f(x) \neq g(x)$ for every $x \in X$, there is a set $Y \in \mathscr{X}$ such that $Y \subseteq X$ and $f''Y \cap g''Y = 0$.

If I is an ideal on $\varkappa$ and I^+ is a separating collection (or a one-one separating collection), then we will refer to I as a *separating ideal* (or a *one-one separating ideal*). A uniform ultrafilter U on $\varkappa$ that is a separating collection is referred to as a *separating ultrafilter*. The following easy proposition shows that although Definition 1.1 suggests that "separating" is a property that pertains to very general collections of sets, it really is a notion that belongs in the context of ideals.

Proposition 1.2. *Suppose that $\mathscr{X}$ is a separating collection and let $I_{\mathscr{X}}$ be given by*

$$Y \in I_{\mathscr{X}} \quad \text{if } \mathscr{P}(Y) \cap \mathscr{X} = 0\,.$$

Then $I_{\mathscr{X}}$ is an ideal.